Research Article

The Prime Filter Theorem for Multilattices

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The aim of this paper is to establish the prime filter theorem for multilattices.

1. Introduction

Benado’s pioneering work on posets [1] laid the foundation for a new theory called multilattice theory. This theory will be consolidated by several authors including Olga [2] and Hansen [3] who proposed many characterizations of multilattices. Subsequently, the work of Cabrera et al. [4, 5] led to an algebraization of multilattices. Multilattices appear under two inseparable aspects: both as an ordered structure generalizing ordered lattices and as an algebraic hyperstructure generalizing algebraic lattices.

If several authors lean towards this theory, it is precisely because of its numerous applications in decision making, in uncertainty modeling, in fuzzy logic, and so on [6]. To further broaden the scope of multilattices, mathematical concepts such as filter/ideal, congruence, and homomorphism [4, 5, 7] will be studied. However, in this new structure, the notion of associativity is replaced by a stronger notion of m-associativity. We thus lose the interest to define the concepts in a canonical way as well as the notion of distributivity is nontrivial [7], which requires more caution when extending some results such as the prime filter theorem.

I.P. Cabrera et al. proved in [5] that the set of all filters of a multilattice is a lattice with respect to inclusion. We observe that for some multilattices, the derived filter lattice is distributive. This allows us to use the notion of distributivity in multilattices.

The paper is divided into two sections. Section 1 concerns the preliminaries: we recall the definitions and results necessarily to understand the paper and we develop some examples to illustrate the proofs. Section 2 brings out the newly established results.

2. Preliminaries

Let \((P, \leq)\) be a poset, and let \(S \subseteq P\) be a nonempty subset of \(P\). An element \(u \in P\) is said to be an upper bound of \(S\) if it satisfies \(s \leq u\) for all \(s \in S\). Dually, \(l \in P\) is a lower bound of \(S\) if it satisfies \(l \leq s\) for all \(s \in S\).

The set of upper bounds (resp., lower bounds) of \(S\) is denoted by \(\uparrow S\) (resp., \(\downarrow S\)). In particular, if \(S = \{s\}\), then we write \(\uparrow s\) (resp., \(\downarrow s\)) instead of \(\{s\}\) (resp., \(\{s\}\)). We also use the following notations and definitions: \(\uparrow S = \bigcup_{u \in S} \uparrow u\) and \(\downarrow S = \bigcup_{s \in S} \downarrow s\).

**Definition 1** (see [1]). A multisupremum (resp., multininfimum) of \(S\) is a minimal (resp., maximal) element of \(\uparrow S\) (resp., \(\downarrow S\)).

We will denote by \(x \sqcup y\) (resp., \(x \sqcap y\)) the set of multisuprema (resp., multininfima) of \([x, y]\). When \([x, y]\) has a unique multisupremum (resp., multininfimum), it is denoted by \(x \sqcup y\) or \(\sup (x, y)\) (resp., \(x \sqcap y\) or \(\inf (x, y)\)). We will also use the following notations and definitions: for all nonempty subsets \(X\) and \(Y\) of \(M\),

\[
X \sqcup Y = \bigcup_{(x, y) \in X \times Y} x \sqcup y; \quad X \sqcap Y = \bigcup_{(x, y) \in X \times Y} x \sqcap y.
\]

(1)
Definition 2 (see [1]). Let \((M; \leq)\) be a poset. \(M\) is said to be an ordered multilattice if the following conditions hold for all \(x, y, a \in M\):

\[
\begin{align*}
M_1: & \quad x \leq a \text{ and } y \leq a \text{ imply that there exists } z \in x \sqcup y \text{ such that } z \leq a \\
M_2: & \quad a \leq x \text{ and } a \leq y \text{ imply that there exists } z \in x \sqcap y \text{ such that } a \leq z
\end{align*}
\]

The previous definition gives rise to two hyperoperations which allow us to look at multilattice from an algebraic perspective.

\[
\begin{align*}
\sqcup: & \quad M^2 \rightarrow 2^M \text{ and } \sqcap: & \quad M^2 \rightarrow 2^M \\
(a, b) & \mapsto a \sqcup b \text{ and } (a, b) \mapsto a \sqcap b.
\end{align*}
\]

These hyperoperations satisfy the following properties called axioms of multilattices. In [2, 3], many characterizations are proposed.

**AM1:** For all \(x \in M\), \(x \sqcup x = \{x\}, \forall x \in x\); 
**AM2:** For all \(x, y \in M\), \(x \sqcup y = y \sqcup x, \forall x \in x\); 
**AM3:** For all \(x, y, z \in M\), \(x \leq y \Rightarrow (x \sqcup y) \sqcup z \sqsubseteq (x \sqcup z) \sqcup (y \sqcup z), \forall x \in x\); 
**AM4:** For all \(x, y \in M\), \(x \sqcup (x \sqcap y) = x \sqcap (x \sqcup y) = \{x\}, \forall x \in x\); 
**AM5:** For all \(x, y \in M\), \(x \leq y \Rightarrow x \sqcup y = \{y\} = x \sqcap y = \{x\}, \forall x \in x\).

Definition 3 (see [5]). An algebraic multilattice is any triple \((M; \sqcup, \sqcap)\) which satisfies the properties AM1-AM5. 
Ordered multilattices and algebraic multilattices are connected as follows.

**Theorem 1.** The following assertions are satisfied:

I. If \((M; \sqcap)\) is an ordered multilattice, set

\[
x \sqcup y = \text{multisup}(x, y) \text{ and } x \sqcap y = \text{multinf}(x, y) \text{ for all } x, y \in M.
\]

Then, \((M; \sqcup, \sqcap)\) is an algebraic multilattice.

II. Conversely if \((M; \sqcup, \sqcap)\) is a multilattice, set

\[
x \leq y \iff x \sqcap y = \{y\} = x \sqcup y = \{x\}. 
\]

Then, \((M; \leq)\) is an ordered multilattice.

As the two aspects mentioned in Theorem 1 are inseparable, it is possible to manipulate multilattices using both the underlined order and the deduced hyperoperations. We can, therefore, use the following three notations to designate multilattice without fear of ambiguity:

\[
M = (M; \leq) = (M; \sqcup, \sqcap) = (M; \leq, \sqcup, \sqcap).
\]

We simply write \(M\) assuming that the underlined order and the corresponding hyperoperations are understood.

**Definition 4.**

1. \(M\) is said to be a complete multilattice if every subset \(\mathcal{L}\) of \(M\) has at least one multisupremum and at least one multinfimum.
2. \(\sqcup\)-full multilattice (resp., \(\sqcap\)-full multilattice) is a multilattice in which \(\forall x \sqcup y \neq \emptyset\) (resp., \(\forall x \sqcap y \neq \emptyset\)) for all \(x, y \in M\). A multilattice is said to be full if it is both \(\sqcup\)-full and \(\sqcap\)-full.

There are several proposals of the definition of homomorphism between hyperstructures. The most used definition was introduced by Benado in [1].

**Definition 5.** (see [5]). A map \(h: M_1 \rightarrow M_2\) between multilattices \((M_1, \sqcup_1, \sqcap_1)\) and \((M_2, \sqcup_2, \sqcap_2)\) is said to be a homomorphism if

\[
h(a \sqcup_1 b) \leq h(a) \sqcup_1 h(b) \text{ and } h(a \sqcap_1 b) \leq h(a) \sqcap_1 h(b) \text{ for all } a, b \in M_1.
\]

(6)

**Remark 1.** If the initial multilattice is full, then the condition to be a multilattice homomorphism can be expressed in terms of equalities as follows:

\[
h(a \sqcup_1 b) = h(a) \sqcup_1 h(b) \text{ and } h(a \sqcap_1 b) = h(a) \sqcap_1 h(b) \text{ for all } a, b \in M_1.
\]

(7)

**Notation 1.** Any lattice \((L; \vee, \wedge)\) induces a multilattice \((L; \sqcup, \sqcap)\) given by \(x \sqcap y = \{x \vee y\}\) and \(x \sqcup y = \{x \wedge y\}\) for all \(x, y \in L\).

In the framework of multilattices, the notions of distributivity and associativity are not quite canonical as stated in [7].

**Proposition 1.** (see [7]). Let \((M, \sqcup, \sqcap)\) be a multilattice. Then, the following conditions hold:

1. If \(\sqcap\) is canonically associative, then \(M\) collapses to a lattice
2. If \(\sqcup\) (resp., \(\sqcap\)) is canonically distributive to \(\sqcup\) (resp., \(\sqcap\)), then \(M\) collapses to a distributive lattice

The notion of filter is studied in [5] in relation to homomorphism and congruences.

**Definition 6.** (see [5]). A nonempty subset \(F\) of \(M\) is said to be a filter if it satisfies the following conditions:

F-1: For all \(x, y \in F, x \sqcap y \subseteq F\);
F-2: For all \(a \in M\), \(x \sqcup a \subseteq F\);
F-3: For all \(x, y \in M\) such that \(x \sqcap y \cap \emptyset \neq \emptyset\) or \(x \sqcup y \subseteq F\).

Dually, a nonempty subset \(I\) of \(M\) is said to be an ideal if it satisfies the following conditions:

I-1: For all \(x, y \in I, x \sqcup y \subseteq I\)
**I-2:** For all $a \in M$ and $x \in I$, $a \sqcap x \subseteq I$

**I-3:** For all $x, y \in M$ such that $(x \sqcap y) \cap I \neq \emptyset$, $x \sqcap y \subseteq I$

**Notation 2.** The set of all filters of $(M, \sqcap, \sqcup)$ will be denoted by $\mathcal{F}(M)$.

**Proposition 2** (see [5]). Let $M$ be an $\sqcup$-full multilattice. Then, $\mathcal{F}(M)$ is a lattice under the set inclusion.

The lattice $\langle \mathcal{F}(M), \sqcap, \sqcup \rangle$ is given by $F_1 \sqcap F_2 = F_1 \cap F_2$ and $F_1 \sqcup F_2$ is the smallest filter of $M$ containing both $F_1$ and $F_2$. When $M$ is not $\sqcup$-full, it is necessary to add the empty set to $\mathcal{F}(M)$ (lifting) in order to obtain a lattice. That is, $\mathcal{F}(M) = \mathcal{F}(M) \cup \{\emptyset\}$ is always a lattice ordered by set inclusion.

**Example 1.** The following diagrams give two multilattices and their filter lattices.

The smallest filter of $M$ containing $\emptyset \neq X \subseteq M$ is the intersection of all filters of $M$ containing $X$. It is denoted by $\langle X \rangle$, and if $X = \{x\}$, we write $\langle x \rangle$ instead of $\langle \{x\} \rangle$.

**3. Main Results**

For every nonempty subset $X$ of $M$, set

$$X_\ast = \bigcup \{a \cup b \mid (a \sqcup b) \cap (\uparrow X) \neq \emptyset, a, b \in X\}.$$  \hfill (8)

When $X = \{x\}$, we denote $x_\ast$ instead of $\{x\}_\ast$. We shall offer a description of the filter generated by $X$. We first list some properties of the operator $(-)_\ast$.

**Proposition 3.** For every nonempty subsets $X$ and $Y$ of $M$, the following conditions hold:

(i) $X \subseteq \uparrow X \subseteq X_\ast$.

(ii) $(X \cap Y)_\ast \subseteq X_\ast \cap Y_\ast$.

(iii) $(X \cup Y)_\ast = X_\ast \cup Y_\ast$. We have the following characterization of filters.

**Lemma 1.** Let $F$ be a nonempty subset of $M$. Then, $F$ is a filter of $M$ if and only if $\forall n \in \mathbb{N}: F \cap F_\ast = F$ and $F_\ast = F$.

**Proof.** Suppose that $F$ is a filter of $M$.

Let $z \in M$. If $z \in F \cap F_\ast$, then $z \in x \sqcap y$ for some $x, y \in F$. Since $F$ is a filter of $M$, $x \sqcap y \subseteq F$ and so $z \in F$. Thus, $F \cap F_\ast = F$. If $z \in F$, then $z \in x \sqcap y \subseteq F$. Thus, $F \subseteq F \cap F_\ast = F$, and it follows that $F \cap F_\ast = F$.

Clearly, $F \subseteq F_\ast$ (see Proposition 3). If $z \in F$, then there is $x, y \in M$ such that $z \in x \cup y$ and $(x \cup y) \cap (\uparrow F) \neq \emptyset$, but $F = \uparrow F$ (any filter is upward closed) and then $(x \cup y) \cap F \neq \emptyset$, so $x \cup y \subseteq F$ and consequently $z \in F$. Therefore, $F_\ast = F$.

Conversely, suppose that $F \cap F_\ast = F$ and $F_\ast = F$.

Let $a, b \in M$. If $a, b \in F$, then $a \cup b \subseteq F \cap F_\ast = F$; if $a \in F$, then $a \sqcap x \subseteq F \cap F_\ast = F$; and if $b \in F$, then $b \sqcap x \subseteq F \cap F_\ast = F$; finally, if $(a \cup b) \cap F \neq \emptyset$, then $a \cup b \subseteq F$. Hence, $F$ is a filter of $M$.

We define a sequence $(X_n)_{n \in \mathbb{N}}$ recursively as follows:

$$X_0 = X, \quad X_1 = X_\ast, \text{and } \forall n \geq 1, X_{n+1} = (X_n \cap X_\ast)_\ast.$$ \hfill (9)

**The following summarizes the main properties of this sequence.**

**Proposition 4.** Let $X$ be a nonempty subset of $M$. Then, the sequence $(X_n)_{n \in \mathbb{N}}$ is increasing with respect to the inclusion.

**Proof.** Clearly, $X_n \subseteq X_n \cap X_{n+1}$ and also $X_n \cap X_{n+1} \subseteq (X_n \cap X_{n+1})_\ast = X_{n+1}$. Therefore, $X_n \subseteq X_{n+1}$.

**Theorem 2.** Let $X$ be a nonempty subset of $M$. Then, $\langle X \rangle = \cup X_n$.

**Proof.** Let $a, x, y \in M$. If $x \in X_n$ and $y \in X_{n'}$ for some positive integers $n$ and $n'$ with $n' \geq n''$, then $x \sqcap y \in X_{n''}$, and so $x \sqcap y \subseteq X_n \cap X_{n'} \subseteq X_{n+1}$; if $x \in X_n$ and $n \geq 0$, then $a \sqcup x \subseteq X_n \subseteq X_{n+1}$; if $(x \cup y) \cap X_n \neq \emptyset$, then $x \cup y \subseteq X_n \subseteq X_{n+1}$. Hence, $\cup X_n$ is a filter of $M$. Clearly, $\subseteq \cup X_n$ (in fact $X = X_0 \subseteq \cup X_n$).

Let $F$ be a filter of $M$. We will prove by induction that $\cup X_n \subseteq F$. We have $X_0 = X_\ast \subseteq F$. If $X_n \subseteq F$, then $X_n \cap X_{n+1} \subseteq F \cap F = F$ and then $X_{n+1} = (X_n \cap X_{n+1})_\ast \subseteq F$, (see Lemma 1). Hence, $X_n \subseteq F$ for all $n \in \mathbb{N}$ and it follows that $\cup X_n \subseteq F$.

**Example 2.** We consider the multilattices of Figure 1. Let $X^C = \{a_7\}$, $X^D = \{b_1, b_{10}\}$. We apply Theorem 2 to obtain the filters of $C$ and $D$, respectively, generated by $X^C$ and $X^D$.

| $X_0^C$ | $\{a_7\}$ |
|        | $X_1^C$ = $\{a_{11}, a_{12}, a_{10}, a_7, a_6\}$ |
| $X_2^C$ = $\{a_{11}, a_{12}, a_{10}, a_9, a_7, a_6, a_4, a_3\}$ |
| $X_3^C$ = $\{a_{11}, a_{12}, a_{10}, a_9, a_7, a_6, a_4, a_3, a_1\}$ |
| $X_n^C = X_3^C \uparrow a_i$, for all $n \geq 3$ |
| $X_0^D$ = $\{b_1, b_{10}\}$ |
| $X_1^D$ = $\{a_{11}, a_{12}, a_9, a_7, b_7, b_6, b_4, b_3, b_1\}$ |
| $X_n^D = X_1^D \uparrow b_1$, for all $n \geq 1$ |

Therefore, $\langle X^C \rangle = \uparrow a_7$ and $\langle X^D \rangle = \uparrow b_1$.

**Figure 1:** Multilattices $C$ and $D$.
Lemma 3. Let $\alpha \in \langle x \rangle \land \langle y \rangle$. Then, the following assertions hold:

1. $z \in \cap \langle x \rangle \Rightarrow \langle z \rangle = \langle x \rangle \lor \langle y \rangle$;
2. $z \in \cup \langle x \rangle \Rightarrow \langle z \rangle \subseteq \langle x \rangle \land \langle y \rangle$;
3. $z, z' \in \cup \langle x \rangle \Rightarrow \langle z \rangle \neq \langle z' \rangle$.

The inclusion of (2) will generally be strict. For instance, in the multilattice $C$ of Figure 1, we have $\langle a_4 \rangle \land \langle a_2 \rangle = \uparrow a_1$ but $a_4 \cup a_2 = \not\subseteq a_{12}$ and $\langle a_{12} \rangle = \uparrow a_{12}$ with $\downarrow a_{12} \subseteq \not\subseteq a_1$.

Figure 2 presents the filter lattice of the multilattices $C$ and $D$ of Figure 1. We can easily claim that $\mathcal{F}(C)$ is not distributive since it contains a copy of $N_2$, while $\mathcal{F}(D)$ is distributive. This allows us to use the following concept of distributivity.

Definition 7. A multilattice $M$ is said to be a distributive multilattice if $\mathfrak{B}(M) = \mathcal{F}(M) \cup \{\emptyset\}$ is a distributive lattice.

Lemma 4. Let $F_1, F_2 \in \mathcal{F}(M)$. Then, the following conditions hold:

1. $F_1 \lor F_2 = \bigcup_{x \in F_1 \lor F_2} \langle x \rangle \lor \langle y \rangle$;
2. $F_1 \land F_2 = \bigcap_{x \in F_1 \land F_2} \langle x \rangle \land \langle y \rangle$.

Proof. For (1), we claim that $\mathcal{F}(M) = \bigcup_{x \in \mathcal{F}(M)} \langle x \rangle \lor \langle y \rangle$. We easy to see that $F_1 \subseteq \mathcal{F}(M)$ and $F_2 \subseteq \mathcal{F}(M)$, and every filter of $M$ containing $F_1 \lor F_2$ necessarily contains $\mathcal{F}(M)$.

For (2), let $F_1 \lor F_2 = \bigcup_{x \in \mathcal{F}(M)} \langle x \rangle \lor \langle y \rangle$. We claim that $\mathcal{F}(M)$ is a lattice homomorphism from $\mathcal{F}(M)$ to the lattice $2 = (\{0, 1\}; \lor, \land)$.

Proof. The first implication follows directly from Definition 8. For the converse, suppose that $F_1 \land F_2 \subseteq P$ but $F_1 \not\subseteq P$ and $F_2 \not\subseteq P$. Let $x \in F_1 \land F_2$ and $y \in F_1 \land P$. Then, neither $x \in P$ nor $y \in P$. From (2), we have $\langle x \rangle \land \langle y \rangle \not\subseteq P$. However, from Lemma 5, $\langle x \rangle \land \langle y \rangle \subseteq F_1 \land F_2 \subseteq P$ which is contradiction.

Corollary 1. Let $F_1, F_2 \in \mathcal{F}(M)$. Then, the following conditions are equivalent:

1. $F_1$ is a prime filter
2. For all $x, y \in M, \langle x \rangle \land \langle y \rangle \subseteq P$ implies $x \in P$ or $y \in P$.

Proof. The first implication follows directly from Definition 8. For the converse, suppose that $F_1 \land F_2 \subseteq P$ but $F_1 \not\subseteq P$ and $F_2 \not\subseteq P$. Let $x \in F_1 \land F_2$ and $y \in F_1 \land P$. Then, neither $x \in P$ nor $y \in P$. From (2), we have $\langle x \rangle \land \langle y \rangle \not\subseteq P$. However, from Lemma 5, $\langle x \rangle \land \langle y \rangle \subseteq F_1 \land F_2 \subseteq P$ which is contradiction.

Theorem 3. Let $P \subseteq M$ be a nonempty subset of $M$. Then, $P$ is a filter of $M$ if and only if there exists a lattice homomorphism $f$ from $\mathcal{F}(M)$ to the lattice $2 = (\{0, 1\}; \lor, \land)$ such that $(f^\gamma)^{-1}(0) = P$.

Proof. Suppose that $p$ is prime, then $Ds(P) = \{F \in \mathcal{F}(M) | F \not\subseteq P\}$ is also prime in $\mathcal{F}(M)$ and then there is a lattice homomorphism $f$ from $\mathcal{F}(M)$ to the lattice $2$ such that $Ds(P) = f^{-1}(0)$. However, we observe that $y^{-1}(Ds(P)) = P$. It follows that $(f^\gamma)^{-1}(0) = P$.

Conversely, if $f$ is a lattice homomorphism from $\mathcal{F}(M)$ to $2$, then $f^{-1}(0)$ is a prime filter of $\mathcal{F}(M)$. This implies $(f^\gamma)^{-1}(0) = y^{-1}(f^{-1}(0))$ is a prime filter of $M$.

Corollary 2. Let $F_1, F_2 \in \mathcal{F}(M)$. Then, $F_1$ is a prime filter if and only if there exists a multilattice homomorphism $\varphi$ from $M$ to $2$ such that $\varphi^{-1}(0) = P$.

Proof. Let $f$ be an homomorphism from $\mathcal{F}(M)$ to $2$. Let $x, y \in M$, then on the one hand, we have $f(\langle x \lor y \rangle) = f(\langle x \rangle \lor f(\langle y \rangle)) = f(\langle x \rangle \lor f(\langle y \rangle))$ since $\langle x \rangle \lor y = \langle x \rangle \lor \langle y \rangle$. Hence, $f(\langle x \lor y \rangle) = f(\langle x \rangle \lor f(\langle y \rangle))$. On the other hand, we have $f(\langle x \land y \rangle) = f(\langle x \rangle \land f(\langle y \rangle)) = f(\langle x \rangle \land f(\langle y \rangle))$ which implies $f(\langle x \land y \rangle) \subseteq f(\langle x \rangle \lor f(\langle y \rangle))$. This proves that $f^\gamma$ is a multilattice homomorphism. Then, we use Theorem 3 to obtain the existence of $\varphi$.
Lemma 4, some $Y \subseteq x \cap F(P) \neq \emptyset$. Clearly, filters which strictly contain $X \subseteq x$ with top element, therefore, $\emptyset \neq x \cup y \subseteq (F \cap F_2) \cap I$.

A. Rahanamai–Barghi established the prime ideal theorem for meet-hyperlattices [8] and for distributive hyperlattices [9]. In the following theorem, we state and prove the prime filter theorem for multilattices.

**Theorem 4** (prime filter theorem). Let $M$ be a distributive multilattice. Let $F$ be a proper filter of $M$ and $I$ a proper ideal of $M$ such that $F \cap I = \emptyset$. Then, there exists a prime filter $P$ of $M$ such that $F \subseteq P$ and $P \cap I = \emptyset$.

**Proof.** Let $\mathcal{X}$ be the set of all filters of $M$ containing $F$ and disjoint from $I$.

$$\mathcal{X} = \{G \mid F \subseteq G \subseteq M, G \cap I = \emptyset\}.$$ \hspace{1cm} (12)

We claim that $\mathcal{X}$ satisfies Zorn’s lemma. Since $F \in \mathcal{X}$, the set $\mathcal{X}$ is not empty. Let $\mathcal{G}$ be a chain in $\mathcal{X}$, and let $N = \bigcup \mathcal{G}$. Let $x, y \in N$. Firstly, if $x, y \in M$, then $x \in X_1$ and $y \in X_2$ for some $X_1, X_2 \in \mathcal{G}$; since $\mathcal{G}$ is a chain, either $x \subseteq X_1$ or $x \subseteq X_2$. If $x \subseteq X_2$, then $x, y \in X_2$ and so $x \cap y \subseteq X_2 \subseteq N$. Secondly, if $x \in M$ and $y \in N$, then $y \in Y$ for some $Y \in \mathcal{G}$; since $Y$ is a filter, $x \cap y \subseteq Y \subseteq N$. Thirdly, if $(x \cap y) \cap N \neq \emptyset$, then $(x \cap y) \cap Z \neq \emptyset$ for some $Z \in \mathcal{G}$; since $Z$ is a filter, $(x \cap y) \cap Z \subseteq N$. Thus, $N$ is a filter of $M$. It is obvious that $F \subseteq N$ and $N \cap I = \emptyset$, verifying that $N \in \mathcal{X}$. Therefore, Zorn’s lemma $\mathcal{X}$ has a maximal element, say $P$. We claim that $P$ is prime. Indeed suppose to the contrary that $P$ is not prime. Then, there exists $F_1, F_2 \in \mathcal{F}(M)$ such that $F_1 \wedge F_2 \subseteq P$, $F_1 \not\subseteq P$, and $F_2 \not\subseteq P$. $P \lor F_1$ and $P \lor F_2$ both are filters which strictly contain $P$. Because of the maximality of $P$, clearly $(P \lor F_1) \cap I \neq \emptyset$ and $(P \lor F_2) \cap I \neq \emptyset$. Thus, from Lemma 4, $(P \lor F_1) \wedge (P \lor F_2) \cap I \neq \emptyset$. Since $\mathcal{F}(M)$ is distributive, we have $(P \lor F_1) \wedge (P \lor F_2) \cap I \neq \emptyset$. This implies $P \cap I \neq \emptyset$, a contradiction.

**Corollary 3.** Let $(M, \cup, \cap, \tau)$ be a distributive multilattice with top element, $\tau$. If $I$ is a proper ideal of $M$, then there exists a prime filter $P$ such that $P \cap I = \emptyset$.

**Proof.** Since $I$ is proper, we have $\tau \not\in I$. Hence, we apply Theorem 4 with $F = \{\tau\}$ to obtain the desired conclusion.

**4. Conclusion**

We have established the prime filter theorem and proposed some characterizations.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest.