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Research Article

Numerical Solution of In-Viscid Burger Equation in the Application of Physical Phenomena: The Comparison between Three Numerical Methods

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In this paper, upwind approach, Lax–Friedrichs, and Lax–Wendroff schemes are applied for working solution of In-thick Burger equation in the application of physical phenomena and comparing their error norms. First, the given solution sphere is discretized by using an invariant discretization grid point. Next, by using Taylor series expansion, we gain discretized nonlinear difference scheme of given model problem. By rearranging this scheme, we gain three proposed schemes. To verify validity and applicability of proposed techniques, one model illustration with subordinated to three different original conditions that satisfy entropy condition are considered, and solved it at each specific interior grid points of solution interval, by applying all of the techniques. The stability and convergent analysis of present three techniques are also worked by supporting both theoretical and numerical fine statements. The accuracy of present techniques has been measured in the sense of average absolute error, root mean square error, and maximum absolute error norms. Comparisons of numerical gets crimes attained by these three methods are presented in table. Physical behaviors of numerical results are also presented in terms of graphs. As we can see from numerical results given in both tables and graphs, the approximate solution is good agreement with exact solutions. Therefore, the present systems approaches are relatively effective and virtually well suited to approximate the solution of in-viscous Burger equation.

1. Introduction

Burgers equation is considered one of the most common nonlinear PDEs [1] and it arises in multitudinous physical problems including one-dimensional turbulence, sound swells and shock swells during a thick medium, swells in fluid-filled thick elastic tubes, and magneto hydrodynamic swells in a medium with finite electrical conductivity [2–6] and fluid dynamics [35]. Burger equations were first introduced by Bateman in 1915 [8–12] and he proposed the steady-state result of the matter [5]. It also arises in numerous physical problems, including shock overflows, business overflows [1], nonlinear surge propagation [1, 13], swells of impact and sound swells in a thick medium [1], gas dynamics, hydrodynamics [14, 15], heat conduction [14], isentropic inflow [16], solid-state drugs and optic filaments [15], boundary sub caste gets, shock surge conformation and mass transport [12]. Nonlinear kind of equation wich represents the Burgers equation, the warmth conduction equation, the nonlinear Schrödinger equation, the Navier–Stokes equation [5, 14]. Thus, Burgers equation belongs to the class of Navier–Stokes equation and its fundamental PDEs, which find in different model, like fluid mechanics analyses [17]. This Navier–stoke equation is solved by using iteration free backward semi-Lagrangian scheme (see [36]). In this study, the spacial structure for system of equation associated with Ronaldo number is produced. In general, the simplification of this Navier–Stoke equation produces different types of nonlinear equation. Hence, one-dimensional Burger equations was introduced by the scientist Burgers in 1948, and he has attained this equation by simplify the Navier–Stokes equation with dropping its pressure term. So that, this equation, contains the simplest forms of nonlinear advection term and dispersions term [19]. The reason of these studies is, to capture some features of turbulence fluid in a channel caused by the commerce of the contrary goods of convection and diffusion [5, 9, 17]. In his study, he also illustrates the proposition of turbulences describe by using this two commerce. This condition arises in the proposition of shock swells, in turbulence problems, and non-stop stochastic processes [20], also see [21]. In addition, one-dimensional Burgers equation arises in oil force simulation, modeling of gas dynamics and business flux, turbulence [2, 3, 20, 22], fluid mechanics, tube drugs, biology [15]. Most explicit type methods are best for overcoming nonlinearity but poor for stability [38]. It means that all implicit type methods require lots of computational costs due to nonlinearity but are good in the sense of stability and accuracy [38]. So that, studying result of Burger equation has been carried out for the last half-century and still, it is an active area of disquisition.

Accordingly, more effective and simpler numerical ways are demanded to break Burgers equation. In the literature review, several numerical ways have been applied to probe the numerical results of 1D Burgers equation like [8, 11, 24, 25]. In addition, multitudinous researchers with different numerical approaches answer the coupled and 2D Burgers equations. Alam Khan et al. [12] Presented numerical results of time-fractional Burger equations. In this study, they present comparison between generalized conversion ways/techniques with homogony perturbation manner and take the results of model problem from coincident series terms by using GDTM. In GDTM, the equation and related original conditions are converted into a rush relation that eventually leads to the result of a system of algebraic equations as portions of a power series result. HPM requires a homotopy with an embedding parameter, which is accounted as a frail parameter.

Correspondingly, Yağmurluel.at. [15] presented numerical result of nonlinear coupled system thick burger's equation by using advanced order splitting system/techniques. Since they use this system/techniques within expression finite element scheme by unyoking the system of coupled-Burger equation into two subequations. Hence, the confluence of this system is based on grid point. Again, Khan et al. [13] presented numerical result of one-way surge propagations in Kerr-media. This study is also use to probe, an operation of Burger equations for surge propagations in rail-life marvels. Azaria and Borhanifar [19] also presented the result of coupled and 2D Burgers equations by using the DTM. DTM is a semi-discretized numerical tactics of logical fashion that formalizes the Taylor series differently. Taylor series system is computationally time-consuming if we use it with large orders. In addition, it is high weakened round out and truncation crimes when its order increases. Ahmad et al. [26] presented numerical results of coupled Burgers' equations. Khater et al. [27] also presents the Chebyshev spectral ways for working the coupled Burgers equations. This pseudo-spectral techniques watch must be taken with the round-off error issue when advanced derivatives or a large several points Mx and Nx is involved, see [5] and the edge of Chebyshev and wavelate wavelet fashion is worked [28]. In all Chebyshev family techniques, error are generated with the number of grid point are increases (see [35, 37]). Mohamed [1] presented numerical result of one and two-dimensional unsteady Burgers' equation by using completely implicit finite difference schemes. Gowrisankar and Natesan [17] are also presented numerical result of singularly perturbed original-boundary Burgers equations by using an effective robust numerical approach. This fashion (techniques) also give A-uniformly convergent numerical results for singularly perturbed Burger's equation, due to actuality of small parameter [5, 17]. Özis et al. [21] presented Numerical result of Burgers' equation by quadratic B-spline finite elements. However, this system/technique has produced high truncation error, if the number of grid point is veritably large. Indeed this, the delicacy of those numerical ways needs to prove; because the treatments of the ways used to break the Burger equation aren't trivial distributions. Indeed though the delicacy of the forenamed numerical techniques is promising, they bear large memory and long computational time. Therefore, the treatments of these approaches' present severe difficulties that have to be addressed to ensure the accurateness of the numerical result.

To this end, the aims of this paper is, to apply three numerical techniques that are able of working solution of inviscid Burgers equation in the application of physical phenomena. The convergences of present numerical results have been shown (measured) in sense of average absolute error, pointwise absolute error and root mean square error norms, and so that original gets of the numerical result is captured exactly. The stability and confluence (convergency) of the present techniques are also delved by using Von-Neumann stability analysis ways.

1.1. Statement of the Problem.

Consider that the following inviscid Burger equation given by:

$$U_t + UU_x = 0,\ (x,t) \in (a,b) \times (0,T). \tag{1}$$

Subjected to initial and boundary conditions respectively

$$\begin{aligned} &U(x,0) = f(x), \quad a \le x \le b, \\ &U(a,t) = g(t),\ U(b,t) = q(t),\ 0 \le t \le T, \end{aligned} \tag{2}$$

where $f(x)$, $g(t))$ and $q(t)$ are assumed to be sufficiently smooth functions for actuality and existence of result [30, 31]. Result U may represent a temperature for heat transfer or a species attention for mass transfer at position x and time t with the advection haste U [14]. Now we define constant of grid points by drawing horizontal and vertical lines of distance 'h' and 'k' singly in 'x' and 't' direction respectively. These lines are called refinement of mesh line and the point at which they interacting is known as mesh point. Mesh point that lies at end of sphere is called boundary point. The points that lie inside region are called interiors points. Therefore, the aim is to compute numerical results of U at each these interior mesh points by applying present three methods and comparing those numerical results obtained at corresponding interior points. Hence to do this, first we discretized the result sphere (interval) as

$$\begin{aligned} &a = x_0 < x_1 < x_2 < \ldots < x_{Mx} = b, \\ &0 = t_0 < t_1 < t_2 < \ldots < t_{Nt} = T, \end{aligned} \tag{3}$$

where $x_{j+1} = x_j + h$ and $t_{n+1} = t_n + k$, $j = 0(1)Mx$, $n = 0(1)Nx$. Both Mx and Nt are maximum numbers of grid points in x and t direction, respectively. In addition, the present paper is organized as follows. Section 2 is a description of numerical styles/techniques. Section 3 is Stability and confluence (convergence) analysis. Section 4 is Numerical results. Section 5 is Discussions of numerical experiments and Section 6 is the conclusion.

2. Description of Numerical Methods

In this paper, upwind, Lax–Friedrichs, and Lax–Wendroff styles are used to break (solve) nonlinear in-viscid parabolic partial differential equation given in (1) and comparing the result with each other. These nonlinear parabolic partial differential equations have the first-order outgrowth in time and first-order outgrowth in the spatial variable are called in thick (in-viscous) Burger equation. Now assuming that $U(x, t)$ has non-stop advanced order partial outgrowth on the region $(0,1)x$ $(0, T$. For the sake of simplicity, consider that $U(x_j, t_n) = U_{jn}$, $\partial^p U/\partial x^p = \partial_x^p U_{jn}$ and $\partial^p U/\partial t^p = \partial_t^p U_{jn}$ for $p \geq 1$ is p^{th} order partial derivatives. By using Taylor series expansion, we have

$$\begin{aligned} U_{j+1n} &= U_{jn} + h\partial_x U_{jn} + \frac{h^2}{2!}\partial_x^2 U_{jn} + \frac{h^3}{3!}\partial_x^3 U_{jn} + \dots \\ U_{j-1n} &= U_{jn} - h\partial_x U_{jn} + \frac{h^2}{2!}\partial_x^2 U_{jn} - \frac{h^3}{3!}\partial_x^3 U_{jn} + \dots \\ U_{jn+1} &= U_{jn} + k\partial_t U_{jn} + \frac{k^2}{2!}\partial_t^2 U_{jn} + \frac{k^3}{3!}\partial_t^3 U_{jn} + \dots \\ U_{jn-1} &= U_{jn} - k\partial_t U_{jn} + \frac{k^2}{2!}\partial_t^2 U_{jn} - \frac{k^3}{3!}\partial_t^3 U_{jn} + \dots \end{aligned} \tag{4}$$

2.1. Upwind Methods. Considering that, nonlinear Burger equation in (1) and using the Taylor expansion in (4) we obtain finite difference discretized scheme of Burger equation in (1) given as:

$$U_{j,n+1} = U_{j,n} - \alpha U_{j,n}\left[U_{j,n} - U_{j-1,n}\right], \tag{5}$$

where $\alpha = k/h$ and $\tau_1 = 1/2\,(h\partial_x^2 + k\partial_t^2)U_{jn}$ is their principal local truncation error respectively in spatial and time discretization of given partial differential equation. Now we refer to (5) as the upwind nonconservative scheme. Although this method is consistent and is adequate for smooth solutions see [16]. But it will not converge in general to a discontinuous weak solution as the grid is refined. Hence, to more understand how this scheme is consistent and convergent simultaneously, you can refer [16]. Now, a time in this work, let us consider that

$$UU_x = [\psi(U)]_x, \quad \psi(U) = \frac{U^2}{2}. \tag{6}$$

We obtain the general upwind nonconservative scheme of the form:

$$\begin{aligned} U_{j,n+1} = U_{j,n} - \alpha\Big[&\psi\big(U_{j-p,n}, U_{j-p+1,n}, \dots, U_{j-q+1,n}\big) \\ &- \psi\big(U_{j-p-1,n}, U_{j-p,n}, \dots, U_{j+q-1,n}\big)\Big], \end{aligned} \tag{7}$$

where ψ some function of $p+q+1$ argument is called numerical flux function. Methods that conform to this scheme are called conservative methods. If we use the standard finite difference discretizations, we obtain the conservation method called upwind conservative, namely,

$$U_{j,n+1} = U_{j,n} - \alpha\left[\psi\left(U_{j,n}\right) - \psi\left(U_{j-1,n}\right)\right]. \tag{8}$$

For $p=0$, $q=1$, (8) becomes (7). Hence, for $\psi(U) = U^2/2$, the upwind conservative scheme for the Burger equation becomes

$$U_{j,n+1} = U_{j,n} - \frac{\alpha}{2}\left[\left(U_{j,n}\right)^2 - \left(U_{j-1,n}\right)^2\right]. \tag{9}$$

Thus, general local truncation error for upwind scheme is

$$\tau_{j,n} = \frac{1}{2}\left[h\partial_x^2 - k\partial_t^2\right]U_{jn}. \tag{10}$$

And it is first-order accuracy which means $O(h+k)$.

2.2. Lax–Friedrichs Scheme. Considering that, nonlinear Burger equation in (1) and using the Taylor series expansion in (4) with (6), we obtain new finite difference discretized scheme of Burger equation given as

$$U_{j,n+1} = U_{j,n} - \beta\left[\psi\left(U_{j+1,n}\right) - \psi\left(U_{j-1,n}\right)\right], \tag{11}$$

where $\beta = k/2h$, $\tau_2 = k/2\partial_t^2 U_{jn}$ and $\tau_3 = h^2/6\partial_x^3 U_{jn}$ are local truncation error term in temporally and spatial direction respectively. In (11), replacing $U_{j,n}$ into the average value of $U_{j-1,n}$ and $U_{j+1,n}$ and using the definition of $\psi(U)$ given in (6), the Lax–Friedrichs scheme is

$$U_{j,n+1} = \frac{1}{2}\left(U_{j-1,n} + U_{j+1,n}\right) - \frac{\beta}{2}\left[\left(U_{j+1,n}\right)^2 - \left(U_{j-1,n}\right)^2\right]. \tag{12}$$

Thus the general local truncation error for upwind scheme is

$$\tau_{j,n} = \frac{1}{2}\left[\frac{1}{3}h^2\partial_x^3 + k\partial_t^2\right]U_{jn}. \tag{13}$$

And, it is first-order accuracy in time variable and second-order accuracy in the spatial variable which means $O(h^2+k)$.

2.3. Lax–Wendroff Scheme. Lax–Wendroff's method is an explicit method that uses for the improvement of accuracy in time, see [16]. This method is same times called an explicit of time integration techniques. Now considering in-viscous Burger equation in (1) and using the Taylor series expansion in (4) with refinement of weak solution term (a flux function) given in (6), we obtain new finite difference discretized scheme of Burger equation given as

$$\begin{aligned} U_{j,n+1} &= U_{j+1\,n} + k\partial_x U_{j\,n} + \frac{k^2}{2!}\partial_x^2 U_{j\,n} + \tau_5 + \tau_6 \\ U_{j,n+1} &= U_{j+1\,n} - \beta\left(\psi\left(U_{j+1,n}\right) - \psi\left(U_{j-1,n}\right)\right) + \\ &\quad \gamma\left[H_{j+1/2}\left(\psi\left(U_{j+1}\right) - \psi\left(U_{j\,n}\right)\right) - H_{j-1/2}\left(\psi\left(U_{j,n}\right) - \psi\left(U_{j-1\,n}\right)\right)\right] + \tau_5 + \tau_6 \end{aligned} \tag{14}$$

where $\gamma = k^2/2h^2$ and $H_{j\pm1/2}$ is the Jacobean matrix $H(U) = \psi'(u)$ evaluated at $1/2\,(U_{j,n} + U_{j\pm1,n})$, $\tau_4 = h^2/6\partial_x^3 U_{j\,n}$ and $\tau_5 = k^3/3!\partial_t^3 U_{j\,n}$. Since for given in-viscous Burgers equation in (1), we have $\psi'(U) = U$. Hence, the required Lax–Wendroff's for given equation becomes

$$U_{j,n+1} = U_{j+1\,n} - \frac{\beta}{2}\left(\left(U_{j+1,n}\right)^2 - \left(U_{j-1,n}\right)^2\right) + \frac{\gamma}{4}\begin{bmatrix} \left(U_{j,n} + U_{j+1,n}\right)\left(\left(U_{j+1}, n\right)^2 - \left(U_{j\,n}\right)^2\right) - \\ \left(U_{j,n} + U_{j-1,n}\right)\left(\left(U_{j,n}\right)^2 - \left(U_{j-1\,n}\right)^2\right) \end{bmatrix} + \tau_{j,n}, \tag{15}$$

where

$$\tau_{j,n} = \frac{1}{6}\left(h^2\partial_x^3 + k^3\partial_t^3\right)U_{j\,n}. \tag{16}$$

is local truncation error of the proposed scheme. Now, truncating this local truncation error Lax–Wendroff's is as follows:

$$\begin{aligned} U_{j,n+1} &= U_{j+1\,n} - \frac{\beta}{2}\left(\left(U_{j+1,n}\right)^2 - \left(U_{j-1,n}\right)^2\right) \\ &\quad + \frac{\gamma}{4}\begin{bmatrix} \left(U_{j,n} + U_{j+1,n}\right)\left(\left(U_{j+1}, n\right)^2 - \left(U_{j\,n}\right)^2\right) - \\ \left(U_{j,n} + U_{j-1,n}\right)\left(\left(U_{j,n}\right)^2 - \left(U_{j-1\,n}\right)^2\right) \end{bmatrix}. \end{aligned} \tag{17}$$

Therefore, this method is the third-order accuracy in the time variable and the second-order accuracy in the spatial variable, and this order is given by $O(h^2 + k^3)$.

3. Stability Analysis and Convergence of the Presented Method

Consistency is one of the most important criteria to measure the quantity of numerical computational methods. If the truncation error of the discretized equation is closed to zero, when time and special step sizes are very small, this shows that the discretization equation is consistent. In the numerical discretization scheme, if truncation error contains the expression $\Delta t^p/\Delta x^q$ (i.e., in our case k^p/h^q , $k = \Delta t$ and $h = \Delta x$ where p, q are constantly greater than zero), consistency is satisfied when Δt^p is a higher infinitesimal of Δx^q . This kind of discrete scheme is the Du–Fort–Frankel scheme [32] and is regarded to be conditional consistency. When truncation error contains $\Delta x^q/\Delta t^p$, the Lax–Friedrich scheme [33] is conditional consistency under the circumstance that Δx^q is the higher-order infinitesimal of Δt^p. Hull and White in [34] observe that explicit schemes do not require upper and lower boundary conditions to achieve convergence. This property makes explicit schemes desirable from a computational point of view, see [22]. Again the consistency of Lax–Wondroff and Lax–Friedrich scheme is briefly worked in [16].

> Definition: we say the discretization scheme in (9), (12), and (17) are consistent if the limit of truncation error in (10), (11), and (16) is going to zero for which the grid size $h, k \longrightarrow 0$. Hence, the considerations of the scheme for the given problem that worked in this paper with their local truncation error are consistence. It means obliviously we can see that for all proposed schemes $\lim|\tau_{j,n}| \longrightarrow 0$ as both mesh-size $h, k \longrightarrow 0$.

3.1. Stability of Scheme. The stability of the proposed numerical method is investigated by using Von-Neumann stability analysis. To avoid nonlinear in the proposed scheme, we assumed that $V_{j\,n} = U_{j,n}U_{j,n}$. Then, without losing generality, we obtain the linear system of a difference equation. Now, we can know to inquire about the eigenvalues of the Mx by Mx linear system of a difference equation. To obtain this eigenvalue, as the working of sterility analysis in references [35–38], we first assume that a trial solution is as follows:

$$U(x,t) = \varphi(t)\phi(x). \tag{18}$$

Further, to investigate the stability of the proposed method by Von Neumann techniques, we assume that the trial function $\phi(x)$ defined by the following:

$$\varphi(t_n) = \lambda_a^n, \quad \phi(x) = e^{ijhK_a}, \tag{19}$$

where $\lambda_a \in \mathbb{C}$, $i = \sqrt{-1}$, $K_a = a\pi$ and $a = 1(1)Mx$, and K is a Fourier number or amplification factor. Now, substituting (18) and (19) into in, and (9) and (12), (17), we obtain the criteria of stability, respectively, as follows, for $V_{jn} = U_{j,n}U_{j,n}$:

$$\begin{aligned}
\lambda_a^{n+1}e^{ijhK_a} &= \lambda_a^n e^{ijhK_a} - \frac{\alpha}{2}\left[\lambda_a^n e^{ijhK_a} - \lambda_a^n e^{ihK_a(j-1)}\right] \\
\lambda_a &= 1 - \frac{\alpha}{2}\left[1 - e^{-ihK_a}\right] \\
\lambda_a &= 1 - \frac{\alpha}{2}\left[1 - \cos(hK_a) + i\,\sin(hK_a)\right].
\end{aligned} \tag{20}$$

Hence, from (20), we obtain: $|\lambda_a| \ll 1$. This shows that the proposed up-wind scheme in (9) is stable with first-order accuracy. Again from (12), we obtain the following:

$$\begin{aligned}
\lambda_a^{n+1}e^{ijhK_a} &= \frac{1}{2}\left(\lambda_a^n e^{ihK_a(j-1)} + \lambda_a^n e^{ihK_a(j+1)}\right) - \frac{\beta}{2}\left[\lambda_a^n e^{ihK_a(j+1)} - \lambda_a^n e^{ihK_a(j-1)}\right] \\
\lambda_a &= \frac{1}{2}\left(e^{-ihK_a} + e^{ihK_a}\right) - \frac{\beta}{2}\left[e^{ihK_a} - e^{-ihK_a}\right] \\
\lambda_a &= \frac{1}{2}\left(\cos(hK_a) - i\,\sin(hK_a) + \cos(hK_a) + i\,\sin(hK_a)\right) - \frac{\beta}{2}\left[\cos(hK_a)\right. \\
&\quad \left. + i\,\sin(hK_a) - \cos(hK_a) + i\,\sin(hK_a)\right] \\
\lambda_a &= \cos(hK_a) - \beta i\,\sin(hK_a).
\end{aligned} \tag{21}$$

From (21), we have $-1 \le \cos(hK_a), sin(hK_a) \le 1$. Hence, $|\lambda_a| = |\cos(hK_a) - \beta isin(hK_a)|$ $\le |\cos(hK_a)| + \beta|sin(hK_a)| < 1$. Therefore, the correspondence Lax-Friedrichs scheme is stable with first-order accuracy in the time variable and second-order accuracy in the spatial variable. At last From (17), we obtain the following:

$$\begin{aligned}
\lambda_a^{n+1}e^{ijhK_a} &= \lambda_a^n e^{ihK_a(j+1)} - \frac{\beta}{2}\left(\lambda_a^n e^{ihK_a(j+1)} - \lambda_a^n e^{ihK_a(j-1)}\right) \\
&\quad + \frac{\gamma}{4}\begin{bmatrix} \left(\lambda_a^n e^{ijhK_a} + \lambda_a^n e^{ihK_a(j+1)}\right)\lambda_a^n e^{ijhK_a(j+1)} - \quad \lambda_a^n e^{ijhK_a} \\ \left(\lambda_a^n e^{ijhK_a} + \lambda_a^n e^{ihK_a(j-1)}\right)\left(\lambda_a^n e^{ijhK_a} - \lambda_a^n e^{ijhK_a(j-1)}\right) \end{bmatrix} \\
\lambda_a &= e^{ihK_a} - \frac{\beta}{2}\left(e^{ihK_a} - e^{-ihK_a}\right) + \frac{\gamma}{4}\begin{bmatrix} \left(e^{ihK_a} - 1\right) \\ -\left(1 - e^{-ihK_a}\right) \end{bmatrix} \\
\lambda_a &= \cos(hK_a) + i\,\sin(hK_a) - \beta i\,\sin(hK_a) + \frac{\gamma}{4}\left[2\cos(hK_a) - 1\right] \\
\lambda_a &= \left(1 + \frac{\gamma}{2}\right)\cos(hK_a) + (1-\beta)i\,\sin(hK_a) - \frac{\gamma}{4} \\
\lambda_a &= \left[\frac{\gamma}{4} - \left(1 + \frac{\gamma}{2}\right)\cos(hK_a)\right] - i(\beta - 1)\sin(hK_a).
\end{aligned} \tag{22}$$

Since for any value of amplification factor K_a, $-1 \le \cos(hK_a), sin(hK_a) \le 1$. Hence from (22), $|[\gamma/4 - (1 + \gamma/2)\cos(hK_a)]| \ll 1$ and $|(\beta - 1)sin(hK_a)| \ll 1$. Therefore, from this equation, we obtain

$|\lambda_a| < |\gamma/4 - (1+\gamma/2)\cos(hK_a)| + |(\beta-1)sin(hK_a)| \ll 1$. Thus, the correspondence lax-Wendorff scheme is stable with third-order accuracy in the time variable and second-order accuracy in the spatial variable. Generally, in all cases, the proposed schemes are convergent. Therefore, in this paper, the average absolute error (AAE), root mean square error (L_2), and maximum absolute error (L_∞) norms are used to measure the accuracy of the proposed method, and they are calculated as follows:

$$L_2 = \sqrt{\frac{1}{Mx}\sum_{j=1}^{Mx}\left|U_{2j2n} - U_{j\,n}\right|^2}, L_\infty = \max_{1\le j\le Mx}\left|U_{2j2n} - U_{j\;N}\right|, AAE = \frac{1}{N}\sum_{j=1}^{Mx}\left|U_{2j2n} - U_{jn}\right|, \tag{23}$$

where $U_{2j\,2n}$ and $U_{j\,n}$ are numerical solutions of given in-viscose Burger equation obtained at the grid point (x_j, t_n) where $U_{2j\;2n}$ is a numerical solution obtained at the selected point by doubling the maximum number of grid-point in solution interval.

4. Results of Numerical Experiments

To test the validity of proposed methods, we have considered one model problem subjected to three different initial conditions associated with different desperation parameters given as follows.

Example 1. Consider the Riemann problem or classical in-viscosity Burger equation given in (1), which is subjected to discontinuous (shock) initial condition of the following form:

$$f(x,0) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases} \tag{24}$$

Example 2. Consider the Riemann problem or classical in-viscosity Burger equation given in (1), which is subjected to the piecewise continuous initial condition of the following form:

$$f(x,0) = \begin{cases} 0 & x \le 0, \\ 1 - x, & 0 \le x \le 1, \\ 1, & x > 1. \end{cases} \tag{25}$$

Example 3. Consider the Riemann problem or classical in-viscosity Burger equation given in (1), which is subjected to the Gaussian initial condition of the following form:

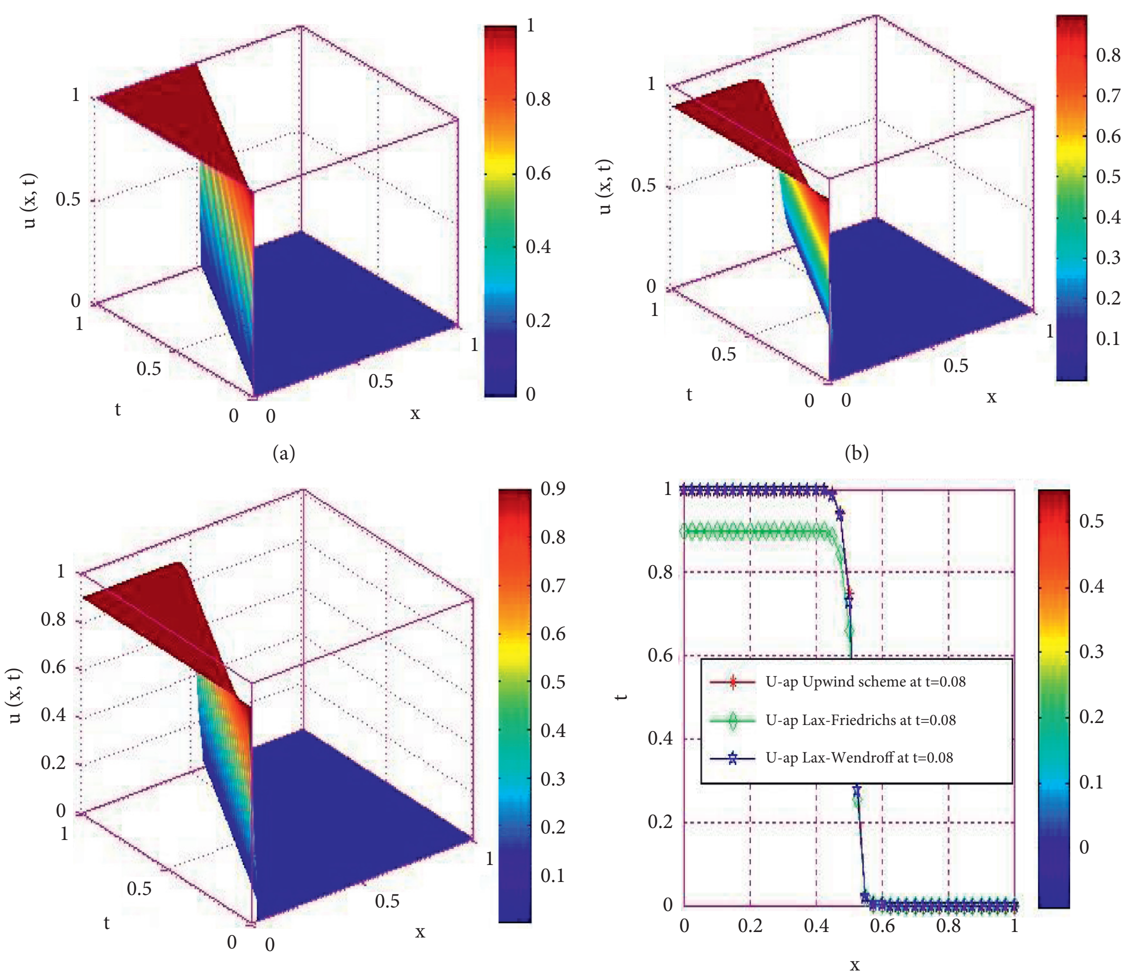
$$f(x,0) = e^{2(x-1)^2} \quad 0 \le x \le 1. \tag{26}$$

5. Discussion

In this paper, the upwind method, Lax–Friedrichs scheme, and Lax–Wendroff scheme are applied to solving one-dimensional first-order in-voices Burger equation in the application of physical phenomena. To demonstrate competence of these three methods, one model example subjected to three different initial conditions is solved by taking different values for step size h, and time step k. The computation of numerical results obtained by present methods has been presented in terms of average absolute error, root mean square error, and pointwise maximum absolute error that calculated the result obtained from between two successive iterated numerical solutions, and the results are summarized in tables and graphs. In the present numerical computation, the result presented in Table 1 shows that average absolute error (AAE), roots mean square error (L_2), and point wise maximum absolute error norm (L_∞) decrease as the number of mesh points Mx and Nt increases in both directions. However, the accuracy of the Lax-Wendroff method rapidly increases and its superior to the accuracy of both the up-wind method and the Lax-Friedrichs scheme. Thus, an accurate solution is obtained after the breaking solution interval (either time or spatial direction). It means that a solution after the breaking time is to allow refinement of discontinuities of solution U. Such discontinuity is called a shock. For a better understanding about discontinuities (shock) of solution for nonlinear partial differential equation, you can see reference in [16]. Figure 1 shows the physical background of discontinuity solution within the traditional form of a conservation law (the rate of change of the total amount of U in any sections of $0 \le x \le 1$ must be balanced by the net inflow across 0 to 1). To obtain the satisfaction of this condition, we need some shock conditions relating to the jumps of U across the discontinuity, see [16]. Again the result presented in Table 2 also shows that all average absolute error (AAE), roots mean square error norm (L_2), and maximum absolute error norm (L_∞) decrease rapidly as step length and time step decreases (one of them is constant wise verse) when the model problem is solved subjected to the piecewise continuous initial condition. In addition, in this case, the accuracy of the present method is rapidly increased. However, when we compare the method, the accuracy of Lax-Wendroff is superior to the accuracy of both up-wind method and Lax-Friedrichs scheme. Further, as shown in Figure 2, the proposed method approximates the solution very well for different values of step length h and time step. The result presented in Table 3 also shows that all average absolute error (AAE), roots mean square error norm (L_2), and maximum absolute error norm (L_∞) decrease rapidly as step length and time step decrease (one of them is constant wise verse) when the model problem is solved with being subjected to Gaussian initial condition. In addition, in this case the accuracy of present method is rapidly increased.

Table 1: Comparison of point average absolute error, root mean square error, and pointwise maximum absolute error for problem give in example one with computations domain carried out until final time $T = 1$ with different mesh size h and time step k.

Applied numerical scheme	Specified mesh size		Estimated errors by using specified mesh size		
	h	k	L_∞	L_2	AAE
Upwind methods	0.025	0.01	$2.5000E-03$	$1.1291E-03$	$5.1980\ E-04$
	0.0125	0.002	$1.2500E-03$	$3.5557E-04$	$1.0230E-04$
	0.01	0.01	$1.0000E-03$	7.0898E − 4	$5.0495E-04$
	0.01	0.005	$1.0000E-03$	$5.0187E-04$	$2.5373E-04$
Lax–Friedrichs scheme	0.025	0.01	$2.5000E-03$	$1.1117E-03$	$5.5122E-04$
	0.0125	0.02	$1.2500E-03$	$3.4930E-04$	$1.1185E-04$
	0.01	0.01	$1.0000E-03$	$7.0736E-04$	$5.0775e-04$
	0.01	0.005	$1.0000E-03$	$4.9905E-04$	$2.5837E-04$
Lax–Wendroff scheme	0.025	0.01	$3.2196E-04$	$1.3774E-04$	$6.2141E-05$
	0.0125	0.02	$1.7711E-03$	$4.6476E-04$	$1.3230E-04$
	0.01	0.01	$1.1205E-03$	$7.8888E-04$	$5.5965E-04$
	0.01	0.005	$1.2586E-03$	$6.0864E-04$	$3.0534E-04$



(a) (b) (c) (d)

Figure 1: Solution profile of Example one subjected to discontinuous (shock) initial condition on the uniform mesh of maximum number grid point is $Mx = 100$ & time step is $\Delta t = 0.001$. (a) Solution obtained by the upwind scheme; (b) solution obtained by Lax–Friedrichs scheme; (c) solution obtained by Lax–Wendroff scheme; (d) comparison of numerical solution at selected time value.

But when we compare the accuracy of present three methods, the accuracy of all three schemes are uniformly increasing as step length and time step decreases (one of them is constant wise verse), and no superior between the accuracy of this three methods. Hence, also, in this case, Figure 3 shows that the proposed method approximates the

Table 2: Comparison of point average absolute error, root mean square error, and pointwise maximum absolute error for problem give in example two with computations domain carried out until final time $T = 1$ with different mesh size h and time step k.

Applied numerical scheme	Specified mesh size		Estimated error with specified mesh size		
	h	k	L_∞	L_2	AAE
Upwind methods	0.025	0.01	$2.5000E-03$	$1.5636E-03$	$9.9388E-04$
	0.0125	0.01	$1.2500E-03$	$1.1102E-03$	$9.9280E-04$
	0.01	0.01	$1.0000E-03$	$9.9420E-04$	$9.9277E-04$
	0.01	0.005	$6.0371E-04$	$7.0294E-05$	$4.9779E-05$
Lax–Friedrichs scheme	0.025	0.01	$2.4310E-03$	$1.4800E-03$	$9.1592E-04$
	0.0125	0.01	$1.2580E-03$	$1.0934E-03$	$8.6834E-04$
	0.01	0.01	$5.230E-04$	$9.8502E-04$	$5.0675E-04$
	0.01	0.005	$4.7900E-05$	$7.8787E-05$	$9.8185E-05$
Lax–Wendroff scheme	0.025	0.01	$2.0481E-03$	$1.3029E-03$	$1.0086E-04$
	0.0125	0.01	$1.3046E-03$	$1.1242E-03$	$1.0305E-04$
	0.01	0.01	$8.1917E-04$	$2.0071E-04$	$6.0012E-05$
	0.01	0.005	$4.0910E-05$	$7.1302E-05$	$5.0288E-05$

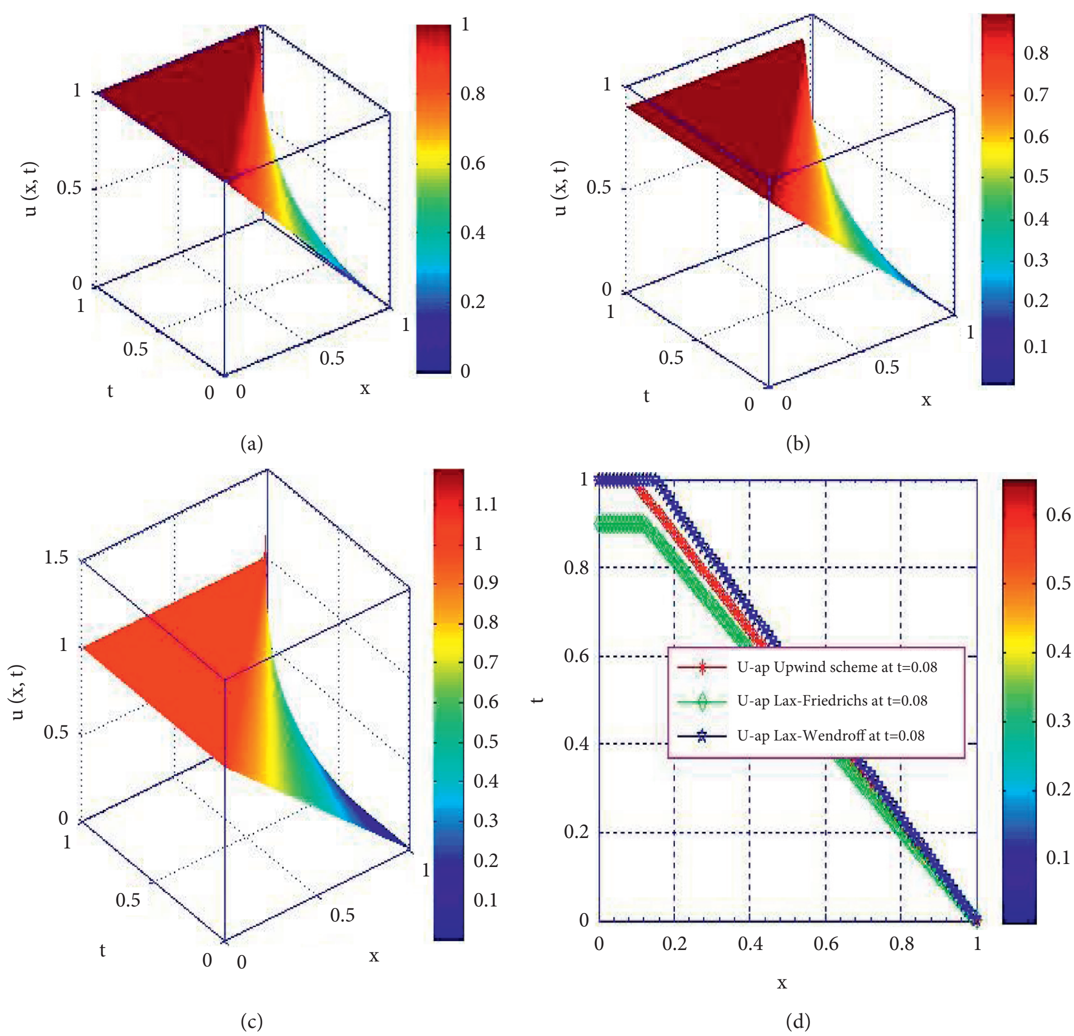
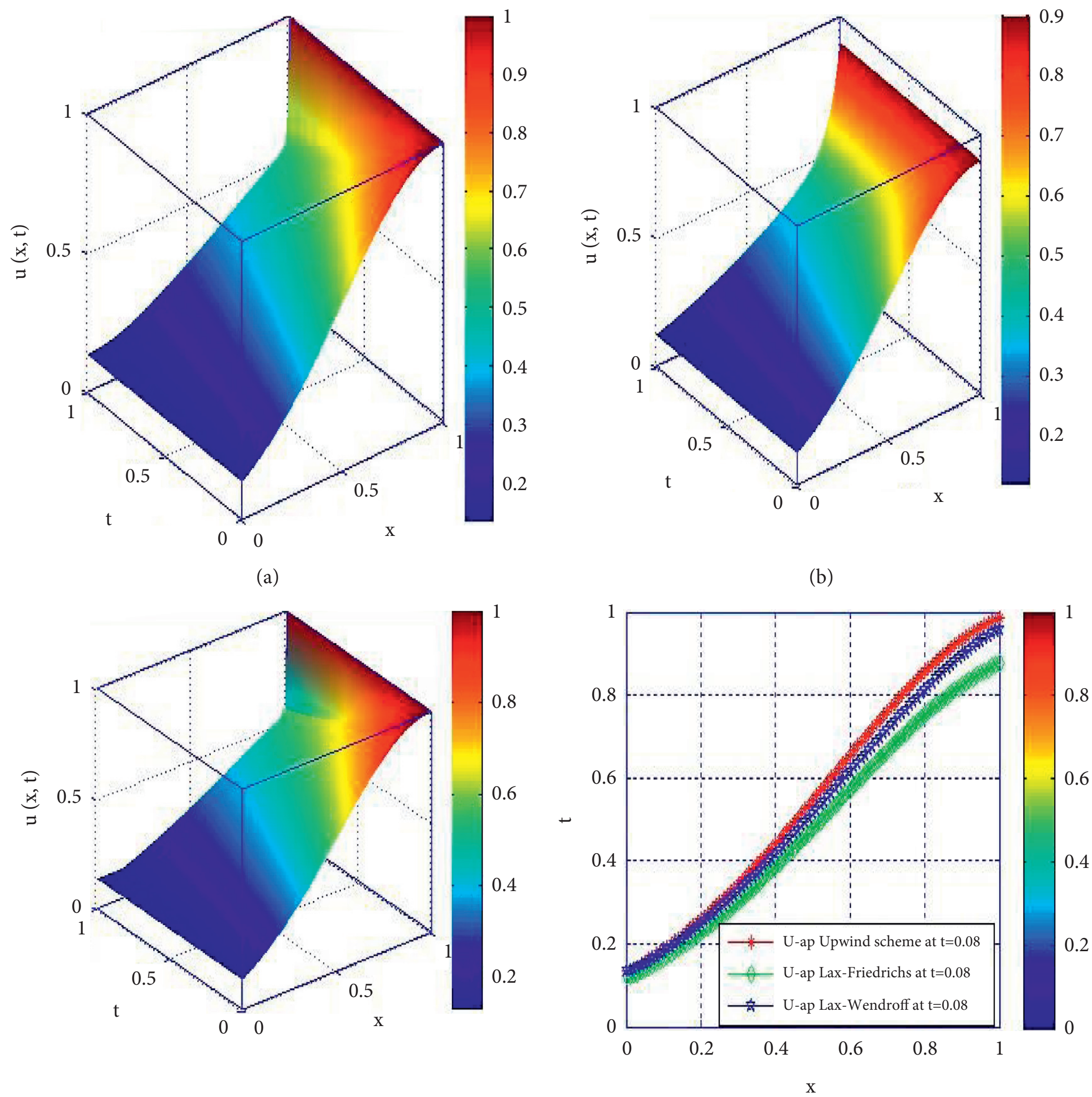


Figure 2: Solution profile of Example two subjected to piecewise continuous initial condition on the uniform mesh of maximum number grid point is $Mx = 100$ & time step is $\Delta t = 0.01$. (a) Solution obtained by Upmind scheme; (b) solution obtained by Lax-Friedrichs scheme; (c) solution obtained by Lax Wendroff scheme; (d) comparison of numerical solution at the selected time.

Table 3: Comparison of point average absolute error, root mean square error, and pointwise maximum absolute error for problem given in example three with computations domain carried out until final time $T = 1$ with different mesh size h and time step k.

Applied numerical scheme	Specified mesh size		Estimated error with the specified mesh size		
	h 0.025	k 0.01	L_∞ 2.5000$E-03$	L_2 1.0598$E-03$	AAE 6.0625$E-03$
Upwind methods	0.0125	0.02	1.2500$E-03$	1.0472$E-03$	1.1867$E-03$
	0.01	0.01	1.0000$E-03$	6.6462$E-04$	5.9784$E-04$
	0.01	0.005	1.0000$E-03$	4.7112$E-04$	3.0041$E-04$
Lax–Friedrichs scheme	0.025	0.01	2.5000$E-03$	1.0598E -03	6.0625E e-03
	0.0125	0.02	1.2500$E-03$	1.0472$E-03$	1.1867$E-03$
	0.01	0.01	1.0000$E-03$	6.6462E -04	5.9784E -04
	0.01	0.005	1.0000$E-03$	4.7112$E-04$	3.0041$E-04$
Lax–Wendroff scheme	0.025	0.01	2.5000$E-03$	1.0598$E-03$	6.0625$E-04$
	0.0125	0.02	1.2500$E-03$	7.4412$E-04$	5.9924$E-04$
	0.01	0.01	1.0000$E-03$	6.6462E -04	5.9784$E-04$
	0.01	0.005	1.0000$E-03$	4.7112E -04	3.0041$E-04$



(a) (b) (c) (d)

Figure 3: Solution profile of Example three subjected to Gaussian initial condition on the uniform mesh of maximum number grid point is $Mx = 100$ & time step is $\Delta t = 0.001$. (a) Profile of solution obtained by Upwind scheme; (b) profile of solution obtained by Lax-Friedrichs scheme; (c) profile of solution obtained by Lax-Wendroff scheme; (d) comparison of numerical solution at the selected time.

solution very well for $Mx = 100$ & the time step is $\Delta t = 0.001$. Generally, the solution satisfies the entropy condition and specifying the behavior solution obtained by present methods and the effects of mesh sizes in the solution domain are acceptable. Hence, in all cases the entropy of the conservation laws of system is nondecreasing. Thus, to further accomplish the accuracy of all three present methods, both the theoretical and numerical error bounds have been established. Then, the established error bound for all three method is acceptable. Because the results in the Tables and graphs are further confirmed, computational rate of convergence and this established error bound for all three cases of the applied initial condition. Therefore, applications of in-viscous-Burger equations are further simulated by these three methods.

6. Conclusion

Approaches, the upwind, Lax–Friedrichs and Lax–Wendroff schemes are applied to solve in-viscous-Barger equation numerically presented in this study. Generally, the comparison of the accuracy of the numerical solution obtained by these three presents' methods with each other reveals that the Lax-Wendroff method is more convenient, reliable, and effective than both the upwind method, Lax-Friedrichs scheme. As it can be seen from comparisons of the numerical result obtained by all those three numerical techniques in terms of error norm listed in tables, the accuracy of the numerical result obtained by the Lax–Wendroff scheme is faster increase either both Mx and Nt increase or one of them is fixed (either Mx or Nt is fixed) and it is superior to the accuracy of both upwind method and Lax–Friedrichs scheme. Because either both Mx and Nt increase or one of them is fixed (either Mx or Nt is fixed), the error norms of numerical result obtained by Lax-Wendroff scheme rapidly decrease. This shows that the accuracy of the numerical result increases and the accuracy of this numerical result is improved as we compare the accuracy of the numerical result obtained by both upwind method and the Lax–Friedrichs scheme. In summary, the upwind, Lax–Friedrichs, and Lax–Wendroff schemes are capable of solving in-viscous Barger equations. Based on the findings, Lax–Wendroff scheme gives well approximate solutions for the given model problem and gives better accuracy of the numerical solution with a fixed step size h, and decreasing time step k.

Abbreviations

L_2: Root mean square error
L_∞: Maximum point wise error
AAE: Average absolute error
Nt: Maximum number of grid point in temporal direction
h: Step length
k: Time step
Mx: Maximum number grid point in spatial direction.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Authors' Contributions

The author planned to apply the present three schemes, initiated developing the Research paper, suggested the experiment, conducted the experiments, and analyzed the empirical results by using MATLAB. The author also reviewed the results and approved the final version of the manuscript.

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