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Research Article

Chromatic Polynomial of Intuitionistic Fuzzy Graphs Using (α, β)-Levels

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The article describes a new thought on the chromatic polynomial of an intuitionistic fuzzy graph which is illustrated based on (α, β)-level graphs. Besides, the alpha-beta fundamental set of an intuitionistic fuzzy graph is also defined with a vivid description. In addition to that, some characterizations of the chromatic polynomial of an intuitionistic fuzzy graph are specified as well verified. Furthermore, some untouched properties of the (α, β)-level graph are also projected and proved.

1. Introduction

Graph coloring has a lot of applications in areas such as wireless radio channel assignment [1], timetable scheduling [2], and job scheduling [3]. Graph coloring is described as vertex coloring, edge coloring, and map coloring. The concept of the chromatic polynomial was introduced as the number of distinct ways of performing map coloring [4]. The chromatic polynomials of many families of graphs have been computed so far [5, 6]. In addition, the computations of the chromatic polynomials of certain graphs using the Mobius inversion theorem were reported by Srinivasa Rao et al. [7].

The fuzzy set theory was introduced by Zadeh [8]. A fuzzy set communicates an element of a given set with a membership value in [0, 1]. The introduction of fuzzy graph theory was given based on the fuzzy set introduced by Kauffman [9]. The traditional fuzzy set cannot be used to completely describe all evidence in the problems where someone wants to know how much nonmembership values. Such a problem got a solution by Atanassov who introduced an intuitionistic fuzzy set (IFS) [10]. IFS is an extension of Zadeh's set theory and is described by a membership function, a nonmembership function, and a hesitation function [10]. The concept of an intuitionistic fuzzy graph (IFG) was introduced by Atanassov [11]. Atanassov and Shannon developed certain properties of IFGs [12]. And also, Akram and Davvaz discussed more properties of IFGs [13]. Akram et al. presented strong IFGs and intuitionistic fuzzy line graphs [14]. Akram et al. presented an algorithm for computing the sum distance matrix, eccentricity, radius, and diameter of IFG [15]. Electoral systems [16], human cells clustering [17], and water supply systems [18] are some application areas of IFGs. Also, IFG digraph is applied in medical diagnosis, gas pipelines, and decision-making systems [19].

The chromatic polynomials of fuzzy graphs using α-cuts and their algebraic properties were developed by Abebe and Srinivasa Rao Repalle [20]. The coloring of IFG using (α, β)-cuts (levels) was introduced by Mohideen and Rifayathali [21]. Some properties of IFG by (α, β)-levels were presented by Akram [22]. *To fill the gap in these articles, the* authors aimed to find the chromatic polynomial of IFG using (α, β)*-levels, define* (α, β)-fundamental set, and *discuss more untouched properties of* (α, β)*-level graphs.* This paper presents the chromatic polynomial of an IFG using (α, β)-levels by illustrating it based on (α, β)-level graphs. In addition, it defines the alpha-beta fundamental set of an IFG. Furthermore, it states and proves some properties of the (α, β)-level graph and the chromatic polynomial of IFG.

This manuscript is organized as follows: Section 2 is the preliminaries that are essential for understanding the article. Section 3*develops some properties of* (α, β)*-levels of IFGs and* *proves wherever verification is required.* Section 4 defines the notion of the chromatic polynomial of an IFG using (α, β)-levels and also develops some related concepts on a chromatic polynomial of an IFG. Section 5 is the summary of the article.

2. Preliminaries

This section defines related concepts such as chromatic polynomial, IFG, and level graph of IFG. In addition, it provides some useful propositions.

Definition 1 (see [20]). The number of distinct ways of attaining a proper vertex coloring of a simple graph G with at most k colors is called chromatic polynomial of G and is denoted by $P(G, k)$.

Proposition 1 (see [23]). *The chromatic polynomial of a simple graph* G *may be found by the formula:* $P(G, k) = P(G- e, k) - P(G/e, k)$, *where* $P(G - e, k)$ *is the chromatic polynomial of a subgraph of* G *whose one of its edges* e *has been removed, and* $P(G/e, k)$ *is the chromatic polynomial of a subgraph of* G *whose one of its edge e has been contracted.*

Proposition 2 (see [23]). *Let* G *be a graph containing nonadjacent vertices* u *and* v, *and let* H *be the graph obtained from* G *the contracting edge* (u, v). *Then,* $P(G, k) = P(G+ (u, v), k) + P(H, k)$.

Proposition 3 (see [23]). *If* $G_1, G_2, \ldots, G_n$ *are pairwise disjoint components of a simple graph* G *whose union is* G, *then its chromatic polynomial is computed as*

$$P(G, k) = (P(G_1, k)) \times (P(G_2, k)) \times \cdots \times (P(G_n, k)), \quad (1)$$

Definition 2 (see [24]). An intuitionistic fuzzy graph is of the form $G = (V, E)$, where

(i) $V = \{v_1, v_2, \ldots, v_n\}$ such that $\mu_1 \colon V \longrightarrow [0, 1]$ and $\gamma_1 \colon V \longrightarrow [0, 1]$ denote the degree of membership and the degree of nonmembership of the element v_i in V, respectively, and $0 \le \mu_1(v_i) + \gamma_1(v_i) \le 1$, for every $v_i \in V$, $(i = 1, 2, \ldots, n)$

(i) $E \subseteq V \times V$ where $\mu_2 \colon V \times V \longrightarrow [0, 1]$ and $\gamma_2 \colon V \times V \longrightarrow [0, 1]$ are such that $\mu_2(v_i, v_j) \le \min\{\mu_1(v_i), \mu_1(v_j)\}$

$$\gamma_2(v_i, v_j) \le \max\{\gamma_1(v_i), \gamma_1(v_j)\}. \quad (2)$$

Satisfy the constraint, $0 \le \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \le 1$ for every $(v_i, v_j) \in E$. Where $\mu_2(v_i, v_j)$ and $\gamma_2(v_i, v_j)$ are the degree of membership and the degree of nonmembership of the element (v_i, v_j) in E respectively.

Definition 3 (see [21]). The chromatic number of IFG: $G = (V, E)$ is the intuitionistic fuzzy number $\chi(G) = \{(x, m(x), n(x)) | x \in X\}$, where $X = \{1, 2, \ldots, |V|\}$, $m(x) = \sup\{\alpha \in [0, 1] | x \in A_{\alpha,\beta}\}$, $n(x) = \inf\{\beta \in [0, 1] | x \in A_{\alpha,\beta}\}$, and $A_{\alpha,\beta} = \{\chi_{1,0}, \ldots, \chi_{\alpha,\beta}\}, \alpha, \beta \in [0, 1]$.

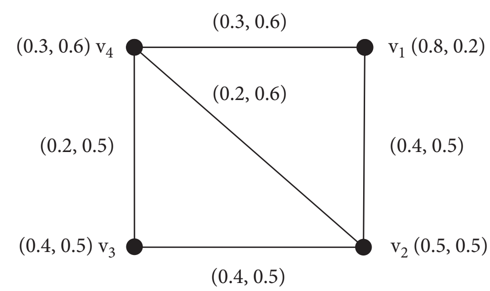

Figure 1: Intuitionistic fuzzy graph.

Definition 4. [22]. Let $G = (V, E)$ be an IFG. Then, for any $\alpha, \beta \in [0, 1]$, (α, β)-level graph of G is defined as the crisp graph $G_{\alpha,\beta} = (V_{\alpha,\beta}, E_{\alpha,\beta})$, where $V_{\alpha,\beta} = \{u \in V | \mu_1(u) \ge \alpha\}$, $\{\gamma_1(u) \le \beta\}$ and $E_{\alpha,\beta} = \{(u, v) \in V \times V | \mu_2(u, V) \ge \alpha\}$, $\{\gamma_2(u, v) \le \beta\}$.

3. The (α, β)-Fundamental Set and the Properties of (α, β)-Level Graph of IFGs

This section defines the (α, β)-fundamental set and illustrates the (α, β)-level graph of IFGs. In addition, it states and proves some character of the (α, β)-level graph of IFGs.

Definition 5. Let $G = (V, E)$ be an IFG. If $L = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_m, \beta_m)\}$ is such that $\alpha_i \le \alpha_{i+1}$ and $\beta_i \ge \beta_{i+1}$ for each $i = 1, 2, \ldots, m - 1$, then $F = \{(0, 1)\} \cup L$ is said to be a fundamental set of G.

Example 1. Consider an IFG with four intuitionistic fuzzy vertices and five intuitionistic fuzzy edges given in Figure 1.

The fundamental set of an IFG in Figure 1 is F = {(0, 1), (0.2, 0.6), (0.3, 0.6), (0.4, 0.5), (0.5, 0.5), (0.8, 0.2)}. To show that the reason why $(0.2, 0.5) \notin F$, consider the levels (0.2, 0.6), (0.2, 0.5), and (0.3, 0.6). Now, let us determine if a level (0.2, 0.5) satisfies the condition in the definition of fundamental set. It is forward that $0.2 \le 0.2$, 0.6 0.5, $0.2 \le 0.3$ but $0.5 \ngeq 0.6$. This shows that $(0.2, 0.5) \notin F$.

The level graphs along with their chromatic number are illustrated in Figure 2.

$$G_{0.8,0.2} = N_1, \chi(G_{0.8,0.2}) = 1. \quad (3)$$

Remark 1.

(i) Table 1 contains the vertex set, edge set, and chromatic number of $G_{\alpha,\beta}$ for each (α, β) in F

(ii) If there is a vertex v with $\mu_1(v) = 1, \gamma_1(v) = 0$, then (1, 0)-level graph exists

(iii) $G_{0,1} = (V_{0,1}, E_{0,1})$ is K_4 as $V_{0,1} = \{v \in V | \mu_1 \ge 0$ and $\gamma_1 \le 1\}$ contains all the vertices listed in V and

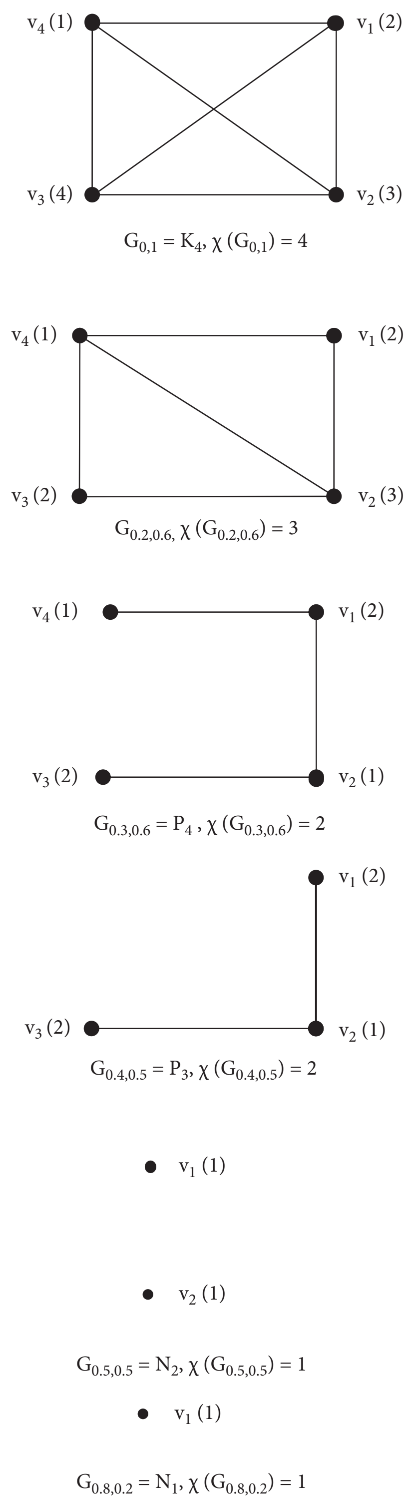

Figure 2: Illustration on the $G_{\alpha,\beta}$ of the graph in Figure 1 along with their chromatic numbers.

Table 1: The comparison of $|V_{\alpha,\beta}|$, $|E_{\alpha,\beta}|$, $\chi(G_{\alpha,\beta})$, and $P^I_{\alpha,\beta}(G,k)$ of IFG in Figure 1.

$\alpha = 0, \beta = 1$	4	6	4	$k(k-1)(k-2)(k-3)$
$\alpha = 0.2, \beta = 0.6$	4	5	3	$k(k-1)(k-2)^2$
$\alpha = 0.3, \beta = 0.6$	4	3	2	$k(k-1)^3$
$\alpha = 0.4, \beta = 0.5$	3	2	2	$k(k-1)^2$
$\alpha = 0.5, \beta = 0.5$	2	0	1	k^2
$\alpha = 0.8, \beta = 0.2$	1	0	1	k
Fundamental sets	s	$\lvert E_{\alpha,\beta}\rvert$	$\chi(G_{\alpha,\beta})$	$P^I_{\alpha,\beta}(G,k)$

$E_{0,1} = \{(u,v) \in V \times V \mid \mu_2 \geq 0 \text{ and } \gamma_2 \leq 1\}$ implies that all the vertices are joined

(iv) The chromatic number of an IFG in Figure 1 is given by

$$\chi(G) = \chi(G_{\alpha,\beta}) = \begin{bmatrix} (4, (0,1)) \\ (3, (0.2, 0.6)) \\ (2, (0.4, 0.5)) \\ (1, (0.8, 0.2)) \end{bmatrix}^T. \tag{4}$$

3.1. The Properties of (α, β)-Level Graph of IFGs

Theorem 1. *Let* G *be an IFG and* G_{α_1,β_1} *and* G_{α_2,β_2} *be level graphs of* G *with* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ *and* $1 \geq \beta_1 \geq \beta_2 \geq 0$*. Then,*

$$V_{\alpha_1,\beta_1} \supseteq V_{\alpha_2,\beta_2}, \quad E_{\alpha_1,\beta_1} \supseteq E_{\alpha_2,\beta_2}. \tag{5}$$

Proof. Let G be an IFG. Suppose G_{α_1,β_1} and G_{α_2,β_2} are level graphs of G with $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$. Now, G_{α_1,β_1} and G_{α_2,β_2} have vertex sets; $V_{\alpha_1,\beta_1} = \{v \in V | \mu_1(v) \geq \alpha_1 \text{ and } \gamma_1(v) \leq \beta_1\}$ and $V_{\alpha_2,\beta_2} = \{v \in V | \mu_1(v) \geq \alpha_2 \text{ and } \gamma_1(v) \leq \beta_2\}$, respectively. To figure out the relation V_{α_1,β_1} has with V_{α_2,β_2}, we draw them on the same plane as follows.

Based on Figure 3, V_{α_1,β_1} is an area of a polygon EFGB and V_{α_2,β_2} is an area of a polygon ABCD. This shows that V_{α_2,β_2} is contained in V_{α_1,β_1}. Thus, $V_{\alpha_1,\beta_1} \supseteq V_{\alpha_2,\beta_2}$.

Similarly, G_{α_1,β_1} and G_{α_2,β_2} have the edge sets; $E_{\alpha_1,\beta_1} = \{(u,v) \in V \times V | \mu_2(u,v) \geq \alpha_1, \gamma_2(u,v) \leq \beta_1\}$ and $E_{\alpha_2,\beta_2} = \{(u,v) \in V \times V | \mu_2(u,v) \geq \alpha_2, \gamma_2(u,v) \leq \beta_2\}$, respectively. For comparison, these sets can be interpreted on the same plane as follows.

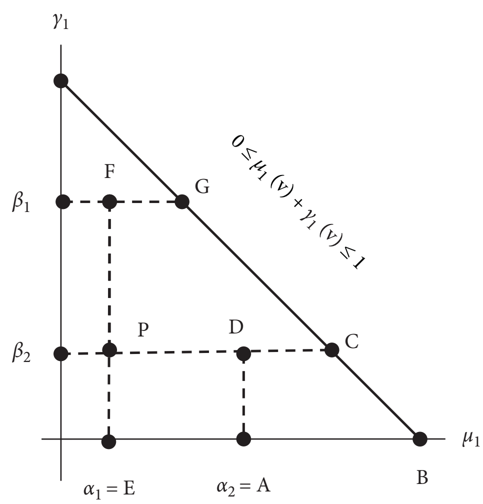

FIGURE 3: The comparison of V_{α_1,β_1} with V_{α_2,β_2} of an IFG.

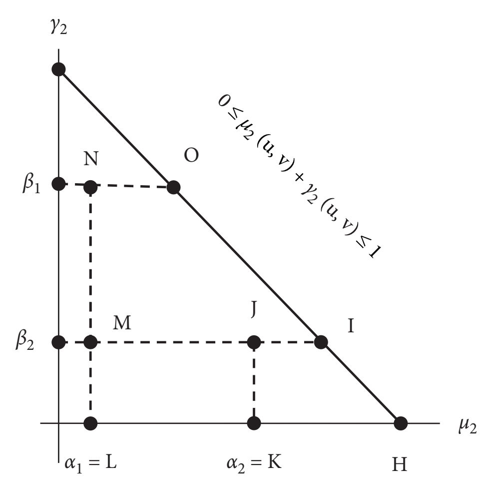

FIGURE 4: The comparison of E_{α_1,β_1} and E_{α_2,β_2} of an IFG.

Based on Figure 4, E_{α_1,β_1} is an area of a polygon HLNO and E_{α_2,β_2} is an area of a polygon HIJK. This indicates that E_{α_2,β_2} is contained in E_{α_1,β_1}. Thus, $E_{\alpha_1,\beta_1} \supseteq E_{\alpha_2,\beta_2}$.

Alternate proof: Let G_{α_1,β_1} and G_{α_2,β_2} be level graphs of G with $\alpha_1 \leq \alpha_2$ and $\beta_2 \leq \beta_1$. Then,

(i) The vertex sets to these levels are $V_{\alpha_1,\beta_1} = \{v \in V | \mu_1(v) \geq \alpha_1 \text{ and } \gamma_1(v) \leq \beta_1\}$ and $V_{\alpha_2,\beta_2} = \{v \in V | \mu_1(v) \geq \alpha_2 \text{ and } \gamma_1(v) \leq \beta_2\}$. Now, let $v \in V_{\alpha_2,\beta_2}$. Then, $\mu_1(v) \geq \alpha_2$ and $\gamma_1(v) \leq \beta_2$, and since $\alpha_2 \geq \alpha_1$ and $\beta_2 \leq \beta_1$, it implies $\mu_1(v) \geq \alpha_1$ and $\gamma_1(v) \leq \beta_1$. This indicates $v \in V_{\alpha_1,\beta_1}$. Thus, $V_{\alpha_1,\beta_1} \supseteq V_{\alpha_2,\beta_2}$

(ii) The edge sets of G_{α_1,β_1} and G_{α_2,β_2} are $E_{\alpha_1,\beta_1} = \{(u,v) \in V \times V | \mu_2(u,v) \geq \alpha_1, \gamma_2(u,v) \leq \beta_1\}$ and $E_{\alpha_2,\beta_2} = \{(u,v) \in V \times V | \mu_2(u,v) \geq \alpha_2, \gamma_2(u,v) \leq \beta_2\}$, respectively.

Now, let $(u,v) \in E_{\alpha_2,\beta_2}$. Then, $\mu_2(u,v) \geq \alpha_2$ and $\gamma_2(u,v) \leq \beta_2$ and $\alpha_2 \geq \alpha_1$ and $\beta_2 \leq \beta_1$, and it implies $\mu_2(u,v) \geq \alpha_1$ and $\gamma_2(u,v) \leq \beta_1$. This shows $(u,v) \in E_{\alpha_1,\beta_1}$. Hence, $E_{\alpha_1,\beta_1} \supseteq E_{\alpha_2,\beta_2}$. □

Corollary 1. *Let* G_{α_1,β_1} *and* G_{α_2,β_2} *be level graphs of an IFG,* G *with* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ *and* $1 \geq \beta_1 \geq \beta_2 \geq 0$. *Then,*

(i) $|V_{\alpha_2,\beta_2}| \leq |V_{\alpha_1,\beta_1}|$

(ii) $|E_{\alpha_2,\beta_2}| \leq |E_{\alpha_1,\beta_1}|$

Proof. Assume an IFG, G. Let G_{α_1,β_1} and G_{α_2,β_2} be two-level graphs of G with $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ and $1 \geq \beta_1 \geq \beta_2 \geq 0$. Then, by applying Theorem 1: $V_{\alpha_1,\beta_1} \supseteq V_{\alpha_2,\beta_2}$ and $E_{\alpha_1,\beta_1} \supseteq E_{\alpha_2,\beta_2}$. Thus, $|V_{\alpha_2,\beta_2}| \leq |V_{\alpha_1,\beta_1}|$ and $|E_{\alpha_2,\beta_2}| \leq |E_{\alpha_1,\beta_1}|$ hold. □

Theorem 2. *Let* G_{α_1,β_1} *and* G_{α_2,β_2} *be level graphs of an IFG;* G *with* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ *and* $1 \geq \beta_1 \geq \beta_2 \geq 0$. *Then,* G_{α_1,β_1} *is a super graph of* G_{α_2,β_2}.

Proof. Let G be an IFG and $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ and $1 \geq \beta_1 \geq \beta_2 \geq 0$ be given. Then, by Theorem 1, $V_{\alpha_1,\beta_1} \supseteq V_{\alpha_2,\beta_2}$ and $E_{\alpha_1,\beta_1} \supseteq E_{\alpha_2,\beta_2}$. From the fact that if the vertex set and edge set of a certain graph J are a subset of a vertex set and edge set of another graph H, respectively, then H is a super graph of J. Thus, G_{α_1,β_1} is a super graph of G_{α_2,β_2}. □

4. The Chromatic Polynomial of IFG Using (α, β)-Levels

Here, we define a new idea of the chromatic polynomial of IFG using (α, β)-levels, and also, we compute the chromatic polynomial of an IFG using (α, β)-levels for illustration. Furthermore, we state and prove some properties of the chromatic polynomial of IFG using (α, β)-levels.

Definition 6. Let G be IFG and F be its fundamental set. Then, the chromatic polynomial of G using (α, β)-level with at most k given colors is denoted by $P^I_{\alpha,\beta}(G,k)$ and is defined as the chromatic polynomial of $G_{\alpha,\beta}$, where (α, β). That is, $P^I_{\alpha,\beta}(G,k) = P(G_{\alpha,\beta}, k)$ for each $(\alpha, \beta) \in F$.

Example 2. Consider the IFG, G in Figure 1. All the level graphs of G have been put in Figure 2, and by Definition 6, the chromatic polynomial of G is the chromatic polynomial of these level graphs. Thus, the chromatic polynomial of G is the chromatic polynomial of the level graphs combined in a single equation and is computed as follows:

(i) When $\alpha = 0$ and $\beta = 1$, we have $P^I_{0,1}(G,k) = P(G_{0,1}, k)$, and as $G_{0,1} = K_4$, $P(G_{0,1}, k)$ is the chromatic polynomial of K_4. Hence, $P(G_{0,1}, k) = P(K_4, k) = (k)(k-1)(k-2)(k-3)$.

(ii) When $\alpha = 0.2$ and $\beta = 0.6$, we compute $P^I_{0.2,0.6}(G,k) = P(G_{0.2,0.6}, k)$ by the addition-contraction method as in Figure 5.

Thus, $P(G_{0.2,0.6}, k) = P(K_4, k) + P(K_3, k) = k(k-1)(k-2)^2$

(iii) When $\alpha = 0.3$ and $\beta = 0.6$, we have $P^I_{0.3,0.6}(G,k) = P(G_{0.3,0.6}, k)$, and it is the chromatic polynomial

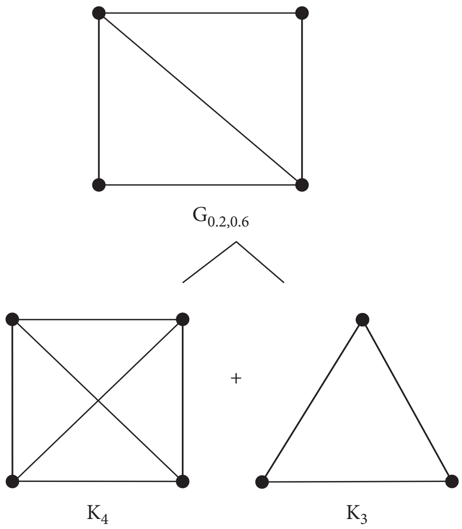

FIGURE 5: The computation of $P^I_{0.2,0.6}(G,k)$.

of P_4. Therefore, $P(G_{0.3,0.6},k) = P(P_4,k) = k(k-1)^3$.

(iv) When $\alpha = 0.4$ and $\beta = 0.5$, we obtain $P^I_{0.4,0.5}(G,k) = P(G_{0.4,0.5},k)$ and $P(G_{0.4,0.5},k)$ is the chromatic polynomial of P_3. Therefore, $P(G_{0.4,0.5},k) = P(P_3,k) = k(k-1)^2$.

(v) When $\alpha = 0.5$ and $\beta = 0.5$, we obtain $P^I_{0.5,0.5}(G,k) = P(G_{0.5,0.5},k)$ and $P(G_{0.5,0.5},k)$ is the chromatic polynomial of N_2. Therefore, $P(G_{0.5,0.5},k) = P(N_2,k) = (k)^2$.

(vi) When $\alpha = 0.8$ and $\beta = 0.2$, we obtain $P^I_{0.8,0.2}(G,k) = P(G_{0.8,0.2},k)$ and $P(G_{0.8,0.2},k)$ is the chromatic polynomial of N_1. Therefore, $P(G_{0.8,0.2},k) = P(N_1,k) = k$.

In general, the chromatic polynomial of IFG in Figure 1 is summarized in the following equation:

$$P^I_{\alpha,\beta}(G,k) = \begin{cases} k(k-1)(k-2)(k-3), & \alpha = 0, \beta = 1, \\ k(k-1)(k-2)^2, & \alpha = 0.2,\ \beta = 0.6, \\ k(k-1)^3, & \alpha = 0.3,\ \beta = 0.6, \\ k(k-1)^2, & \alpha = 0.4,\ \beta = 0.5, \\ k^2, & \alpha = 0.5,\ \beta = 0.5, \\ k, & \alpha = 0.8,\ \beta = 0.2. \end{cases} \tag{6}$$

Remark 2

(i) As α increases the degree of the chromatic polynomial, the chromatic number, the number of edges, and the number of vertices decrease.

(ii) As β decreases the degree of the chromatic polynomial, the chromatic number, the number of edges, and the number of vertices also decrease.

4.1. Properties of the Chromatic Polynomial of IFGs Using (α,β)-Levels

Theorem 3. *Let* G *be IFG on n intuitionistic fuzzy vertices and let* $G_{\alpha,\beta}$ *be* (α,β)*-level graph of* G. *If* $\alpha = 0$ *and* $\beta = 1$, *then*

$$P^I_{\alpha,\beta}(G,k) = P(K_n,k). \tag{7}$$

Proof. Let G be an IFG. Then, when $\alpha = 0$ and $\beta = 1$, $G_{0,1}$ possesses $V_{0,1} = \{x \in V \mid \mu_1(w) \geq 0 \text{ and } \gamma_1(w) \leq 1\}$ and $E_{0,1} = \{(w,v) \in V \times V \mid \mu_2(x,y) \geq 0 \text{ and } \gamma_2(w,v) \leq 1\}$.

For every edge in G, $\mu_2(w,v) \geq 0$ and $\gamma_2(w,v) \leq 1$. So, every edge in G satisfies $E_{0,1}$. In addition to edges in $G_{0,1}$, $E_{0,1}$ contains edge(s) correspond to edge(s) labeled $(0,0)$ of G. Therefore, every pair of vertices is joined. $|V_{0,1}| = n$, as every vertex in G, could satisfy $V_{0,1}$. This implies that $G_{0,1}$ and K_n represent the same graph. Hence, $P^I_{0,1}(G,k) = P(G_{0,1},k) = P(K_n,k)$. □

Theorem 4. *Let* G *be IFG. If* G_{α_1,β_1} *and* G_{α_2,β_2} *are intuitionistic fuzzy levels of* G *with* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ *and* $1 \geq \beta_1 \geq \beta_2 \geq 0$, *then*

$$\deg\left(P^I_{\alpha_1,\beta_1}(G,k)\right) \geq \deg\left(P^I_{\alpha_2,\beta_2}(G,k)\right). \tag{8}$$

Proof. Assume that G is an IFG. Let G_{α_1,β_1} and G_{α_2,β_2} be the level graphs of G with $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ and $1 \geq \beta_1 \geq \beta_2 \geq 0$. Then, $\deg(P^I_{\alpha_1,\beta_1}(G,k)) = \deg(P(G_{\alpha_1,\beta_1},k))$ and $\deg(P^I_{\alpha_2,\beta_2}(G,k)) = \deg(P(G_{\alpha_2,\beta_2},k))$. It is obvious to realize that the degree of the chromatic polynomial of a simple graph equals the number of its vertex set. That is, $\deg(P(G_{\alpha_1,\beta_1},k) = |V_{\alpha_1,\beta_1}|$ and $\deg(P(G_{\alpha_2,\beta_2},k) = |V_{\alpha_2,\beta_2}|$, and also, since $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$, $|V_{\alpha_1,\beta_1}| \geq |V_{\alpha_2,\beta_2}|$ (see Corollary 1). Hence, $\deg(P^I_{\alpha_1,\beta_1}(G,k)) \geq \deg(P^I_{\alpha_2,\beta_2}(G,k))$. □

Theorem 5. *Let* H *be an IFG and* H_c *be its underlying graph. If* $\alpha_j \leq \alpha_i$ *and* $\beta_i \leq \beta_j$ *for each* $i = 1, 2, \ldots, m$ *and* $\{(\alpha_i,\beta_i)\} \in I$, *then* $P^I_{\alpha_j,\beta_j}(H,k) = P(H_c,k)$ *for some* $j = 1, 2, \ldots, m$.

Proof. Suppose that H_c is an underlying graph of IFG H. Let $\{(0,1)\} \cup (\alpha_1,\beta_1), (\alpha_2,\beta_2), \ldots, (\alpha_m,\beta_m)$ be a fundamental set of H and let $\alpha_j \leq \alpha_i$ and $\beta_i \leq \beta_j$ for each $i = 1, 2, \ldots, m$ and for some $j = 1, 2, \ldots, m$. By Theorem 11, H_{α_j,β_j} is a super graph of H_{α_i,β_i} for each $i = 1, 2, \ldots, m$. This shows that H_{α_j,β_j} contains all vertices and all edges of H_c as H_c is the underlying crisp graph of H. Hence, H_{α_j,β_j} and H_c represent the same graph, and both have the same chromatic polynomial. That is, $P^I_{\alpha_j,\beta_j}(H,k) = P(H_c,k)$. □

Theorem 6. *Let* H *and* J *be components of an IFG,* G, *and let* (α,β) *be any level such that* $\alpha > 0$ *and* $\beta < 1$. *Then,*

$$P^I_{\alpha,\beta}(G,k) = P^I_{\alpha,\beta}(H,k) \times P^I_{\alpha,\beta}(J,k). \tag{9}$$

Proof. Let G be IFG and (α,β) be any of its levels such that $\alpha > 0$ and $\beta < 1$. Also, let H and J be an intuitionistic fuzzy component of G. Now, since H and J are components of G,

$H_{\alpha,\beta}$ and $J_{\alpha,\beta}$ are also components of $G_{\alpha,\beta}$. So, $P(G_{\alpha,\beta}, k) = P(H_{\alpha,\beta}, k) \times P(J_{\alpha,\beta}, k)$.

Thus, $P^I_{\alpha,\beta}(G,k) = P^I_{\alpha,\beta}(H,k) \times P^I_{\alpha,\beta}(J,k)$. □

Corollary 2. *Let* $F_1, F_2, \ldots, F_n$ *be intuitionistic fuzzy components of an IFG,* F, *and* (α,β)*-be any level of* F. *Then,*

$$P^I_{\alpha,\beta}(F,k) = P^I_{\alpha,\beta}(F_1,k) \times P^I_{\alpha,\beta}(F_2,k) \times \ldots \times P^I_{\alpha,\beta}(F_n,k). \tag{10}$$

Proof of this corollary is omitted as it can be forwarded by extending Theorem 6.

Theorem 7. *If* H *and* J *are intuitionistic fuzzy component graphs of an IFG,* G *with* n *intuitionistic fuzzy vertices and m intuitionistic fuzzy vertices, respectively, and* K_{n+m} *is a complete graph on* $n+m$ *vertices, then* $P^I_{0,1}(G,k) = P(K_{n+m}, k)$.

Proof. Consider an IFG, $G = (V, E)$. Now, let H and J be intuitionistic fuzzy component graphs of G containing *n* and m intuitionistic fuzzy vertices, respectively. Then, (0, 1)-level graph of G, $G_{0,1}$ possesses

(i) $V_{0,1} = \{x \in V | \mu_1(w) \geq 0, \gamma_1(w) \leq 1\}$. $|V_{0,1}| = n + m$, as every vertex in both H and J, satisfy $V_{0,1}$

(ii) $E_{0,1} = \{(w,v) \in V \times V | \mu_2(x,y) \geq 0, \gamma_2(w,v) \leq 1\}$

Every edge joining any two vertices w and v in both component graphs H and J satisfy $\mu_2(w,v) \geq 0$ and $\gamma_2(w,v) \leq 1$. And also, in addition to the edges in both H and J, $E_{0,1}$ contains an edge(s) corresponding to edge(s) labeled $(0,0)$ of G which are actually neither in the crisp graph of H nor in the crisp graph of J.

From (i), $(0,1)$-level graph of G has n + m vertices, and from (ii), every pair of vertices of G is joined. In other words, the $(0,1)$-level graph of G is K_{n+m}. Hence, $P^I_{0,1}(G,k) = P(K_{n+m}, k)$. □

Corollary 3. *Let* G *be IFG. If* H *and* J *are intuitionistic fuzzy component graphs of* G *with n intuitionistic fuzzy vertices and* m *intuitionistic fuzzy vertices, respectively, then*

$$\mathrm{Deg}\left(P^I_{0,1}(G,k)\right) = n + m. \tag{11}$$

Proof. Based on Theorem 7, $(0,1)$-level graph forms a complete crisp graph; K_{n+m} and $P^I_{0,1}(G,k) = P(K_{n+m}, k)$. Thus, the corollary holds. □

5. Conclusions

To conclude, the coloring of the IFG using (α,β)-levels is applied in traffic light control, register allocation, clustering analysis, and decision-making system. The chromatic polynomial of IFG using (α,β)-levels provides alternative solutions in such application areas. In this work, the notion of the (α,β)-fundamental set of an IFG has been clearly stated with relevant examples. Apart from that, some characteristics (α,β)-level graphs have been detailed and proved. Also, the chromatic polynomial of IFG, by using (α,β)-levels, has been presented. Moreover, the chromatic polynomials of IFG of the different (α,β)-levels have been derived and compared. Furthermore, certain properties of the chromatic polynomial of an IFG have also been discussed and illustrated in a detailed note.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest concerning the publication of this paper.

Authors' Contributions

All the authors contributed an equal share in the preparation of this article. The authors read and approved the final manuscript.

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