

Research Article

On the Inequality Theorem for a Wider Class of Analytic Functions with Hadamard Product

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In this paper, we discuss a well known class studied by Ramesha in 1995 and later by Janteng in 2006, and we then extend the class to a wider class of functions f denoted by $n_{\alpha}^{\beta,\gamma}$ which are normalized and univalent, in the unit disk $\mathcal{D} = \{z \in \mathbb{C}: |z| < 1\}$ satisfying the condition $\operatorname{Re}((\alpha z^2 f''(z)/g(z)) + (z f'(z)/g(z))) > \beta$, $0 \leq \alpha < 1$, $0 \leq \beta < 1$, where g is analytic function in \mathcal{D} , such that $g(z) \neq 0$, with a new condition that is introduced. The main purpose of this paper is to give an estimate for the same $|a_3 - \mu a_2^2|$ when f belongs to the class $n_{\alpha}^{\beta,\gamma}$.

1. Introduction and Definition

Let S denote the class of normalized analytic univalent functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \text{ is complex number}, \quad (1)$$

where $z \in \mathcal{D} = \{z: |z| < 1\}$. Also, let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $\phi(z) = z + \sum_{n=2}^{\infty} k_n z^n$, and $\psi(z) = z + \sum_{n=2}^{\infty} d_n z^n$ be analytic functions in \mathcal{D} where $b_n, k_n, d_n > 0$ and $k_n > d_n$. We define the Hadamard product as follows:

$$\begin{aligned} g(z) * \phi(z) &= z + \sum_{n=2}^{\infty} b_n k_n z^n, \\ g(z) * \psi(z) &= z + \sum_{n=2}^{\infty} b_n d_n z^n. \end{aligned} \quad (2)$$

Earlier in 1933, Fekete and Szego [1] states that for $f \in S$ and given by (1),

$$|a_3 - \mu a_2^2| \leq 1 + 2 \exp\left(\frac{-2\mu}{1-\mu}\right), \quad (3)$$

for $0 \leq \mu \leq 1$ and the inequality is sharp.

The Fekete-Szego problems for the subclass of S consisting of the families, convex functions C , starlike functions S^* , and close-to-convex functions CC have been completely solved in the literature. Among others are Keogh and Merkes [2], Koepf [3], Darus and Thomas [4, 5], Frasin and Darus [6], Ebadian et al. [7], and Mohammed et al. [8]. In particular, for $f \in CC$ and be given by (1), Keogh and Merkes [2] showed that

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq \frac{1}{3}, \\ \frac{1}{3} + \frac{4}{9\mu}, & \text{if } \frac{1}{3} \leq \mu \leq \frac{2}{3}, \\ 1, & \text{if } \frac{2}{3} \leq \mu \leq 1, \\ 4\mu - 3, & \text{if } \mu \geq 1, \end{cases} \quad (4)$$

and for each μ , there is a function in CC for which equality holds.

Moreover, an estimate is given for the same functional for the new class $n_{\alpha}^{\beta, \gamma}$ defined as follows.

Definition 1. For $0 \leq \alpha < 1$, $0 \leq \beta < 1$, and $0 \leq \gamma < 1$, let the function f be given by (1). Then, the function $f \in n_{\alpha}^{\beta, \gamma}$ if and only if there exists g analytic function, $g(z) \neq 0$, such that for $z \in \mathcal{D}$,

$$\operatorname{Re} \left(\frac{\alpha z^2 f''(z)}{g(z)} + \frac{z f'(z)}{g(z)} \right) > \beta, \tag{5}$$

$$\operatorname{Re} \left(\frac{g(z) * \phi(z)}{g(z) * \psi(z)} \right) > \gamma, \quad g(z) * \psi(z) \neq 0, \quad \text{for } z \in \mathcal{D}. \tag{6}$$

This class is extended from Ramesha et al. [10] and Janteng [11], for suitable choices of ϕ, ψ , we easily obtain the various subclasses of S .

For example, if $\phi(z) = z/(1-z)^2$ and $\psi(z) = z/1-z$ satisfy (6), we have the class of starlike functions g of order γ ($0 \leq \gamma < 1$) denoted by $S^*(\gamma)$. Also if $\phi(z) = z + z^2/(1-z)^3$ and $\psi(z) = z/(1-z)^2$ satisfy (6), we get the class of convex functions g of order γ ($0 \leq \gamma < 1$) denoted by $C(\gamma)$.

To establish our main theorems in this paper, we first state some preliminary lemmas, required in proving of our theorem.

2. Preliminary Results

Lemma 1 (see [9]). *Let h be analytic in \mathcal{D} with $\operatorname{Re} h(z) > 0$ and be given by*

$$h(z) = 1 + c_1 z + c_2 z^2 + \dots, \text{ for } z \in \mathcal{D}, \quad \text{then,} \tag{7}$$

$$|c_n| \leq 2 \text{ where } n \geq 1, \quad \left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

Lemma 2 (see [2]). *Let $g \in S^*$, the starlike function with $g(z) = z + b_2 z^2 + b_3 z^3 \dots$. Then, for μ real,*

$$|b_3 - \mu b_2^2| \leq \max\{1, |3 - 4\mu|\}. \tag{8}$$

The first result for the class $n_{\alpha}^{\beta, \gamma}$ is as follows.

Lemma 3. *Let the function f given by (1) belong to the class $n_{\alpha}^{\beta, \gamma}$. Then,*

$$(\alpha + 1)|a_2| \leq \frac{(k_2 - d_2)(1 - \beta) + 1 - \gamma}{k_2 - d_2}, \tag{9}$$

$$3(2\alpha + 1)|a_3| \leq \frac{4(1 - \gamma)^2 d_2}{(k_3 - d_3)(k_2 - d_2)} + \frac{4(1 - \gamma)(1 - \beta)}{k_2 - d_2} + \frac{2(1 - \gamma)}{k_3 - d_3} + 2(1 - \beta),$$

where $0 \leq \alpha < 1$.

Proof. From (6), we get

$$g(z) * \phi(z) = (p(z)(1 - \gamma) + \gamma)(g(z) * \psi(z)), \tag{10}$$

$$z + b_2 k_2 z^2 + b_3 k_3 z^3 + \dots = (1 + p_1(1 - \gamma)z + p_2(1 - \gamma)z^2 + p_3(1 - \gamma)z^3 + \dots)(z + b_2 d_2 z^2 + b_3 d_3 z^3 + \dots). \tag{11}$$

Now, equating coefficients, we get

$$b_2(k_2 - d_2) = p_1(1 - \gamma), \tag{12}$$

$$b_3(k_3 - d_3) = b_2 d_2 p_1(1 - \gamma) + p_2(1 - \gamma). \tag{13}$$

It also follows from (5) that

$$\alpha z^2 f''(z) + z f'(z) = g(z)(h(z)(1 - \beta) + \beta), \tag{14}$$

where $\operatorname{Re} h(z) > 0$, and writing $h(z) = 1 + c_1 z + c_2 z^2 \dots$, where $c_1, c_2, \dots \in \mathbb{C}$, and equating coefficients give

for $z \in \mathcal{D}$, with $\operatorname{Re} p(z) > 0$ given by $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots$, where $p_1, p_2, \dots \in \mathbb{C}$.

From (10), we have

$$2(\alpha + 1)a_2 = c_1(1 - \beta) + b_2, \tag{15}$$

$$3(2\alpha + 1)a_3 = (1 - \beta)(c_2 + b_2 c_1) + b_3. \tag{16}$$

The result now follows after applying the classical inequalities: $|p_1| \leq 2, |p_2| \leq 2, |c_1| \leq 2, |c_2| \leq 2$, and the inequalities $|b_2| \leq 2(1 - \gamma)/k_2 - d_2$ and $|b_3| \leq 4d_2(1 - \gamma)^2 / (k_2 - d_2)(k_3 - d_3) + 2(1 - \gamma)/k_3 - d_3$ which follow from (12) and (13).

Now we will display the main result for the class $n_{\alpha}^{\beta, \gamma}$. \square

3. Main Result

Theorem 1. Let the function f be given by (1) and belong to the class $n_{\alpha}^{\beta, \gamma}$. Then, for $0 \leq \alpha < 1$,

$$3(2\alpha + 1)|a_3 - \mu a_2^2| \leq \begin{cases} \frac{4(1-\gamma)^2 d_2}{(k_3-d_3)(k_2-d_2)} + \frac{2(1-\gamma)}{k_3-d_3} + 2(1-\beta) + \frac{4(1-\gamma)(1-\beta)}{k_2-d_2} - \frac{2\mu(1-\gamma+(1-\beta)(k_2-d_2))^2}{\mu_1(k_2-d_2)^2} & \text{if } \mu \leq \mu_{\bullet}; \\ \frac{4(1-\gamma)^2 d_2}{(k_3-d_3)(k_2-d_2)} - \frac{4(1-\gamma)^2}{(k_2-d_2)^2} + \frac{2(1-\gamma)}{k_3-d_3} + 2(1-\beta) + \frac{2\mu_1(1-\gamma)^2}{\mu(k_2-d_2)^2} & \text{if } \mu_{\bullet} \leq \mu \leq \mu_1; \\ \frac{4(1-\gamma)^2 d_2}{(k_3-d_3)(k_2-d_2)} - \frac{2(1-\gamma)^2}{(k_2-d_2)^2} + \frac{2(1-\gamma)}{k_3-d_3} + 2(1-\beta) & \text{if } \mu_1 \leq \mu \leq \mu_2; \\ \frac{4(1-\gamma)^2 d_2}{(k_3-d_3)(k_2-d_2)} - \frac{2(1-\gamma)}{k_3-d_3} - 2(1-\beta) - \frac{4(1-\gamma)(1-\beta)}{k_2-d_2} + \frac{2\mu(1-\gamma+(1-\beta)(k_2-d_2))^2}{\mu_1(k_2-d_2)^2} & \text{if } \mu_2 \leq \mu; \end{cases} \quad (17)$$

where

$$\begin{aligned} \mu_{\bullet} &= \frac{\mu_1(1-\gamma)}{1-\gamma+(1-\beta)(k_2-d_2)}, \\ \mu_1 &= \frac{2(\alpha+1)^2}{3(2\alpha+1)}, \\ \mu_2 &= \frac{\mu_{\bullet}(k_2-d_2)^2}{1-\gamma+(1-\beta)(k_2-d_2)} \left(\frac{4(1-\gamma)d_2}{(k_3-d_3)(k_2-d_2)} + \frac{2}{k_3-d_3} + \frac{2(1-\beta)}{1-\gamma} + \frac{2(1-\beta)}{k_2-d_2} - \frac{1-\gamma}{(k_2-d_2)^2} \right). \end{aligned} \quad (18)$$

The inequalities are sharp for all cases. However, the proofs for the case $\mu = \mu_2$ are still unsolved.

Proof. Write

$$3(2\alpha + 1)(a_3 - \mu a_2^2) = 3(2\alpha + 1)a_3 - 3(2\alpha + 1)\mu a_2^2. \quad (19)$$

From (15) and (16), we have $3(2\alpha + 1)(a_3 - \mu a_2^2) = c_2(1 - \beta) + b_2 c_1(1 - \beta) + b_3 - 3(2\alpha + 1)\mu [c_1^2(1 - \beta)^2 + b_2^2 + 2c_1 b_2(1 - \beta)/4(\alpha + 1)^2]$. Then, we get

$$\begin{aligned} 3(2\alpha + 1)(a_3 - \mu a_2^2) &= b_3 - \frac{3(2\alpha + 1)\mu b_2^2}{4(\alpha + 1)^2} + c_2(1 - \beta) \\ &+ c_1^2(1 - \beta)^2 \left[\frac{2(\alpha + 1)^2 - 3(2\alpha + 1)\mu}{4(\alpha + 1)^2} - \frac{1}{2} \right] \\ &+ b_2 c_1(1 - \beta) \left[1 - \frac{3(2\alpha + 1)\mu}{2(\alpha + 1)^2} \right]. \end{aligned} \quad (20)$$

From (20), we have

$$\begin{aligned}
 3(2\alpha + 1)|a_3 - \mu a_2^2| \leq & \left| b_3 - \frac{3(2\alpha + 1)\mu b_2^2}{4(\alpha + 1)^2} \right| + \left| c_2(1 - \beta) - \frac{1}{2}c_1^2(1 - \beta)^2 \right| \\
 & + \frac{1}{4(\alpha + 1)^2} |2(\alpha + 1)^2 - 3(2\alpha + 1)\mu| |c_1|^2 (1 - \beta)^2 \\
 & + \frac{|b_2||c_1(1 - \beta)|}{2(\alpha + 1)^2} |2(\alpha + 1)^2 - 3(2\alpha + 1)\mu|.
 \end{aligned} \tag{21}$$

Now, consider the first case for all

$$\mu \leq \frac{2(\alpha + 1)^2}{3(2\alpha + 1)}, \tag{22}$$

that is, having $b_3 - 3(2\alpha + 1)\mu b_2^2/4(\alpha + 1)^2 > 0$ and $2(\alpha + 1)^2 - 3(2\alpha + 1)\mu > 0$.

Inequalities

$$|b_2| \leq \frac{2(1 - \gamma)}{k_2 - d_2}, \quad |b_3| \leq \frac{4(1 - \gamma)^2 d_2}{(k_3 - d_3)(k_2 - d_2)} + \frac{2(1 - \gamma)}{k_3 - d_3}, \tag{23}$$

and

$$\left| c_2(1 - \beta) - \frac{c_1^2(1 - \beta)^2}{2} \right| \leq 2(1 - \beta) - \frac{|c_1|^2(1 - \beta)^2}{2}, \tag{24}$$

follow immediately from Lemma 1. Now we get

$$\begin{aligned}
 3(2\alpha + 1)|a_3 - \mu a_2^2| \leq & \frac{4(1 - \gamma)^2 d_2}{(k_3 - d_3)(k_2 - d_2)} + \frac{2(1 - \gamma)}{k_3 - d_3} + 2(1 - \beta) \\
 & - \frac{3(2\alpha + 1)(1 - \gamma)^2 \mu}{(\alpha + 1)^2 (k_2 - d_2)^2} - \frac{3(2\alpha + 1)(1 - \beta)^2 \mu |c_1|^2}{4(\alpha + 1)^2} \\
 & + \frac{(1 - \gamma)(1 - \beta)(2(\alpha + 1)^2 - 3(2\alpha + 1)\mu) |c_1|}{(\alpha + 1)^2 (k_2 - d_2)} \\
 = & \phi(x), \text{ say with, } x = |c_1|.
 \end{aligned} \tag{25}$$

Now, after doing some operations, the function ϕ achieves its maximum value at

$$x^* = \frac{2[2(\alpha + 1)^2 - 3\mu(2\alpha + 1)](1 - \gamma)}{3\mu(2\alpha + 1)(k_2 - d_2)(1 - \beta)}. \tag{26}$$

Next, we find

$$\begin{aligned}
 \phi(x^*) = & \frac{4(1 - \gamma)^2 d_2}{(k_3 - d_3)(k_2 - d_2)} - \frac{4(1 - \gamma)^2}{(k_2 - d_2)^2} + \frac{2(1 - \gamma)}{k_3 - d_3} \\
 & + 2(1 - \beta) + \frac{4(1 - \gamma)^2(\alpha + 1)^2}{3(2\alpha + 1)(k_2 - d_2)^2 \mu}.
 \end{aligned} \tag{27}$$

Finally, we get our results:

$$\begin{aligned}
 3(2\alpha + 1)|a_3 - \mu a_2^2| \leq & \frac{4(1 - \gamma)^2 d_2}{(k_3 - d_3)(k_2 - d_2)} - \frac{4(1 - \gamma)^2}{(k_2 - d_2)^2} \\
 & + \frac{2(1 - \gamma)}{k_3 - d_3} + 2(1 - \beta) \\
 & + \frac{4(1 - \gamma)^2(\alpha + 1)^2}{3(2\alpha + 1)(k_2 - d_2)^2 \mu}.
 \end{aligned} \tag{28}$$

Now since we know $|x^*| \leq 2$, we get the interval

$$\frac{2(\alpha + 1)^2(1 - \gamma)}{3(2\alpha + 1)(1 - \gamma + (1 - \beta)(k_2 - d_2))} \leq \mu. \tag{29}$$

Hence, result (28) concludes for the case

$$\frac{2(\alpha + 1)^2(1 - \gamma)}{3(2\alpha + 1)(1 - \gamma + (1 - \beta)(k_2 - d_2))} \leq \mu \leq \frac{2(\alpha + 1)^2}{3(2\alpha + 1)} \quad (30)$$

We shall be certain that the result of this case is sharp. Let $c_1 = 2(1 - \gamma)[2(\alpha + 1)^2 - 3(2\alpha + 1)\mu] / 3(1 - \beta)(2\alpha + 1)\mu(k_2 - d_2)$, $c_2 = 2$, $p_1 = 2$, $p_2 = 2$, $b_2 = 2(1 - \gamma)/k_2 - d_2$ and $b_3 = 4(1 - \gamma)^2 d_2 / (k_3 - d_3)(k_2 - d_2) + 2(1 - \gamma)/k_3 - d_3$, in (20).

Secondly, we consider the case $\mu \leq \mu_\bullet = 2(\alpha + 1)^2(1 - \gamma) / 3(2\alpha + 1)(1 - \gamma + (1 - \beta)(k_2 - d_2))$.

Write

$$a_3 - \mu a_2^2 = a_3 - \mu_\bullet a_2^2 + (\mu_\bullet - \mu) a_2^2. \quad (31)$$

Now we say that with the aid of the inequality $|a_2| \leq 1 - \gamma + (1 - \beta)(k_2 - d_2) / (k_2 - d_2)(\alpha + 1)$, we get the following:

$$\begin{aligned} 3(2\alpha + 1)|a_3 - \mu a_2^2| &\leq 3(2\alpha + 1)|a_3 - \mu_\bullet a_2^2| + 3(2\alpha + 1)|\mu_\bullet - \mu||a_2|^2, \\ &= \frac{4(1 - \gamma)^2 d_2}{(k_3 - d_3)(k_2 - d_2)} + \frac{2(1 - \gamma)}{k_3 - d_3} + \frac{4(1 - \gamma)(1 - \beta)}{k_2 - d_2}, \\ &\quad + 2(1 - \beta) - \frac{3(2\alpha + 1)\mu(1 - \gamma + (1 - \beta)(k_2 - d_2))^2}{(\alpha + 1)^2(k_2 - d_2)^2}. \end{aligned} \quad (32)$$

Thus, the proof for this case is complete.

Here, to find $3(2\alpha + 1)|a_3 - \mu_\bullet a_2^2|$ we use the previous result of (28), for $\mu = \mu_\bullet = 2(\alpha + 1)^2(1 - \gamma) / 3(2\alpha + 1)(1 - \gamma + (1 - \beta)(k_2 - d_2))$.

After that, the result is sharp for this case. Upon choosing $c_1 = c_2 = p_1 = p_2 = 2$, $b_2 = 2(1 - \gamma) / (k_2 - d_2)$, and $b_3 = 4(1 - \gamma)^2 d_2 / (k_3 - d_3)(k_2 - d_2) + 2(1 - \gamma) / k_3 - d_3$ and substituting in (20), we get the result sharp.

Next we consider the case $2(\alpha + 1)^2 / 3(2\alpha + 1) \leq \mu \leq \mu_2$.

The results for this case are correct and sharp, but we have problems to solve for the value $|a_3 - \mu_2 a_2^2|$. As for $\mu_1 = 2(\alpha + 1)^2 / 3(2\alpha + 1)$, it is very easy and can be directly obtained from the previous result (28).

Without any doubt, we can assure that

$$3(2\alpha + 1)|a_3 - \mu_2 a_2^2| \leq \frac{4(1 - \gamma)^2 d_2}{(k_3 - d_3)(k_2 - d_2)} - \frac{2(1 - \gamma)^2}{(k_2 - d_2)^2} + \frac{2(1 - \gamma)}{k_3 - d_3} + 2(1 - \beta), \quad (33)$$

which is left as conjecture for the readers to solve. Without loss of generality, we would suggest that in the case $\mu_1 \leq \mu \leq \mu_2$,

$$3(2\alpha + 1)|a_3 - \mu a_2^2| \leq \frac{4(1 - \gamma)^2 d_2}{(k_3 - d_3)(k_2 - d_2)} - \frac{2(1 - \gamma)^2}{(k_2 - d_2)^2} + \frac{2(1 - \gamma)}{k_3 - d_3} + 2(1 - \beta). \quad (34)$$

This is sharp on choosing $c_1 = p_1 = 0$, $p_2 = 2$, $c_2 = 4(1 - \gamma)^2 d_2 / (k_3 - d_3)(k_2 - d_2)(1 - \beta) - 2(1 - \gamma)^2 /$

$(k_2 - d_2)^2(1 - \beta) + 2$, $b_2 = 0$, and $b_3 = 2(1 - \gamma) / k_3 - d_3$ in (20). Thus, we get the desired equality.

Finally, for the case $\mu_2 \leq \mu$, we achieve the following easily.

Write

$$a_3 - \mu a_2^2 = a_3 - \mu_2 a_2^2 + (\mu_2 - \mu) a_2^2, \quad (35)$$

and by using the inequality $|a_2| \leq 1 - \gamma + (1 - \beta)(k_2 - d_2)/(k_2 - d_2)(\alpha + 1)$, we obtain the following:

$$\begin{aligned} 3(2\alpha + 1)|a_3 - \mu a_2^2| &\leq 3(2\alpha + 1)|a_3 - \mu_2 a_2^2| + 3(2\alpha + 1)|\mu_2 - \mu||a_2|^2 \\ &= \frac{-4(1 - \gamma)^2 d_2}{(k_3 - d_3)(k_2 - d_2)} - \frac{2(1 - \gamma)}{k_3 - d_3} - 2(1 - \beta) \\ &\quad - \frac{4(1 - \gamma)(1 - \beta)}{k_2 - d_2} + \frac{3(2\alpha + 1)\mu(1 - \gamma + (1 - \beta)(k_2 - d_2))^2}{(\alpha + 1)^2(k_2 - d_2)^2}, \end{aligned} \quad (36)$$

and hence the proof is complete for this case.

Here

$$3(2\alpha + 1)|a_3 - \mu_2 a_2^2| \leq \frac{4(1 - \gamma)^2 d_2}{(k_3 - d_3)(k_2 - d_2)} - \frac{2(1 - \gamma)^2}{(k_2 - d_2)^2} + \frac{2(1 - \gamma)}{k_3 - d_3} + 2(1 - \beta). \quad (37)$$

We used result (34) for

$$\mu = \mu_2. \quad (38)$$

At last, it remains to show that the result is sharp, upon choosing $c_1 = p_1 = 2i$, $c_2 = p_2 = -2$, $b_2 = 2i(1 - \gamma)/k_2 - d_2$, and $b_3 = -4(1 - \gamma)^2 d_2 / (k_3 - d_3)(k_2 - d_2) - 2(1 - \gamma) / k_3 - d_3$ and substituting in (20). \square

4. Conclusion

This article aims at finding a new class of analytic univalent functions on the open unit disc defined by Hadamard product. Further to study their inequality theorem, one of the main requirements needed to satisfy certain classes. This approach, for example, can provide several many fascinating features.

Data Availability

No data were used or available upon request or included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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