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Research Article

Horadam Polynomials and a Class of Biunivalent Functions Defined by Ruscheweyh Operator

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In this paper, we introduce and investigate a class of biunivalent functions, denoted by $\mathscr{H}(n, r, \alpha)$, that depends on the Ruscheweyh operator and defined by means of Horadam polynomials. For functions in this class, we derive the estimations for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. Moreover, we obtain the classical Fekete–Szegö inequality of functions belonging to this class.

1. Introduction

Let $\mathscr{A}$ be the family of all analytic functions f that are defined on the open unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Any function $f \in \mathscr{A}$ has the following Taylor–Maclaurin series expansion:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{where } z \in \mathbb{D}. \tag{1}$$

Let $\mathscr{S}$ denote the class of all functions $f \in \mathscr{A}$ that are univalent in $\mathbb{D}$. Let the functions f and g be analytic in $\mathbb{D}$, we say the function f is subordinate by the function g in $\mathbb{D}$, denoted by $f(z) \prec g(z)$ for all $z \in \mathbb{D}$, if there exists a Schwarz function w, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$, such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent over $\mathbb{D}$, then $f(z) \prec g(z)$ equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. For more information about the subordination principle, we refer the readers to the monographs [1–3].

It is well known that univalent functions are injective functions. Hence, they are invertible, and the inverse functions may not be defined on the entire unit disk $\mathbb{D}$. In fact, the Koebe one-quarter theorem tells us that the image of $\mathbb{D}$ under any function $f \in \mathscr{S}$ contains the disk $D(0, 1/4)$ of center 0 and radius 1/4. Accordingly, every function $f \in \mathscr{S}$ has an inverse $f^{-1} = g$ which is defined as

$$\begin{aligned} &g(f(z)) = z, \quad z \in \mathbb{D}, \\ &f(g(\omega)) = \omega, \quad |\omega| < r(f); r(f) \geq \frac{1}{4}. \end{aligned} \tag{2}$$

Moreover, the inverse function is given by

$$\begin{aligned} g(\omega) = \omega - a_2\omega^2 &+ \left(2a_2^2 - a_3\right)\omega^3 \\ &- \left(5a_2^3 - 5a_2a_3 + a_4\right)\omega^4 + \cdots. \end{aligned} \tag{3}$$

For this reason, we define the class Σ as follows. A function $f \in \mathscr{A}$ is said to be biunivalent if both f and f^{-1} are univalent in $\mathbb{D}$. Therefore, let Σ denotes the class of all biunivalent functions in $\mathscr{A}$ which are given by equation (1). For example, the following functions belong to the class Σ:

$$\frac{z}{1-z}, -\log(1-z), \log\sqrt{\frac{1+z}{1-z}}. \tag{4}$$

However, Koebe function, $2z - z^2/2$ and $z/1 - z^2$, does not belong to the class Σ. For more information about univalent and biunivalent functions, we refer the readers to the articles [4–6], the monographs [7–9], and the references therein.

Recently, many researchers have studied the geometric function theory in complex analysis, and the typical problem in this field is studying a functional made up of combinations of the initial coefficients of the functions $f \in \mathscr{A}$. For a function in the class $\mathscr{S}$, it is well known that $|a_n|$ is bounded by n. Moreover, the coefficient bounds give information about the geometric properties of those functions. For instance, the bound for the second coefficients of the class $\mathscr{S}$ gives the growth and distortion bounds for the class. In addition, the Fekete–Szegö functional arises naturally in the investigation of univalency of analytic functions. In the year 1933, Fekete and Szegö [10] found the maximum value of $|a_3 - \lambda a_2^2|$, as a function of the real parameter $0 \leq \lambda \leq 1$ for a univalent function f. Since then, the problem of dealing with the Fekete–Szegö functional for $f \in \mathscr{A}$ with any complex λ is known as the classical Fekete–Szegö problem. There are many researchers investigated the Fekete–Szegö functional and the other coefficient estimates problems, for example, see the articles [4–6, 10–20] and the references therein.

2. Preliminaries

In this section, we present some information that are curial for the main results of this paper. In the year 1965, for $a, b, p, q \in F$, Horadam [21] introduced the sequence $W_n = W_n(a, b; p, q)$ that is defined by the following recurrence relation:

$$W_{n+2} = pW_{n+1} + qW_n, \quad \text{for } n \geq 2, \tag{5}$$

with the initial values $W_0 = a$ and $W_1 = b$. The characteristic equation of this sequence is given by

$$t^2 - \mathrm{pt} - q = 0. \tag{6}$$

In addition, the generating function of Horadam sequence is

$$f(t) = \frac{a + t(b - \mathrm{ap})}{1 - \mathrm{pt} - \mathrm{qt}^2}. \tag{7}$$

The Horadam sequences generalize many famous sequences such as Fibonacci, Lucas, Pell, Pell-Lucas, and Jacobsthal sequences. These sequences have been studied for a long time. For more information about these sequences, we refer the readers to the articles [22, 23] and the monograph [24].

In the year 1985, Horadam and Mahon defined the Horadam polynomials $h_n(r) = h_n(a, b; p, q)$ by the following recurrence relation:

$$h_n(r) = \mathrm{prh}_{n-1}(r) + \mathrm{qh}_{n-2}(r), \quad \text{for } n \geq 3, \tag{8}$$

with initial values

$$\begin{aligned} &h_1(r) = a, h_2(r) = \mathrm{br}, \\ &h_3(r) = \mathrm{pbr}^2 + \mathrm{qr}. \end{aligned} \tag{9}$$

Moreover, the generating function of Horadam polynomials is given by

$$\Pi(r, z) = \sum_{n=1}^{\infty} h_n(r) z^{n-1} = \frac{a + (b - \mathrm{ap})\mathrm{rz}}{1 - \mathrm{prz} - \mathrm{qz}^2}. \tag{10}$$

For particular values of a, b, p, and q, the Horadam polynomials lead to many known polynomials. In the following, we list some special cases of Horadam Polynomials.

(i) If $a = b = p = q = 1$, we get Fibonacci polynomials $F_n(r)$ whose recurrence relation is

$$F_n(r) = rF_{n-1}(r) + F_{n-2}(r); \text{ with } F_1(r) = 1, F_2(r) = r. \tag{11}$$

(ii) If $a = 2$ and $b = p = q = 1$, we get Lucas polynomials $L_n(r)$ whose recurrence relation is

$$L_{n-1}(r) = rL_{n-2}(r) + L_{n-3}(r); \text{ with } L_0(r) = 2, L_1(r) = r. \tag{12}$$

(iii) If $a = q = 1$ and $b = p = 2$, we get Pell polynomials $P_n(r)$ whose recurrence relation is

$$P_n(r) = 2rP_{n-1}(r) + P_{n-2}(r); \text{ with } P_1(r) = 1, P_2(r) = 2r. \tag{13}$$

(iv) If $a = b = p = 2$ and $q = 1$, we get Pell–Lucas polynomials $Q_n(r)$ whose recurrence relation is

$$Q_{n-1}(r) = 2rQ_{n-2}(r) + Q_{n-3}(r); \text{ with } Q_0(r) = 2, Q_1(r) = 2r. \tag{14}$$

(v) If $a = b = p = r = 1$ and $q = 2y$, we get Jacobsthal polynomials $J_n(y)$ whose recurrence relation is

$$J_n(y) = J_{n-1}(y) + 2yJ_{n-2}(y);\ \text{with}\ J_1(y) = 1, J_2(y) = 1. \tag{15}$$

(vi) If $a = 2$, $b = p = r = 1$ and $q = 2y$, we get Jacobsthal–Lucas polynomials $\mathcal{J}_n(y)$ whose recurrence relation is

$$\mathcal{J}_{n-1}(y) = \mathcal{J}_{n-2}(y) + 2y\mathcal{J}_{n-3}(y);\ \text{with}\ \mathcal{J}_0(y) = 2, \mathcal{J}_1(y) = 1. \tag{16}$$

(vii) If $a = 1$ and $b = p = 2$, and $q = -1$, we get Chebyshev polynomials $H_n(r)$ of the second kind whose recurrence relation is

$$H_{n-1}(r) = 2rH_{n-2}(r) - H_{n-3}(r);\ \text{with}\ H_0(r) = 1, H_1(r) = 2r. \tag{17}$$

(viii) If $a = b = 1$ and $p = 2$, and $q = -1$, we get Chebyshev polynomials $T_n(r)$ of the first kind whose recurrence relation is

$$T_{n-1}(r) = 2rT_{n-2}(r) - T_{n-3}(r);\ \text{with}\ T_0(r) = 1, T_1(r) = r. \tag{18}$$

For more information about Horadam polynomials and its special interesting cases, we refer the readers to the articles [5, 6, 11, 22, 25–29], the monograph [24], and the references therein.

In the year 1975, Ruscheweyh [30] introduced the operator $\mathcal{R}$ which is defined, using the Hadamard product, as follows:

$$\mathcal{R}^{\lambda} f(z) = f(z) * \frac{z}{(1-z)^{1-\lambda}}, \tag{19}$$

where $f \in \mathcal{A}$, $z \in \mathbb{D}$, and real number $\lambda \geq -1$. For $\lambda = n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we get the Ruscheweyh derivative $\mathcal{R}^n$ of order n of the function f:

$$\mathcal{R}^n f(z) = z\frac{\left(z^{n-1} f(z)\right)^{(n)}}{n!}. \tag{20}$$

Moreover, the Taylor–Maclaurin series of $\mathcal{R}^n f$ is given by

$$\mathcal{R}^n f(z) = z + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k, \tag{21}$$

$$\delta(n,k) = \frac{\Gamma(n+k)}{(k-1)!\Gamma(n+1)}. \tag{22}$$

We say that a function $f \in \Sigma$ in the subclass $\mathcal{H}(n,r,\alpha)$ if it fulfills the subordination conditions, associated with the Horadam polynomials, for all z, $w \in \mathbb{D}$ and for $\alpha > 0$:

$$\left(\mathcal{R}^n f(z)\right)' + \alpha z\left(\mathcal{R}^n f(z)\right)'' \prec \Pi(r,z) + 1 - a, \tag{23}$$

and

$$\left(\mathcal{R}^n g(w)\right)' + \alpha w\left(\mathcal{R}^n g(w)\right)'' \prec \Pi(r,w) + 1 - a. \tag{24}$$

The following lemma (see for details, [20]) is a well-known fact, so we omit its proof.

Lemma 1. *Let K, $L \in \mathbb{R}$, and $p, q \in \mathbb{C}$. If $|p| < R$ and $|q| < R$,*

$$|(K+L)p + (K-L)q| \leq \begin{cases} 2|K|R, & \text{if } |K| \geq |L|, \\ 2|L|R, & \text{if } |K| \leq |L|. \end{cases} \tag{25}$$

Our investigation in this paper is motivated by the work of the researchers presented in the papers [31, 32]. In this presenting paper, we investigate a subclass of biunivalent functions Σ in the open unit disk $\mathbb{D}$, which we denote by $\mathscr{H}(n, r, \alpha)$ with $\alpha > 0, r \in \mathbb{R}$, and $n \in \mathbb{N}_0$. For functions in this subclass, we derive upper bounds for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. Furthermore, we examine the corresponding Fekete–Szegö functional problem for functions belong to this subclass.

3. Initial Coefficient Estimates for the Function Class $\mathscr{H}(n, r, \alpha)$

In this section, we provide bounds for the initial Taylor–Maclaurin coefficients for the functions belong to the class $\mathscr{H}(n, r, \alpha)$ which are given by equation (1).

Theorem 2. *Let the function f given by (1) be in the class $\mathscr{H}(n, r, \alpha)$. Then,*

$$|a_2| \le \frac{|\mathrm{br}|\sqrt{2|\mathrm{br}|}}{\sqrt{\left|\left[\left(3(1+2\alpha)^2(n+2)b - 8p(1+\alpha)^2(n+1)\right](n+1)\mathrm{br}^2 - 8\mathrm{qa}(1+\alpha)^2(n+1)^2\right|}}, \tag{26}$$

and

$$|a_3| \le \frac{2(n!)|\mathrm{br}|}{3(1+2\alpha)(n+2)!} + \frac{b^2r^2}{4(1+\alpha)^2(n+1)^2}. \tag{27}$$

Proof. Let f belong to the class $\mathscr{H}(n, r, \alpha)$. Then, using (5) and (6), we can find two analytic functions u and v on the unit disk $\mathbb{D}$ such that

$$\left(\mathscr{R}^n f(z)\right)' + \alpha z\left(\mathscr{R}^n f(z)\right)'' \prec \Pi(r, u(z)) + 1 - a, \tag{28}$$

and

$$\left(\mathscr{R}^n g(w)\right)' + \beta w\left(\mathscr{R}^n g(w)\right)'' \prec \Pi(r, v(w)) + 1 - a, \tag{29}$$

where the analytic functions u and v are given by

$$\begin{aligned} u(z) &= \sum_{n=1}^{\infty} u_n z^n, \quad \text{where } z \in \mathbb{D}, \\ v(w) &= \sum_{n=1}^{\infty} v_n w^n, \quad \text{where } w \in \mathbb{D}, \end{aligned} \tag{30}$$

such that

$$u(0) = v(0) = 0, \tag{31}$$

and for all z, $w \in \mathbb{D}$

$$\begin{aligned} |u(z)| &< 1, \\ |v(z)| &< 1. \end{aligned} \tag{32}$$

Moreover, it is well known that (see, for details, [7]) for all $j \in \mathbb{N}$,

$$\begin{aligned} \left|u_j\right| &\le 1, \\ \left|v_j\right| &\le 1. \end{aligned} \tag{33}$$

Now, upon comparing the coefficients in both sides of (28) and (29), we obtain the following equations:

$$2(1+\alpha)\delta(n,2)a_2 = h_2(r)u_1, \tag{34}$$

$$3(1+2\alpha)\delta(n,3)a_3 = h_2(r)u_2 + h_3(r)u_1^2, \tag{35}$$

$$-2(1+\alpha)\delta(n,2)a_2 = h_2(r)v_1, \tag{36}$$

and

$$3(1+2\alpha)\delta(n,3)\left(2a_2^2 - a_3\right) = h_2(r)v_2 + h_3(r)v_1^2. \tag{37}$$

In view of equations (34) and (36), we get the following equations:

$$u_1 = -v_1, \tag{38}$$

and

$$8(1+\alpha)^2[\delta(n,2)]^2 a_2^2 = \left[h_2(r)\right]^2\left(u_1^2 + v_1^2\right). \tag{39}$$

Moreover, if we add equations (35) and (37), we get

$$6(1+2\alpha)\delta(n,3)a_2^2 = \left[h_2(r)\right]\left(u_2 + v_2\right) + \left[h_3(r)\right]\left(u_1^2 + v_1^2\right). \tag{40}$$

In view of equation (39), equation (40) can be written as

$$a_2^2 = \frac{\left[h_2(r)\right]^3\left(u_2 + v_2\right)}{6(1+2\alpha)\delta(n,3)\left[h_2(r)\right]^2 - 8(1+\alpha)^2[\delta(n,2)]^2 h_3(r)}. \tag{41}$$

Using equations (22) and (41) becomes

$$a_2^2 = \frac{b^3r^3\left(u_2 + v_2\right)}{\left[3(1+2\alpha)(n+2)(n+1)b - 8p(1+\alpha)^2(n+1)^2\right]\mathrm{br}^2 - 8\mathrm{qa}(1+\alpha)^2(n+1)^2}. \tag{42}$$

Using the facts $|u_2|\leq 1$ and $|v_2|\leq 1$, we get inequality (26), which is the desired estimate of a_2.

Next, we turn our attention to find an upper bound for $|a_3|$. Subtracting equation (37) from equation (35), we get

$$6(1+2\beta)\sigma(n,3)\left(a_3-a_2^2\right)=h_2(r)(u_2-v_2)+h_3(r)\left(u_1^2-v_1^2\right). \tag{43}$$

In view of equations (38) and (39), we obtain

$$a_3=\frac{h_2(r)(u_2-v_2)}{6(1+2\alpha)\delta(n,3)}+\frac{[h_2(r)]^2\left(u_1^2+v_1^2\right)}{8(1+\alpha)^2[\delta(n,2)]^2}. \tag{44}$$

Finally, using equation (22) and the facts $|u_2|\leq 1$ and $|v_2|\leq 1$, we get

$$|a_3|\leq\frac{2|\mathrm{br}|}{3(1+2\alpha)(n+2)(n+1)}+\frac{b^2r^2}{4(1+\alpha)^2(n+1)^2}, \tag{45}$$

which gives the estimate (27). This completes the proof of Theorem 2.

Taking $\alpha=1$, we get the following corollary of Theorem 2. This proof is similar to the proof of previous theorem, so we omit the proof's details. ☐

Corollary 3. *Let the function f given by (1) be in the class $\mathscr{H}(n,r,1)$. Then,*

$$\begin{aligned}|a_2|&\leq\frac{|\mathrm{br}|\sqrt{2|\mathrm{br}|}}{\sqrt{\left|\left[\left[27(n+2)b-32p(n+1)\right](n+1)\mathrm{br}^2-32\mathrm{qa}(n+1)^2\right|}},\\|a_3|&\leq\frac{2(n!)|\mathrm{br}|}{9(n+2)!}+\frac{b^2r^2}{16(n+1)^2}.\end{aligned} \tag{46}$$

4. Fekete–Szegö Problem for the Function Class $\mathscr{H}(n,r,\alpha)$

In this section, we consider the classical Fekete–Szegö problem for our presenting class $\mathscr{H}(n,r,\alpha)$.

Theorem 4. *Let the function f given by (1) be in the class $\mathscr{H}(n,r,\alpha)$. Then, for some $\lambda\in\mathbb{R}$,*

$$\left|a_3-\lambda a_2^2\right|\leq\begin{cases}\dfrac{2(1-\lambda)b^4r^4}{\Phi(n,r,\alpha,a,b,p,q)}, & \text{if } |1-\lambda|\geq\Theta,\\[2ex]\dfrac{2|\mathrm{br}|n!}{3(1+2\alpha)(n+2)!}, & \text{if } |1-\lambda|\leq\Theta,\end{cases} \tag{47}$$

where

$$\begin{aligned}\Phi(n,r,\alpha,a,b,p,q)&=\left|\left[\left(3(1+2\alpha)^2(n+2)(n+1)b-8p(1+\alpha)^2(n+1)^2\right]\mathrm{br}^2-8\mathrm{qa}(1+\alpha)^2(n+1)^2\right|,\\\Theta&=\frac{\left|\left[\left(3(1+2\alpha)^2(n+2)b-8p(1+\alpha)^2(n+1)\right]\mathrm{br}^2-8\mathrm{qa}(1+\alpha)^2(n+1)\right|}{3b^2r^2(1+2\alpha)(n+2)}.\end{aligned} \tag{48}$$

Proof. For some real number λ, using equations (39) and (43), we have

$$a_3-\lambda a_2^2=\frac{h_2(r)(u_2-v_2)}{6(1+2\alpha)\delta(n,3)}+(1-\lambda)a_2^2. \tag{49}$$

Now, using equation (41), we obtain

$$a_3-\lambda a_2^2=h_2(r)\left[\frac{u_2-v_2}{6(1+2\alpha)\delta(n,3)}+\frac{(1-\lambda)[h_2(r)]^2(u_2+v_2)}{6(1+2\alpha)\delta(n,3)[h_2(r)]^2-8(1+\alpha)^2[\delta(n,2)]^2h_3(r)}\right]. \tag{50}$$

The last expression can be written as follows:

$$a_3-\lambda a_2^2=h_2(r)\left[\left(\Delta(n,r,\lambda,\alpha)-\frac{n!}{3(1+2\alpha)(n+2)!}\right)u_2+\left(\Delta(n,r,\lambda,\alpha)+\frac{n!}{3(1+2\alpha)(n+2)!}\right)v_2\right], \tag{51}$$

where

$$\Delta(n,r,\lambda,\alpha) = \frac{(1-\lambda)\left[h_2(r)\right]^2}{6(1+2\alpha)\delta(n,3)\left[h_2(r)\right]^2 - 8(1+\alpha)^2\left[\delta(n,2)\right]^2 h_3(r)}. \tag{52}$$

Using Lemma 1, we get the following equation:

$$\left|a_3 - \lambda a_2^2\right| \leq \begin{cases} 2\left|h_2(r)\right|\left|\Delta(n,r,\lambda,\alpha)\right|, & \text{if } |\Delta(n,r,\lambda,\alpha)| \geq \left|\dfrac{n!}{3(1+2\alpha)(n+2)!}\right|, \\[2ex] \dfrac{2\left|h_2(r)\right| n!}{3(1+2\alpha)(n+2)!}, & \text{if } |\Delta(n,r,\lambda,\alpha)| \leq \left|\dfrac{n!}{3(1+2\alpha)(n+2)!}\right|. \end{cases} \tag{53}$$

Simplifying the last inequality, we get the desired inequality (47), and this completes the proof of Theorem 4.

The following corollary is just a consequence of Theorem 4. Taking $\alpha = 1$, we get the following Fekete–Szegö inequality. □

Corollary 5. *Let the function f given by (1) be in the class $\mathscr{H}(n,r,1)$. Then, for some $\lambda \in \mathbb{R}$,*

$$\left|a_3 - \lambda a_2^2\right| \leq \begin{cases} \dfrac{2(1-\lambda)b^4 r^4}{\left|\left[27(n+2)(n+1)b - 32p(n+1)^2\right]\mathrm{br}^2 - 32\mathrm{qa}(n+1)^2\right|}, & \text{if } |1-\lambda| \geq \Theta, \\[2ex] \dfrac{2|\mathrm{br}|n!}{9(n+2)!}, & \text{if } |1-\lambda| \leq \Theta, \end{cases} \tag{54}$$

where

$$\Theta = \frac{\left|[27(n+2)b - 32p(n+1)]\mathrm{br}^2 - 32\mathrm{qa}(n+1)\right|}{9b^2 r^2 (n+2)}. \tag{55}$$

5. Conclusion

This research paper has investigated a new subclass of biunivalent functions, defined in terms of the Ruscheweyh derivative $\mathscr{R}^n$ of order n, associated with the Horadam polynomials. For functions belong to this function class, the author has derived estimates for the Taylor–Maclaurin initial coefficients and Fekete–Szegö functional problem. The work presented in this paper will lead to many different results for subclasses defined by the means of special cases of Horadam polynomials, such as Fibonacci polynomials, Lucas polynomials, Pell polynomials, and Chebyshev polynomials of first and second kinds. Moreover, the presented work in this paper will inspire researchers to extend its concepts to harmonic functions and symmetric q-calculus.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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References

[1] P. Duren, "Lecture notes in mathematics," *Subordination in Complex Analysis*, pp. 22–29, Springer, Berlin, Germany, 1977.

[2] S. Miller and P. Mocabu, *Differential Subordination: Theory and Applications*, CRC Press, New York, NY, USA, 2000.

[3] Z. Nehari, *Conformal Mappings*, McGraw-Hill, New York, NY, USA, 1952.

[4] M. Lewin, "On a coefficient problem for bi-univalent functions," *Proceedings of the American Mathematical Society*, vol. 18, no. 1, pp. 63–68, 1967.

[5] H. M. Srivastava, S. Altınkaya, and S. Yalcin, "Certain subclasses of bi-univalent functions associated with the Horadam polynomials," *Iranian Journal of Science and Technology Transaction A-Science*, vol. 43, no. 4, pp. 1873–1879, 2019.

[6] S. R. Swamy, "Bi-univalent function subclasses subordinate to Horadam polynomials," *Earthline Journal of Mathematical Sciences*, vol. 11, no. 2023, pp. 183–198, 2022.

[7] P. Duren, "Univalent Functions," *Grundlehren der Mathematischen Wissenschaften 259*, Springer-Verlag, New York, NY, USA, 1983.

[8] A. W. Goodman, *Univalent Functions*, Mariner Publishing Co. Inc, Boston, BO, USA, 1983.

[9] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press, John Wiley and Sons, New York, NY, USA, 1984.

[10] M. Fekete and G. Szegö, "Eine bemerkung über ungerade schlichte funktionen," *Journal of the London Mathematical Society*, vol. 1, no. 2, pp. 85–89, 1933.

[11] C. Abirami, N. Magesh, and J. Yamini, "Initial bounds for certain classes of bi-univalent functions defined by Horadam polynomials," *Abstract and Applied Analysis*, vol. 2020, Article ID 7391058, 8 pages, 2020.

[12] W. Al-Rawashdeh, *A Class of Non-bazilevic Functions Subordinate to Gegenbauer Polynomials*, preprint.

[13] W. Al-Rawashdeh, "Coefficient bounds of a class of bi-univalent functions related to Gegenbauer polynomials," *International Journal of Mathematics and Computer Science*, vol. 19, no. 3, pp. 635–642, 2024.

[14] W. Al-Rawashdeh, *A Family of Close-To-Convex Functions Satisfying Subordinate Conditions*, preprint.

[15] Ş. Altinkaya and S. Yalçin, "On the Chebyshev coefficients for a general subclass of univalent functions," *Turkish Journal of Mathematics*, vol. 42, no. 6, pp. 2885–2890, 2018.

[16] M. Cağlar, H. Orhan, and N. Yağmur, "Coefficient bounds for new subclasses of bi-univalent functions," *Filomat*, vol. 27, no. 7, pp. 1165–1171, 2013.

[17] J. H. Choi, Y. C. Kim, and T. Sugawa, "A general approach to the Fekete-Szegö problem," *Journal of the Mathematical Society of Japan*, vol. 59, no. 3, pp. 707–727, 2007.

[18] Q. Hu, T. G. Shaba, J. Younis, B. Khan, W. K. Mashwani, and M. Çağlar, "Applications of q-derivative operator to subclasses of bi-univalent functions involving Gegenbauer polynomials," *Applied Mathematics in Science and Engineering*, vol. 30, no. 1, pp. 501–520, 2022.

[19] M. Kamali̇, M. Çağlar, E. Deni̇z, and M. Turabaev, "Fekete-Szegö problem for a new subclass of analytic functions satisfying subordinate condition associated with Chebyshev polynomials," *Turkish Journal of Mathematics*, vol. 45, no. 3, pp. 1195–1208, 2021.

[20] F. R. Keogh and E. P. Merkes, "A Coefficient inequality for certain classes of analytic functions," *Proceedings of the American Mathematical Society*, vol. 20, no. 1, pp. 8–12, 1969.

[21] A. F. Horadam, "Basic properties of a certain generalized sequence of numbers," *Fibonacci Quarterly*, vol. 3, pp. 161–176, 1965.

[22] A. F. Horadam and J. M. Mahon, "Pell and pell-lucas polynomials," *Fibonacci Quarterly*, vol. 23, pp. 7–20, 1985.

[23] A. F. Horadam, "Jacobsthal representation polynomials," *Fibonacci Quarterly*, vol. 35, pp. 137–148, 1997.

[24] T. Koshy, "Fibonacci and Lucas numbers with applications," *Pure and Applied Mathematics*, John Wiley and Sons, Inc, Hoboken, New Jersey, USA, 2nd edition, 2018.

[25] H. Ö. Güney, G. Murugusundaramoorthy, and J. SokÓł, "Subclasses of bi-univalent functions related to shell-like curves connected with Fibonacci numbers," *Assam College Teachers' Association Universitatis Sapientiae, Mathematica*, vol. 10, no. 1, pp. 70–84, 2018.

[26] T. Horzum and E. G. Kocer, "On some properties of Horadam polynomials," *International Mathematics Forum*, vol. 4, pp. 1243–1252, 2009.

[27] F. Qi, C. Kizilateş, and W. S. Du, "A closed formula for the Horadam polynomials in terms of a tridiagonal Determinant," *Symmetry*, vol. 11, no. 6, p. 782, 2019.

[28] H. M. Srivastava, A. K. Wanas, and G. Murugusundaramoorthy, "Certain family of bi-univalent functions associated with Pascal distribution series based on Horadam polynomials," *Surveys Mathematics and Applications*, vol. 16, pp. 193–205, 2021.

[29] K. Vijaya, "Coefficient estimates of bi-Bazilevič functions defined by q-Ruscheweyh differential operator associated with Horadam polynomials," *Palestine Journal of Mathematics*, vol. 11, no. 2, pp. 352–361, 2022.

[30] S. Ruscheweyh, "New criteria for univalent functions," *Proceedings of the American Mathematical Society*, vol. 49, no. 1, pp. 109–115, 1975.

[31] K. I. Abdullah and N. H. Mohammed, "Bounds for the coefficients of two new subclasses of bi-univalent functions," *Science Journal of University of Zakho*, vol. 10, no. 2, pp. 66–69, 2022.

[32] W. Al-Rawashdeh, "Fekete-Szegö functional of a subclass of bi-univalent functions associated with Gegenbauer polynomials," *European Journal of Pure and applied Mathematics*, 2023.