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Research Article

Radio and Radial Radio Numbers of Certain Sunflower Extended Graphs

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Communication systems including AM and FM radio stations transmitting signals are capable of generating interference due to unwanted radio frequency signals. To avoid such interferences and maximize the number of channels for a predefined spectrum bandwidth, the radio-k-chromatic number problem is introduced. Let $G = (V, E)$ be a connected graph with diameter d and radius ρ. For any integer k, $1 \le k \le d$, radio k−coloring of G is an assignment φ of color (positive integer) to the vertices of G such that $d(a,b) + |\varphi(a) - \varphi(b)| \ge 1 + k$, $\forall a,b \in V(G)$, where $d(a,b)$ is the distance between a and b in G. The biggest natural number in the range of φ is called the radio k−chromatic number of G, and it is symbolized by $r_{ck}(\varphi)$. The minimum number is taken over all such radio k−chromatic numbers of φ which is called the radio k−chromatic number, denoted by $r_{ck}(G)$. For $k = d$ and $k = \rho$, the radio k−chromatic numbers are termed as the radio number $(rn(G))$ and radial radio number $(rr(G))$ of G, respectively. In this research work, the relationship between the radio number and radial radio number is studied for any connected graph. Then, several sunflower extended graphs are defined, and the upper bounds of the radio number and radial radio number are investigated for these graphs.

1. Introduction

The channel frequency assignment problem was first proposed by Griggs and Yeh [1] in 1992 for the amplitude modulation radio stations. Due to the cochannel interference, there is a challenge to fix the transmitters in a particular geographical area. Therefore, studying the channel assignment problem in radio stations is NP-complete. However, Fotakis et al. [2] proved that even for graphs with diameter 2, the problem is NP-hard. Chartrand et al. [3] presented the theoretical graph definition for the radio-k-chromatic number as follows.

Let $G = (V, E)$ be a connected graph with diameter d and radius ρ. For any integer k, $1 \le k \le d$, radio k−coloring of G is an assignment φ of color (positive integer) to the vertices of G such that $d(a,b) + |\varphi(a) - \varphi(b)| \ge 1 + k$, $\forall a,b \in V(G)$, where $d(a,b)$ is the distance between a and b in G. The biggest natural number in the range of φ is called the radio k−chromatic number of G, and it is symbolized by $r_{ck}(\varphi)$. The minimum number is taken over all such radio k−chromatic numbers of φ which is called the radio k−chromatic number, denoted by $r_{ck}(G)$.

Čada et al. [4] proved that, for any distance graph $D(t-1, t)$, we have

$$r_{ck}(D(t-1,t)) \le \begin{cases} \frac{1}{2}k^2 + k - \frac{t+2}{2}, & \text{where } t > 2 \text{ is an integer, } k > 3 \text{ is an odd integer,} \\ \frac{1}{2}tk^2, & k > 0 \text{ is an odd integer, } t > 3 \text{ is an even integer.} \end{cases} \quad (1)$$

Recently, Bantva [5] improved this general lower bound. Based on different k values, the radio $k-$chromatic number is classified into different problems.

For $k = d$, the radio $k-$chromatic number is termed as the radio number problem, and it is symbolized by $rn(G)$. It was introduced by Chartrand et al. [6] for the purpose of determining the maximum number of channels for frequency modulation (FM) radio stations by minimum utilization of spectrum bandwidth. The radio number problem has been studied by several researchers [7, 8]. In 2017, Avadayappan et al. [9] brought in the concept of radial radio labelling. A mapping $\varphi: V(G) \longrightarrow N \cup \{0\}$ for a connected graph $G = (V, E)$ is called a radial radio labelling if this satisfies the inequality $|\varphi(a) - \varphi(b)| + d(a, b) \geq \rho + 1 \forall a, b \in V(G)$, where ρ is the radius of the graph G. Radial radio number of φ symbolized by $rr(\varphi)$ is the maximum number mapped under φ. The radial radio number of G, denoted by $rr(G)$, is equal to $\min\{rr(\varphi)\ /\varphi \text{ is a radial radio labelling of } G\}$. A few number of research articles [10, 11] were published in the area of radial radio labelling. In this paper, we have studied a comparative relation between $rn(G)$ and $rr(G)$. Furthermore, we have defined and determined the radio and radial radio numbers of certain sunflower extended graphs such as $SS(n, ͱ)$, $CS(n, ͱ)$, and $WS(n, ͱ)$.

2. Relation between the Radio Number and Radial Radio Number

This section deals with certain results which connect $rn(G)$ with $rr(G)$ for any connected graph G.

Definition 1. The eccentricity of a vertex z, represented by $e(z)$ in a connected graph G, is the maximum distance from z to any other vertex in G. That is, $e(z) = \max\{d(z, a) \forall a \in V(G)\}$. The maximum eccentricity of the vertices of G is called the diameter of the graph, and it is symbolized by d or $di\ am(G)$. In addition, the radius of graph G, symbolized by ρ or $ra\ d(G)$, is the minimum eccentricity of the vertices of G.

Definition 2. A connected graph $G = (V, E)$ is called a self-centred graph if $e(u) = e(v) \forall u, v \in V(G)$. In other words, $di\ am(G) = ra\ d(G)$.

The following is a straight result from the definitions of the radio number and radial radio number.

Theorem 1. *For any connected graph G, $rn(G) \geq rr(G)$.*

Chartrand et al. [6] proved the following three theorems, which will be used to study the general results for the radial radio number.

Theorem 2. *If G is a connected graph of order n and diameter d, then $n \leq rn(G) \leq (n-1)d$.*

Theorem 3. *For a complete $k-$partite graph G of order n, $rn(G) = n + (k+1)$.*

Theorem 4. *Every connected graph G of order n with $rn(G) = n$ is self-centred.*

Using Theorem 5 and Definition 2, we have attained the equality of Theorem 1 as follows.

Theorem 5. *A connected graph G of order n is self-centred if and only if $rn(G) = rr(G) = n$.*

Theorem 6. *Let $G = (V, E)$ be a complete $k-$partite graph of order n; then, $rr(G) = k$.*

Proof. Let the vertex set of G be partitioned into k disjoint sets $U_1, U_2, \ldots, U_k$ such that $U_i \cap U_j = \emptyset, 1 \leq i \neq j \leq k$, and $V = \cup_{i=1}^{k} U_i$. The radius of the complete $k-$partite graph is 1, and all the vertices in the sets U_i, $1 \leq i \leq k$, are at distance two. Hence, we can label the vertices in each set U_i as $i\,(i = 1, 2, \ldots, k)$. Clearly, the radial radio labelling condition $d(a, b) + |\varphi(a) - \varphi(b)| \geq 2$ is satisfied for any pair of vertices in G. Hence, $rr(G) = k$. □

Theorem 7. *If G is a connected graph of order $n > 1$ and radius ρ, then $2 \leq rr(G) \leq (n-1)\rho$.*

Proof. Given G is a connected graph that contains at least two vertices. Therefore, the lower bound of the theorem attains in the particular case of Theorem 6 which is for the complete bipartite graphs. Furthermore, the upper bound is obtained by replacing d by ρ in Theorem 2. Consequently, $2 \leq rr(G) \leq (n-1)\rho, n > 1$. □

3. Results and Discussion

In this section, we have defined and investigated the radial radio and radio number of some sunflower extended graphs such as star-sun graph $SS(n, ͱ)$, complete-sun graph $CS(n, ͱ)$, wheel-sun graph $WS(n, ͱ)$, and fan-sun graph $FS(n, ͱ)$.

Definition 3. A sunflower graph consists of a wheel with a centre vertex w_n, n-cycle $w_0, w_1, \ldots, w_{n-1}$, and additional n vertices $u_0, u_1, \ldots, u_{n-1}$ where u_i is joined with edges to (w_i, w_{i+1}), $i = 0, 1, 2, \ldots n-1$, and $i+1$ is taken as modulo n. It is represented by SF_n. The radius, diameter, and number of vertices of SF_n are 2, 4, and $2n+1$, respectively.

Definition 4. A star graph, denoted by $S_{ͱ+1}$, is defined as a complete bipartite graph of the form $K_{1,ͱ}, ͱ > 1$. In other words, $S_{ͱ+1}$ is a tree having ͱ leaves and one internal vertex.

Definition 5. A star-sun graph, denoted by $SS(n, ͱ)$, is a graph obtained from the sunflower graph SF_n and n copies of star graph $S_{ͱ+1}$ by merging the internal vertex of the k^{th} star graph $S_{ͱ+1}$ and vertex u_{k-1} of SF_n, $1 \leq k \leq n$, as shown in Figure 1(a).

Remark 1. The cardinality of $V(SS(n, ͱ))$ and $E(SS(n, ͱ))$ in $SS(n, ͱ)$ is $n(ͱ + 2) + 1$ and $2nͱ + 4n$, respectively. Also, the diameter and radius of the graph are 6 and 3, respectively.

Definition 6. A complete-sun graph, denoted by $CS(n,\hbar)$, is a graph obtained from the sunflower graph SF_n and n copies of complete graph $K_\hbar$ by merging a vertex of the m^{th} complete graph $K_\hbar$ and the vertex u_{m-1} of SF_n, $1 \le m \le n$, as shown in Figure 1(b). Here, we have $|V(CS(n,\hbar))| = n(\hbar + 1)$ and $|E(CS(n,\hbar))| = (n\hbar(\hbar - 1)/2) + 2n$.

Remark 2. The diameter and radius of $CS(n,\hbar)$ are 6 and 3, respectively.

Definition 7. A wheel-sun graph, denoted by $WS(n,\hbar)$, is a graph obtained from the sunflower graph SF_n and n copies of wheel graph $W_{\hbar+1}$ by merging the vertex u_{k-1} of SF_n and the centre vertex of the k^{th} wheel, where $1 \le k \le n$ as shown in Figure 1(c).

Remark 3. The number of vertices in $WS(n,\hbar)$ is $n(\hbar + 2) + 1$, while its number of edges is $2n(2n + \hbar)$. Also, its diameter and radius are 6 and 3, respectively.

Definition 8. A fan-sun graph is a graph obtained from the sunflower graph SF_n and n copies of fan graph $F_{\hbar+1} = P_\hbar + K_1$ by merging K_1 of the k^{th} fan and the vertex u_{k-1} of SF_n, $1 \le k \le n$. It is denoted by $FS(n,\hbar)$ as shown in Figure 1(d).

Remark 4. For the graph $FS(n,\hbar)$, the number of edges is $4n + n(2\hbar - 1)$, while the number of vertices is $n(\hbar + 2) + 1$. Moreover, the diameter and radius are 6 and 3, respectively.

In this work, we name the newly included $n\hbar$ vertices of $SS(n,\hbar)$, $WS(n,\hbar)$, and $FS(n,\hbar)$ as $v_1, v_2, \ldots, v_{n\hbar}$ in the clockwise sense.

3.1. Radial Radio Number of Sunflower Extended Graphs.

The following theorems provide the upper bound for the radial radio number of $S(n,\hbar)$, $CS(n,\hbar)$, and $WS(n,\hbar)$.

Theorem 8. *Let G be the sun-star graph $SS(n,\hbar)$. Then, $rr(SS(n,\hbar)) \le 3n + 2\hbar + 1$.*

Proof. First, we define a mapping $\varphi: V(SS(n,\hbar)) \longrightarrow N$ as follows: $\varphi(v_{\hbar(j-1)+i}) = 2i, i = 1,2,\ldots,\hbar, j = 1,2,\ldots,n$, $\varphi(w_{2i}) = 2(\hbar + 1 + i), i = 0,1,\ldots,\lceil n/2\rceil - 1$, $\varphi(w_{2i+1}) = 2(\lceil n/2\rceil + \hbar + 1 + i), i = 0,1,\ldots,\lfloor n/2\rfloor - 1$, $\varphi(u_{2i}) = 2(n + \hbar + 1) + i, i = 0,1,\ldots,\lceil n/2\rceil - 1$, $\varphi(u_{2i+1}) = 2(n + \hbar + 1) + \lceil n/2\rceil + i, i = 0,1,2,\ldots,\lfloor n/2\rfloor - 1$, and $\varphi(w_n) = 1$ as shown in Figure 2.

Since the radius of the graph is 3, we must verify φ satisfies the radial radio labelling condition $d(a,b) + |\varphi(a) - \varphi(b)| \ge 1 + rad(SS(n,\hbar)) = 1 + 3 = 4$ for every pair of vertices $a, b \in V(SS(n,\hbar))$.

Let us choose any two arbitrary vertices a and b in the sun-star graph.

Case 1: suppose a and b are star vertices, then a and b are of the form $a = v_l$ and $b = v_m, 1 \le m \ne l \le n\hbar$.

Case 1.1: if $l = \hbar(j-1) + \alpha$ and $m = \hbar(j-1) + \beta, \alpha \ne \beta$, then the value of a and b under φ is $2p$ and $2q$, respectively. Also, a and b are at a distance two. Hence, the radial radio labelling condition becomes $d(a,b) + |\varphi(a) - \varphi(b)| = 2 + |2(\alpha - \beta)| \ge 4$ since $\alpha \ne \beta$.

Case 1.2: if $l = \alpha + \hbar(s-1)$ and $m = \beta + \hbar(t-1), t \ne s$, then a and b are at a distance at least 4. Hence, the radial radio labelling condition is trivially satisfied.

Case 2: let $a = v_{(j-1)+l}$, $1 \le l \le \hbar$, and $b = w_m, 0 \le m \le n-1$; then, the value of $\varphi(a)$ is $2l$, and $\varphi(b)$ is at least $2(\hbar + 1 + m)$. Furthermore, $d(a,b)$ is at least 2. Therefore, $d(a,b) + |\varphi(a) - \varphi(b)| \ge 2 + |2\hbar - 2(\hbar + 2)| \ge 4$.

Case 3: if we take $a = v_{\hbar(j-1)+s}$, $1 \le s \le \hbar$, and $b = u_t, 0 \le t \le n-1$, then $|\varphi(a) - \varphi(b)| \ge |2(n + \hbar + 1) - (2\hbar)| \ge 2n > 4$ since $n > 2$, which trivially verifies the radial radio labelling condition.

Case 4: suppose $a = v_{\hbar(j-1)+l}$, $1 \le l \le \hbar$, and $b = w_m, 0 \le m \le n-1$, then the value of $\varphi(a)$ is $2l$, and $\varphi(b)$ is at least $2(\hbar + 1 + m)$. Furthermore, $d(a,b)$ is at least 2. Therefore, $d(a,b) + |\varphi(a) - \varphi(b)| \ge 2 + |(2\hbar + 2) - 2\hbar| \ge 4$.

Case 5: if a is the centre vertex of the wheel and b is any other star vertex, then the distance between them is exactly 3. Also, $\varphi(a) = 0$ and $\varphi(b) \ge 1$. Therefore, $d(a,b) + |\varphi(a) - \varphi(b)| \ge 4$.

Case 6: let a and b be the vertices in the n-cycle of the sunflower graph.

Case 6.1: if $a = w_{2l}$ and $b = w_{2m}, 0 \le l \ne m \le \lceil n/2\rceil - 1$, then $|\varphi(a) - \varphi(b)| \ge |2(l - m)|$. Again, $d(a,b) = 2$. Since $l \ne m$, the condition for the radial radio labelling is satisfied.

Case 6.2: suppose a and b are of the form w_{2l+1} and w_{2m+1}, where $0 \le l \ne m \le \lfloor n/2\rfloor - 1$, then the distance between them is exactly two. Also, the function values of a and b are $2(\hbar + l + \lceil n/2\rceil + 1)$ and $2(\hbar + m + \lceil n/2\rceil + 1)$, respectively. Hence, the radial radio labelling condition becomes $d(a,b) + |\varphi(a) - \varphi(b)| = 2 + |2(\lceil n/2\rceil + \hbar + l + 1) - (2(\lceil n/2\rceil + \hbar + m + 1))| \ge 4$.

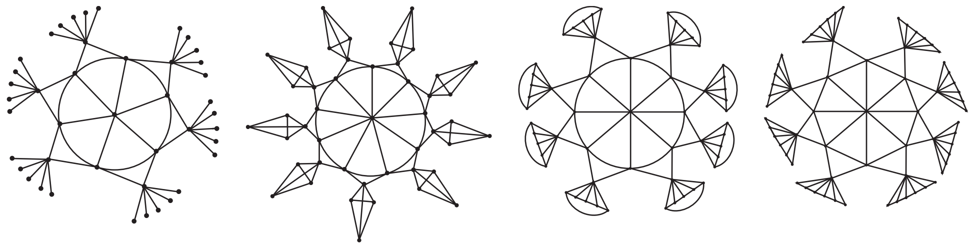
Case 6.3: suppose $a = w_{2l}$, $0 \le l \le \lceil n/2\rceil - 1$, and $b = w_{2m+1}, 0 \le m \le \lfloor n/2\rfloor - 1$, then $\varphi(a) = 2(\hbar + 1 + l)$ and $\varphi(b) = 2(\hbar + m + \lceil n/2\rceil + 1)$.

If $m = 0$ and $l = \lceil n/2\rceil - 1$, then $d(a,b) = 2$; else, $d(a,b) \ge 1$. Hence, in both possibilities, we obtain $d(a,b) + |\varphi(a) - \varphi(b)| = 2 + |2(\lceil n/2\rceil + \hbar + 1) - (2(\hbar + \lceil n/2\rceil - 1 + 1))| \ge 2 + 2 = 4$ and $d(a,b) + |\varphi(a) - \varphi(b)| \ge 1 + |2(\lceil n/2\rceil + \hbar + m) - (2(\hbar + l + 1))| \ge 1 + 4 > 4$.

Case 7: if $a = w_l, 0 \le l \le n-1$, and $b = w_n$, then $|\varphi(a) - \varphi(b)|$ is at least $2\hbar + 1$. Since $n > 2$, the radial radio labelling condition is easily verified.

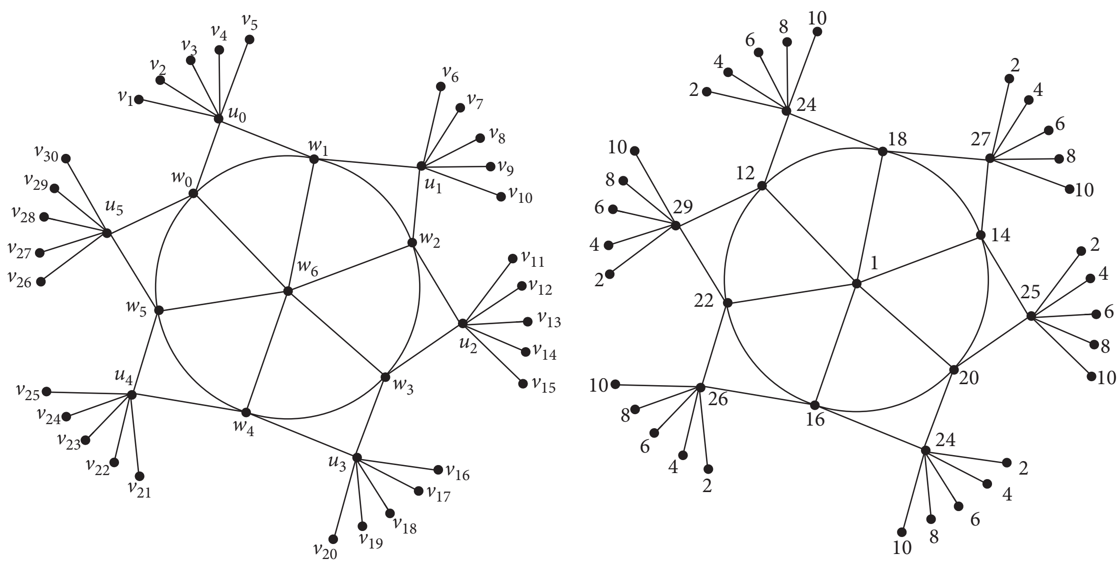
Case 8: suppose $a = u_l$ and $= u_m, 0 \le l \ne m \le n-1$.

Case 8.1: if $l = 2\alpha$ and $m = 2\beta$, $0 \le \alpha \ne \beta \le \lceil n/2\rceil - 1$, respectively, then $\varphi(a) = 2(n + \hbar + 1) + \alpha$ and $\varphi(b) = 2(n + \hbar + 1) + \beta$. Also, the distance between a and b is 2. Hence, we have $d(a,b) + |\varphi(a) - \varphi(b)| = 2 + |2(n + \hbar + 1) + \alpha - (2(n + \hbar + 1) + \beta)| > 3$ since $\alpha \ne \beta$.



(a) (b) (c) (d)

Figure 1: Different sunflower extended graphs.



(a) (b)

Figure 2: A sun-star graph $SS(6,5)$ and its radial radio labelling.

Case 8.2: if $l = 2\alpha + 1$ and $m = 2\beta + 1$, $0 \le \alpha \ne \beta \le \lceil n/2 \rceil - 1$, then $d(a,b) = 2$. Also, φ takes the values of a and b to $2(n + \hbar) + \lceil n/2 \rceil + \alpha + 1$ and $2(n + \hbar + 1) + \lceil n/2 \rceil + \beta + 1$, respectively. Therefore, $d(a,b) + |\varphi(a) - \varphi(b)| = 2 + |2(n + \hbar + 1) + \lceil n/2 \rceil + \alpha + 1 - (2(n + \hbar + 1) + \lceil n/2 \rceil + \beta + 1)| \ge 4$.

Case 8.3: let $a = u_{2l}$, $0 \le l \le \lceil n/2 \rceil - 1$, and $b = u_{2m+1}$, $0 \le m \le \lfloor n/2 \rfloor - 1$; then, a and b are mapped to $2(n + \hbar + 1) + l$ and $2(n + \hbar + 1) + \lceil n/2 \rceil + m + 1$, respectively.

When $l = \lceil n/2 \rceil - 1$ and $m = 0$, $d(a,b) = 2$, and hence, $d(a,b) + |\varphi(a) - \varphi(b)| = 2 + |2(n + \hbar + 1) + \lceil n/2 \rceil - 1 - (2(n + \hbar + 1) + \lceil n/2 \rceil + 1)| \ge 2 + 2 = 4$.

The remaining possibility is obvious since the distance between them is at least 1, and the modulus difference between $\varphi(a)$ and $\varphi(b)$ is at least 4.

Case 9: if $a = w_n$ and b is any vertex of the form u_l, $0 \le m \le n - 1$, then the verification is obvious since the difference in the function values of a and b is at least $2(n + \hbar)$.

Thus, φ is a valid radial radio labelling, and the vertex u_{n-1} receives the maximum number $2(n + \hbar + 1) + \lceil n/2 \rceil + \lfloor n/2 \rfloor - 1 = 3n + 2\hbar - 1$. That is, $rr(\varphi) = 3n + 2\hbar + 1$.

Hence, we conclude that $rr(SS(n,\hbar)) \le 3n + 2\hbar + 1$. □

Theorem 9. *Let G be the complete-sun graph $CS(n,\hbar)$. If $n \equiv 0 \pmod 3$, then the radial radio number of G satisfies $rr(G) \le 3(\hbar + 1) + 2n$.*

Proof. Let us name the newly included $n(\hbar - 1)$ vertices of $CS(n,\hbar)$ as $v_1, v_2 \ldots v_{n(\hbar-1)}$ in the clockwise sense. Now, we define a one-one mapping $\varphi: V(CS(n,\hbar)) \longrightarrow \{1, 2 \ldots\}$ as follows:

$$\begin{aligned}
\varphi\left(v_{(\hbar-1)(j-1)+i}\right) &= 3i-1, \quad i=1,2,\ldots,\hbar-1, j=1,2\ldots n,\\
\varphi\left(u_{3i+2}\right) &= 3\hbar-1, \quad i=0,1,\ldots,\frac{n}{3}-1,\\
\varphi\left(u_{3i}\right) &= 3\hbar+1, \quad i=0,1,\ldots,\frac{n}{3}-1,\\
\varphi\left(u_{3i+1}\right) &= 3\hbar+3, \quad i=0,1,\ldots,\frac{n}{3}-1,\\
\varphi\left(w_{2i}\right) &= 3\hbar+2i+5, \quad i=0,1,\ldots,\left\lceil\frac{n}{2}\right\rceil-1,\\
\varphi\left(w_{2i+1}\right) &= 3\hbar+2\left\lceil\frac{n}{2}\right\rceil+2i+5, \quad i=0,1,\ldots,\left\lfloor\frac{n}{2}\right\rfloor-1,\\
\varphi\left(w_n\right) &= 1.
\end{aligned} \tag{2}$$

This mapping is visible in Figure 3(a).

In the following, we claim that $d(a,b) + |\varphi(a) - \varphi(b)| \geq 4 \forall a,b \in V(CS(n,\hbar))$.

Let $a,b \in V(CS(n,\hbar))$.

Case 1: suppose $a = v_{(\hbar-1)(p-1)+\alpha}$ and $b = v_{(\hbar-1)(q-1)+\beta}, 1 \leq \alpha,\beta \leq \hbar-1$, $1 \leq p,q \leq n$, then $\varphi(v_{(\hbar-1)(p-1)+l}) = 3l-1$ and $\varphi(v_{(\hbar-1)(q-1)+m}) = 3m-1$.

Case 1.1: if $\alpha \neq \beta$, then $d(v_{(\hbar-1)(p-1)+\alpha}, v_{(\hbar-1)(q-1)+\beta}) = 1$ and $|\varphi(a) - \varphi(b)| \geq |3(\alpha-\beta)|$. Therefore, $d(a,b) + |\varphi(a) - \varphi(b)| \geq 1 + |3(\alpha-\beta)| \geq 4$ since $\alpha \neq \beta$.

Case 1.2: if $p = q$, then $d(v_{(\hbar-1)(p-1)+\alpha}, v_{(\hbar-1)(q-1)+\beta}) = 4$, which is enough for verifying the condition.

Case 2: assume that $a = v_{(\hbar-1)(p-1)+\alpha}$, $1 \leq \alpha \leq \hbar-1$, $1 \leq p \leq n$, and $b = u_\beta, 0 \leq \beta \leq n-1$; then, $\varphi(a) \leq 3\hbar-4$ and $\varphi(b) \geq 3\hbar-1$. Also, $d(a,b) \geq 1$. Therefore, $d(a,b) + |\varphi(a) - \varphi(b)| \geq 1+3 \geq 4$.

Case 3: if we take $= v_{(\hbar-1)(p-1)+\alpha}$, $1 \leq \alpha \leq \hbar-1, 1 \leq p \leq n$, and $b = w_\beta, 0 \leq \beta \leq n-1$, then $|\varphi(a) - \varphi(b)| \geq |3\hbar-4-(3\hbar+3)| > 4$ since $n > 2$, which verifies the condition trivially.

Case 4: assume that $a = u_\alpha$ and $b = w_\beta, 0 \leq \alpha,\beta \leq n-1$.

Case 4.1: if $\alpha = n-2$ and $\beta = 1$, then $\varphi(u_\alpha) = 3\hbar+3$ and $\varphi(w_\beta) = 3\hbar+5$. However, $d(u_\alpha, w_\beta) = 2$. Therefore, $d(a,b) + |\varphi(a) - \varphi(b)| \geq 4$. Otherwise, $|\varphi(a) - \varphi(b)|$ is greater than 3.

Case 5: let $a = u_\alpha$ and $b = u_\beta$, $0 \leq \alpha \neq \beta \leq n-1$. In this case, if $\alpha = 3p+2$ and $\beta = 3q+2$ or $\alpha = 3p+1$ and $\beta = 3q+1$ or $\alpha = 3p$ and $\beta = 3q$, $0 \leq p \neq q \leq (n/3)-1$, then $d(a,b) = 4$ and $|\varphi(a) - \varphi(b)| = 0$. Otherwise, $d(a,b) \geq 2$ and $|\varphi(a) - \varphi(b)| \geq 2$. Hence, in both possibilities, the condition for radial radio labelling is satisfied.

Case 6: assume that $a = w_\alpha$ and $b = w_\beta$, $0 \leq \alpha \neq \beta \leq n-1$. If $\alpha = 2p$ and $\beta = 2q$ or $\alpha = 2p+1$ and $\beta = 2q+1$, $0 \leq p \neq q$, $0 \leq p \neq q \leq \lceil n/2 \rceil - 1$, then $d(a,b) = 2$. Thus, $d(a,b) = 2$ and $|\varphi(a) - \varphi(b)| \geq 2$. Otherwise, $d(a,b) \geq 1$ and $|\varphi(a) - \varphi(b)| \geq 4$. Therefore, the condition holds in both of the possibilities.

Case 7: finally, let us assume that $a = w_n$, and b is any vertex in $CS(n,\hbar)$. If $b = v_1$, then $d(a,b) = 3$ and $|\varphi(a) - \varphi(b)| = 1$. Otherwise, the radial radio labelling condition is obviously true. Thus, φ satisfies the condition of radial radio labelling and attains the maximum value $3\hbar + 2\lceil n/2 \rceil + 2\lfloor n/2 \rfloor - 2 + 5 = 3\hbar + 2n + 3$ for the vertex $w_{2\lfloor n/2 \rfloor}$. Therefore, we get $rr(CS(n,\hbar)) \leq 3(\hbar+1) + 2n$.

The proof for the other two cases, namely, $n \equiv 1 \pmod 3$ and $n \equiv 2 \pmod 3$, is left to the reader. □

Theorem 10. *For $n > 2$ and $r > 3$, the radial radio number of the wheel-star graph satisfies*

$$rr(SS(n,\hbar)) \leq \begin{cases} 2(\hbar+n)+6, n \equiv 0 \pmod 3 \\ 2(\hbar+n)+8, n \equiv 1 \pmod 3 \\ 2(\hbar+n)+9, n \equiv 2 \pmod 3 \end{cases}. \tag{3}$$

We omit the proof, but Figure 3(b) illustrates the case $n \equiv 2 \pmod 3$.

3.2. Radio Number of Sunflower Extended Graphs. This section provides the upper bound for the radio number of $S(n,\hbar)$, $SS(n,\hbar)$, and $WS(n,\hbar)$.

Theorem 11. *For $n > 3$ and $n \equiv 0 \pmod 3$, the radio number of the complete-sun graph satisfies $rr(CS(n,\hbar)) \leq 18\hbar + 9n - 30$.*

Proof. We define a 1-1 mapping $\varphi: V(CS(n,\hbar)) \longrightarrow \{1,2\ldots\}$ as follows:

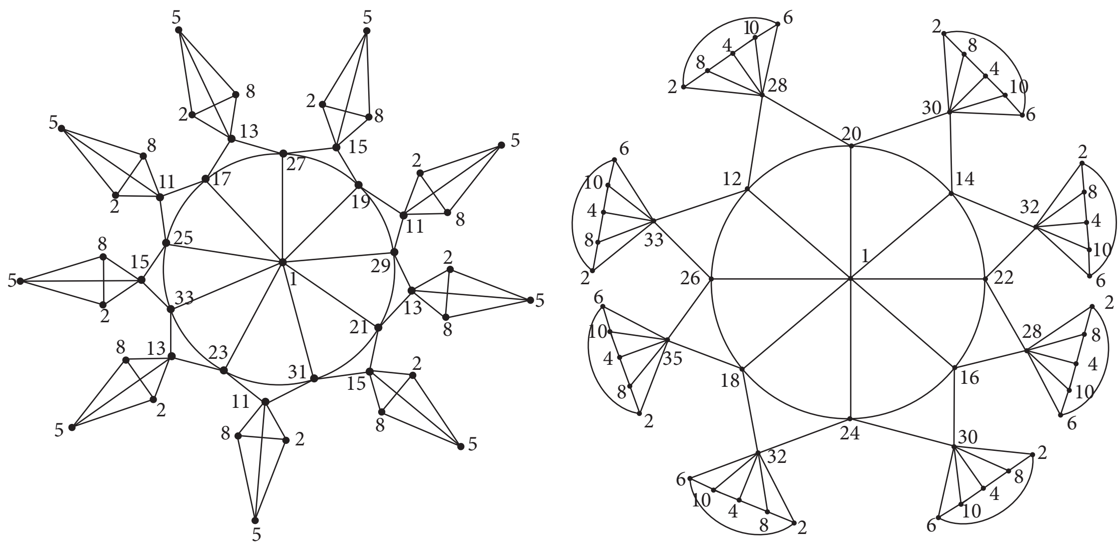


Figure 3: A radial radio labelling of complete-sun graph $CS(n,\hbar)$ for $n=9$ and $\hbar=4$ and wheel-sun graph $WS(8,5)$.

$$
\begin{aligned}
\varphi\left(v_{3(\hbar-1)(j-1)+i}\right) &= 6(i-1)+j+4, \quad i=1,2,\ldots,\hbar-1,\ j=1,2,\ldots\frac{n}{3},\\
\varphi\left(v_{(\hbar-1)(3j-2)+i}\right) &= 6\hbar+\frac{n}{3}+6(i-1)+j-8, \quad i=1,2,\ldots,\hbar-1,\ j=1,2,\ldots,\frac{n}{3},\\
\varphi\left(v_{(\hbar-1)(3j-1)+i}\right) &= 12\hbar+2\frac{n}{3}+6(i-1)+j-20, \quad i=1,2,\ldots,\hbar-1,\ j=1,2,\ldots,\frac{n}{3},\\
\varphi\left(u_{3i}\right) &= 18\hbar+n+3i-28, \quad i=0,1,\ldots,\frac{n}{3}-1,\\
\varphi\left(u_{3i+1}\right) &= 18\hbar+2n+3i-28, \quad i=0,1,\ldots,\frac{n}{3}-1,\\
\varphi\left(u_{3i+2}\right) &= 18\hbar+3(n+i)-28, \quad i=0,1,\ldots,\frac{n}{3}-1,\\
\varphi\left(w_{2i}\right) &= 18\hbar+4n+5i-25, \quad i=0,1,\ldots,\left\lceil\frac{n}{2}\right\rceil-1,\\
\varphi\left(w_{(2i+1)}\right) &= 18\hbar+4n+5\left\lceil\frac{n}{2}\right\rceil+5i-25, \quad i=0,1,\ldots,\left\lfloor\frac{n}{2}\right\rfloor-1,\\
\varphi\left(w_n\right) &= 1.
\end{aligned} \tag{4}
$$

See Figure 3(a).

Then, to show φ is a valid radio labelling, we must verify the inequality

$$d(a,b)+|\varphi(a)-\varphi(b)|\geq 7\,\forall a,b\in V(CS(n,\hbar)). \tag{5}$$

Let $a,b\in V(CS(n,\hbar))$.

Case 1: suppose that $a=v_\alpha$ and $b=v_\beta$, $1\leq\alpha\neq\beta\leq n(\hbar-1)$.

Case 1.1: if $\alpha=3(\hbar-1)(s-1)+p$ and $\beta=3(\hbar-1)(t-1)+q$ or $\alpha=(\hbar-1)(3s-2)+p$ and $\beta=(\hbar-1)(3s-2)+p$ or $\alpha=(\hbar-1)(3s-1)+p$ and $\beta=(\hbar-1)(3s-1)+p$ where $p\neq q$ and $s=t$, then $d(v_\alpha,v_\beta)=1$ and $|\varphi(v_\alpha)-\varphi(v_\beta)|\geq|6(p-q)|\geq 6$. In the same subcase, if $s\neq t$, then $d(v_\alpha,v_\beta)=6$ and $|\varphi(v_\alpha)-\varphi(v_\beta)|\geq 1$. So, $d(a,b)+|\varphi(a)-\varphi(b)|\geq 7$.

Case 1.2: if $\alpha=3(\hbar-1)(s-1)+p$ and $\beta=3(\hbar-1)(t-1)+q$, where $1\leq p,q\leq\hbar-1$, $1\leq s,t\leq(n/3)$, then $\varphi(v_\alpha)=6(p-1)+s+4$ and $\varphi(v_\beta)=6\hbar+(n/3)+6(q-1)+t-8$. In addition, $d(v_\alpha,v_\beta)\geq 4$. Therefore, $d(a,b)+|\varphi(a)-\varphi(b)|\geq 4+|6(p-1)+s+4-(6\hbar+(n/3)+6(q-1)+t-8)|\geq 4+3=7$.

Case 1.3: if $\alpha=3(\hbar-1)(s-1)+p$ and $\beta=(\hbar-1)(3t-1)+q$, where $1\leq p,q\leq\hbar-1$, $1\leq s,t\leq(n/3)$, then $|\varphi(a)-\varphi(b)|=|6(p-1)+s+4-(12\hbar+$

$2(n/3)+6(q-1)+t-20)| > 6$. Consequently, the condition is true.

Case 1.4: if $\alpha = (\hbar-1)(3s-2)+p$ and $\beta = (\hbar-1)(3t-1)+q$, where $1 \le p, q \le \hbar-1$, $1 \le s, t \le$, then $d(v_\alpha, v_\beta) \ge 4$ and $|\varphi(v_\alpha)-\varphi(v_\beta)| = |6\hbar+(n/3)+6(p-1)+s-8-(12\hbar+2(n/3)+6(q-1)+t-20)| \ge 3$. It follows that $d(a,b)+|\varphi(a)-\varphi(b)| \ge 7$.

Case 2: take $a = v_\alpha$ and $b = v_\beta, 0 \le \alpha, \beta \le n-1$. If $\alpha = 3s+p$ and $\beta = 3t+q, 0 \le p = q \le 2$, $0 \le s, t \le (n/3)-1$, then $d(v_{3s+p}, v_{3t+p}) = 4$ and $|\varphi(v_{3s+p})-\varphi(v_{3t+p})| \ge 3$. Otherwise, that is, if $p \ne q$, thus, $d(v_{3s+p}, v_{3t+q}) \ge 2$ and $|\varphi(v_{3s+p})-\varphi(v_{3t+q})| \ge n > 6$ since $n > 6$. Therefore, in both chances, we get $d(a,b)+|\varphi(a)-\varphi(b)| \ge 7$.

Case 3: assume that $a = u_{2s+p}$ and $b = u_{2t+q}$, $0 \le s \le \lceil n/2 \rceil - 1, 0 \le t \le \lfloor n/2 \rfloor - 1, 0 \le p, q \le 1$. If $p = q$, then $|\varphi(u_{2s+p})-\varphi(u_{2t+q})| \ge 4$ and $d(v_{3s+p}, v_{3t+p}) = 3$; else, $d(v_{3s+p}, v_{3t+q}) \ge 2$ and $|\varphi(v_{3s+p})-\varphi(v_{3t+q})| \ge \lceil n/2 \rceil + \lfloor n/2 \rfloor$. Therefore, in both of them, the inequality is satisfied.

Case 4: suppose that $a = v_\alpha$ and $b = u_\beta, 0 \le \alpha \le n(\hbar-1)$, $0 \le \beta \le n-1$; then, either $d(a,b) = 3$ and $|\varphi(a)-\varphi(b)| \ge 4$ or $d(a,b) = 1$ and $|\varphi(a)-\varphi(b)| > 6$. Therefore, $d(a,b)+|\varphi(a)-\varphi(b)| \ge 7$.

Case 5: if we take $a = v_\alpha$, $1 \le \alpha \le n(\hbar-1)$, and $b = w_\beta$, $0 \le \beta \le n-1$, then from the mapping, $|\varphi(a)-\varphi(b)| \ge |12\hbar+2(n/3)+6(\hbar-2)+(n/3)-20-(18\hbar+4n-25)\varphi| > n > 6$, which verifies the condition trivially.

Case 6: assume that $a = u_\alpha$ and $b = w_\beta$, $0 \le \alpha, \beta \le n-1$. If $\alpha = n-1$ and $\beta = 0$, then $|\varphi(u_\alpha)-\varphi(w_\beta)| = |18\hbar+3(n+(n/3)-1)-28-(18\hbar+4n+5\lceil n/2 \rceil-25)| = 6$ and $d(u_\alpha, w_\beta) = 1$. Otherwise, $|\varphi(u_\alpha)-\varphi(w_\beta)| > 6$. So, $d(u_\alpha, w_\beta)+|\varphi(u_\alpha)-\varphi(w_\beta)| > 6$.

Case 7: let a be the centre vertex of the wheel and b be any other vertex in the graph. If $b = v_1$, then $|\varphi(w_n)-\varphi(v_1)| = |1-5| = 4$ and $d(a,b) = 3$. Otherwise, the condition is obviously true. Thus, φ is a valid radio labelling, and the vertex w_{n-1} is labelled with the maximum number $18\hbar+4n+5\lfloor n/2 \rfloor+5(\lceil n/2 \rceil-1)-25 = 18\hbar+9n-30$. Hence, $rn(CS(n,\hbar)) \le 18\hbar+9n-30$, $n > 4$ and $n \equiv 0 \pmod 3$. □

Theorem 12. *For $n > 5$ and $n \equiv 1 \pmod 3$, the radio number of the star-sun graph satisfies $rr(SS(n,\hbar)) \le 20\hbar+12\lfloor n/3 \rfloor+5n-4$.*

Proof. First, we define an injection $\varphi: V(SS(n,\hbar)) \longrightarrow N$ as follows:

$$\begin{aligned}
\varphi\left(v_{3\hbar(j-1)+i}\right) &= 5(i-1)+j+4, \quad i = 1,2,\ldots,\hbar, j = 1,2,\ldots,\left\lfloor \frac{n}{3} \right\rfloor,\\
\varphi\left(v_{\hbar(3j-2)+i}\right) &= 5\hbar+\left\lfloor \frac{n}{3} \right\rfloor+5(i-1)+j-1, \quad i = 1,2,\ldots\hbar, j = 1,2,\ldots,\left\lfloor \frac{n}{3} \right\rfloor,\\
\varphi\left(v_{(\hbar-1)(3j-1)+i}\right) &= 10\hbar+2\left\lfloor \frac{n}{3} \right\rfloor+5(i-1)+j-4, \quad i = 1,2,\ldots,\hbar, j = 1,2,\ldots,\left\lfloor \frac{n}{3} \right\rfloor,\\
\varphi\left(v_{\hbar(n-1)+i}\right) &= 15\hbar+3\left\lfloor \frac{n}{3} \right\rfloor+5(i-1)-6, \quad i = 1,2,\ldots,\hbar,\\
\varphi(u_{3i}) &= 20\hbar+3\left\lfloor \frac{n}{3} \right\rfloor+3i-7, \quad i = 0,1,\ldots,\left\lfloor \frac{n}{3} \right\rfloor,\\
\varphi(u_{3i+1}) &= 20\hbar+6\left\lfloor \frac{n}{3} \right\rfloor+3i-4, \quad i = 0,1,\ldots,\left\lfloor \frac{n}{3} \right\rfloor,\\
\varphi(u_{3i+2}) &= 20\hbar+9\left\lfloor \frac{n}{3} \right\rfloor+3i-1, \quad i = 0,1,\ldots,\frac{n}{3}-1,\\
\varphi(w_{2i}) &= 20\hbar+12\left\lfloor \frac{n}{3} \right\rfloor+5i+1, \quad i = 0,1,\ldots,\left\lceil \frac{n}{2} \right\rceil-1,\\
\varphi(w_{2i+1}) &= 20\hbar+12\left\lfloor \frac{n}{3} \right\rfloor+5\left(\left\lceil \frac{n}{2} \right\rceil+i\right)+1, \quad i = 0,1,\ldots,\left\lfloor \frac{n}{2} \right\rfloor-1,\\
\varphi(w_n) &= 1.
\end{aligned} \tag{6}$$

We omit the rest of the proof as it is similar to the one for Theorem 11.

The other two cases in Theorems 11 and 12 are also left to the reader. □

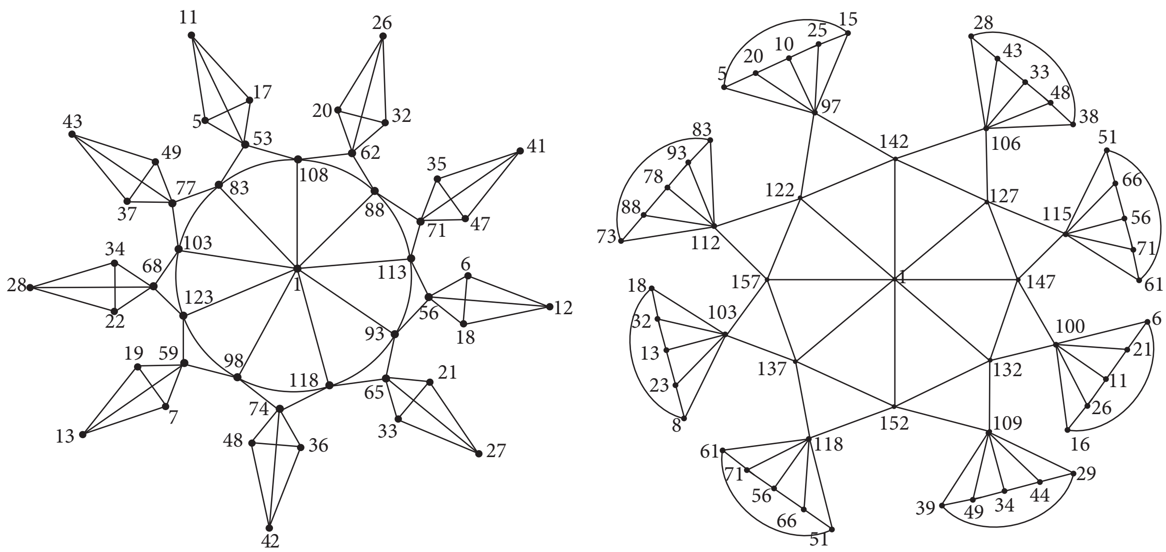


FIGURE 4: A radio labelling of complete-sun graph $CS(9,4)$ and wheel-sun graph $WS(8,5)$.

Theorem 13. *For $n>5$, the radio number of the wheel-sun graph satisfies*

$$rr(WS(n,\hbar)) \leq \begin{cases} 15\hbar + \left\lfloor \frac{n}{3} \right\rfloor + 8n - 4, n \equiv 0 \,(\mathrm{mod}\, 3) \\ 19\hbar + \left\lfloor \frac{n}{3} \right\rfloor + 8n - 4, n \equiv 1 \,(\mathrm{mod}\, 3). \\ 19 + \left\lfloor \frac{n}{3} \right\rfloor + 8n - 5, n \equiv 2 \,(\mathrm{mod}\, 3) \end{cases} \quad (7)$$

Figure 4 illustrates the case $n \equiv 2\,(\mathrm{mod}\,3)$. We omit the proof of this theorem.

4. Conclusion

In this paper, we have presented the relation between the radio number and radial radio number. We have also defined and investigated the bounds for the same problems for the graphs $\mathrm{CS}(n,\hbar)$, $SS(n,\hbar)$, and $WS(n,\hbar)$. For the graph fan-sun graph $SS(n,\hbar)$, the problem is still considered as an open research problem that needs further investigation. Since the method of finding the radial radio number and radio number of the fan-sun graph is similar to the previous theorem, it is still open to the interested researchers to do a further research work that can extend our results to identify more relations between the radio number and radial number by studying the same problem for interconnection networks.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Mohammed K. A. Kaabar contributed to actualization and initial draft, provided the methodology, validated and investigated the study, supervised the original draft, and edited the article. Kins Yenoke validated and investigated the study, provided the methodology, performed formal analysis, and contributed to the initial draft. Both authors read and approved the final version.

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