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Research Article

Connected ϵ-Chainable Sets and Existence Results

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In the setting of ϵ-chainable metric spaces, we introduce $(\epsilon - \rho - \sigma)$ uniformly local weak contraction and obtain some results on the existence of fixed points. To show the veracity of the results, we also constructed some examples.

1. Introduction and Mathematical Preliminaries

In 1883, Cantor introduced the concept of connectedness of some subsets of Euclidean spaces $\mathbb{R}^n$, $n \geq 2$ as follows.

A set is connected, if for any elements u and v of the set and any $\epsilon > 0$, then a finite set of points "$u = u_0, u_1, \cdots, u_n = v$" can be found with the property that $\tau(u_i, u_{i+1}) < \epsilon$ for every $0 \leq i \leq n-1$. A metric space (Y, τ) with the above-mentioned property is said to be chainable and the collection "$u_0, u_1, \cdots, u_n$" is an ϵ-chain of length n from u to v, whereas (Y, τ) is said to be ϵ-chainable if any pair of elements of Y can be connected by a finite length ϵ-chain (see [1]).

The Banach contraction principle is a turning point in the metric fixed point theory. Alber and Guerre-Delabriere [2] used weak contractions to prove the contraction principle in Hilbert space which subsequently established in metric spaces by Rhoades [3]. It is vital to work on the weak contraction since we know that this form of contraction is decomposable [4] and may be used to obtain a more widespread contraction. In 2008, Suzuki [5] generalized the contraction principle which characterizes metric completeness. Since the beginning of the twenty-first century, three topics of chainability have received significant attention, namely, finitely chainable metric spaces, chainable subsets of metric spaces, and chainability through the use of functions. Edelstein [1] extended the contraction principle in the setting of the ϵ-chainable metric space using local contraction. Many researchers extend the local contraction in different ways and prove the contraction principle (see [6–12]). The notion of finitely chainable metric space was introduced by Atsuji [13]. Kundu et al. [14] collected equivalent conditions for finite chainability in metric spaces. In 2002, Shrivastava and Agrawal [15] discovered the concept of ϵ-chainable sets in metric spaces.

Motivated by these papers on ϵ-chainable sets and the contraction principle, we introduce $(\epsilon - \rho - \sigma)$ uniformly local weak contraction and give some existence results using ϵ-chainable sets in the setting of ϵ-chainable metric spaces. For various definitions on the topic, see [1, 7, 15].

Some relevant previous results on metric spaces and generalized metric spaces are given as follows.

Theorem 1 (see [16]). *Let (Y, τ, s) be a complete b-metric space and $\digamma: Y \longrightarrow CB(Y)$ weak quasicontraction for which there exists $\vartheta \in (0,1)$, $l \in [0,1]$, and $L \geq 0$ such that*

$$H(\digamma u, \digamma v) \leq \vartheta \max\{\tau(u,v), l\tau(u, \digamma u), l\tau(v, \digamma v)\} + L\tau(v, \digamma u), \quad (1)$$

for all $u, v \in Y$. Then, there exists a sequence $(u_n)_{n\in N}$ in Y which converges to some point $u^* \in Y$ such that $u_{n+1} \in F(u_n)$ for every $n\in N$. Also, u^* is a fixed point of F if any of the following conditions are satisfied:

(i) F is closed

(ii) τ is $*$-continuous

(iii) $s\vartheta l < 1$

Theorem 2 (see [17]). *Let (Y, τ) be a complete metric space and $F: Y \longrightarrow Y$ such that for all $u, v \in Y$, then $1/2\tau(u, Fu) \leq \tau(u, v)$ implies that*

$$\rho(\tau(Fu, Fv)) \leq \rho\left(\xi_F(u, v)\right) - \varrho\left(\xi_F(u, v)\right), \tag{2}$$

where

(i) ρ: $[0, +\infty) \longrightarrow [0, +\infty)$ is a continuous nondecreasing function and $\rho(\eta) = 0$, if and only if, $\eta = 0$,

(ii) ϱ: $[0, +\infty) \longrightarrow [0, +\infty)$ is lower semicontinuous with $\varrho(\eta) = 0$, if and only if, $\eta = 0$,

and

$$\xi_F(u, v) = \max\left\{\tau(u, v), \tau(u, Fu), \tau(v, Fv), \frac{\tau(u, Fv) + \tau(v, Fu)}{2}\right\}. \tag{3}$$

Then, F has a unique fixed point.

2. Main Results

Throughout the paper, suppose that ρ: $[0, +\infty) \longrightarrow [0, +\infty)$ is a nondecreasing and continuous function, satisfying $\rho(\eta) > 0$ for $\eta > 0$ and $\rho(0) = 0$.

To start with, we give the following definition for uniformly locally weak contraction:

Definition 3. Let (Y, τ) be an ϵ-chainable metric space. A function $F: Y \longrightarrow Y$ is said to be $(\epsilon - \rho - \sigma)$ uniformly local weak contraction if

$$\tau(u, v) < \epsilon \text{ implies } \tau(Fu, Fv) \leq \sigma(u, v) - \rho(\sigma(u, v)), \tag{4}$$

where $\sigma(u, v) = \min\{\tau(u, v), \tau(u, Fu), \tau(v, Fv), (\tau(u, Fv) + \tau(v, Fu))/2\}$.

Theorem 4. *Every $(\epsilon - \rho - \sigma)$ uniformly local weak contraction mapping on a complete ϵ-chainable metric space (Y, τ) has a unique fixed point.*

Proof. Let $u \in Y$ be an arbitrary element. We construct a sequence $\{u_n\}$ such that $u_0 = u$ and $u_i = F^i u$, for all $i \in N$. As Y is ϵ-chainable, $u = \alpha_0, \alpha_1, \ldots, \alpha_n = Fu$ is an τ- chain from u to Fu, such that $\tau(\alpha_i, \alpha_{i+1}) < \epsilon$, for all $i = 0, 1, 2, \cdots, n-1$. Consider

$$\begin{aligned}\tau(F\alpha_i, F\alpha_{i+1}) &\leq \sigma(\alpha_i, \alpha_{i+1}) - \rho(\sigma(\alpha_i, \alpha_{i+1}))\\ &\leq \sigma(\alpha_i, \alpha_{i+1})\\ &= \min\left\{\tau(\alpha_i, \alpha_{i+1}), \tau(\alpha_i, F\alpha_{i+1}), \tau(\alpha_{i+1}, F\alpha_{i+1}), \frac{\tau(\alpha_i, F\alpha_{i+1}) + \tau(\alpha_{i+1}, F\alpha_i)}{2}\right\}\\ &\leq \tau(\alpha_i, \alpha_{i+1}) < \epsilon.\end{aligned} \tag{5}$$

Inductively, we obtain $\tau(F^m\alpha_i, F^m\alpha_{i+1}) < \epsilon$ for any $m \in N$. Suppose that $\mathfrak{R}_m^i = \sigma(F^m\alpha_i, F^m\alpha_{i+1})$. Consider

$$\begin{aligned}\mathfrak{R}_{m+1}^i &= \sigma\left(F^{m+1}\alpha_i, F^{m+1}\alpha_{i+1}\right)\\ &\leq \tau\left(F^{m+1}\alpha_i, F^{m+1}\alpha_{i+1}\right)\\ &\leq \sigma(F^m\alpha_i, F^m\alpha_{i+1}) - \rho(\sigma(F^m\alpha_i, F^m\alpha_{i+1}))\\ &\leq \sigma(F^m\alpha_i, F^m\alpha_{i+1})\\ &= \mathfrak{R}_m^i.\end{aligned} \tag{6}$$

Thus, $\{\mathfrak{R}_m^i\}$ is a nonincreasing sequence and being bounded below (0 is a lower bound), it must be convergent. Suppose that

$$\lim_{m\longrightarrow+\infty} \mathfrak{R}_m^i = \mathfrak{R}^i (\geq 0), \tag{7}$$

for each $i = 0, 1, 2, \cdots, n-1$.

Again, we have $\mathfrak{R}_{m+1}^i \leq \mathfrak{R}_m^i - \rho(\mathfrak{R}_m^i)$. Taking limit $m \longrightarrow +\infty$ and using the continuity of ρ, we get $\mathfrak{R}^i \leq \mathfrak{R}^i - \rho(\mathfrak{R}^i)$. Hence, $\rho(\mathfrak{R}^i) = 0$, that is, $\mathfrak{R}^i = 0$. Now, using triangle inequality, we have

$$\begin{aligned}\tau(u_m, u_{m+1}) &= \tau(F^m u, F^m(Fu))\\ &\leq \sum_{i=0}^{n-1} \tau(F^m\alpha_i, F^m\alpha_{i+1})\\ &= \sum_{i=0}^{n-1} \mathfrak{R}_m^i.\end{aligned} \tag{8}$$

Taking limit $m \longrightarrow +\infty$, we obtain

$$\lim_{m\longrightarrow+\infty} \tau(u_m, u_{m+1}) \leq \lim_{m\longrightarrow+\infty} \sum_{i=0}^{n-1} \mathfrak{R}_m^i = \sum_{i=0}^{n-1} \lim_{m\longrightarrow+\infty} \mathfrak{R}_m^i = 0. \quad (9)$$

So, there exists $k \in \mathbb{N}$ such that

$$\tau(u_k, u_{k+1}) < \min\left\{\frac{\epsilon_0}{2}, \rho\left(\frac{\epsilon_0}{2}\right)\right\}, \quad (10)$$

where $\epsilon_0 = \min\{\epsilon, \epsilon_1\} > 0$, for any $\epsilon_1 > 0$.

We note that if $\tau(v, u_k) < \epsilon_0 \leq \epsilon$, then (4) holds for $u = u_k$.

Case I: Take $v \in B(u_k, \epsilon_0/2)$. Consider

$$\begin{aligned}\tau(\mathsf{F}v, u_k) &\leq \tau(\mathsf{F}v, \mathsf{F}u_k) + \tau(\mathsf{F}u_k, u_k)\\ &\leq \sigma(v, u_k) - \rho(\sigma(v, u_k)) + \tau(u_k, u_{k+1})\\ &\leq \tau(v, u_k) - \rho(\sigma(v, u_k)) + \frac{\epsilon_0}{2}\\ &< \frac{\epsilon_0}{2} - \rho(\sigma(v, u_k)) + \frac{\epsilon_0}{2}\left[\because v \in B\left(u_k, \frac{\epsilon_0}{2}\right)\right]\\ &\leq \epsilon_0 - \rho(\sigma(v, u_k))\\ &\leq \epsilon_0.\end{aligned} \quad (11)$$

Therefore, $\mathsf{F}v \in B(u_k, \epsilon_0)$ for each $v \in B(u_k, \epsilon_0/2)$. Since $\mathsf{F}u_k \in B(u_k, \epsilon_0)$ and $\mathsf{F}v \in B(u_k, \epsilon_0)$, $\sigma(v, u_k) \leq \tau(v, u_k)$.

Case II: Also, if $\epsilon_0/2 \leq \sigma(v, u_k) \leq \tau(v, u_k) \leq \epsilon_0$, by the monotonic property of ρ, we have $\rho(\epsilon_0/2) \leq \rho(\sigma(v, u_k))$. Consider

$$\begin{aligned}\tau(\mathsf{F}v, u_k) &\leq \tau(\mathsf{F}v, \mathsf{F}u_k) + \tau(\mathsf{F}u_k, u_k)\\ &\leq \sigma(v, u_k) - \rho(\sigma(v, u_k)) + \tau(u_k, u_{k+1})\\ &\leq \sigma(v, u_k) - \rho\left(\frac{\epsilon_0}{2}\right) + \rho\left(\frac{\epsilon_0}{2}\right)\\ &\leq \tau(v, u_k)\\ &\leq \epsilon_0.\end{aligned} \quad (12)$$

By the abovementioned two cases, we have $\mathsf{F}v \in B(u_k, \epsilon_0)$ for all $v \in B(u_k, \epsilon_0)$. It implies $u_m \in B(u_k, \epsilon_0)$ for all $m \geq k$. Hence, $\tau(u_m, u_k) < \epsilon_0 \leq \epsilon_1$ for all $m \geq k$, and thus $\{u_n\}$ is a Cauchy sequence. Since Y is complete, $\{u_n\}$ converge to some $\overline{u} \in Y$.

Again for any $\epsilon_2 > 0$, we get $\delta = \min\{\epsilon, \epsilon_2\}$. Now, if $\tau(u, v) < \delta$, then $\tau(u, v) < \epsilon$. Therefore, $\tau(\mathsf{F}u, \mathsf{F}v) \leq \sigma(u, v) - \rho(\sigma(u, v)) \leq \sigma(u, v) \leq \tau(u, v) < \delta \leq \epsilon_2$. It implies F is continuous. Consider

$$\begin{aligned}\overline{u} &= \lim_{n\longrightarrow+\infty} u_{n+1}\\ &= \lim_{n\longrightarrow+\infty} \mathsf{F}(u_n)\\ &= \mathsf{F}\overline{u}.\end{aligned} \quad (13)$$

Therefore, $\overline{u}$ is a fixed point of F.

Now, we prove that $\overline{u}$ is unique. On the contrary, suppose that $\overline{v}(\neq \overline{u}) \in Y$ such that $\overline{v} = \mathsf{F}\overline{v}$. Then, $\tau(\overline{u}, \overline{v}) > 0$.

Choose $\overline{u} = \beta_0, \beta_1, \cdots, \beta_p = \overline{v}$ as an $\epsilon-$ chain from $\overline{u}$ to $\overline{v}$. Thus, $\tau(\beta_i, \beta_{i+1}) < \epsilon$, for all $i = 0, 1, 2, \cdots, p-1$.

Using (8), we get

$$\tau(\overline{u}, \overline{v}) = \tau(\mathsf{F}^m\overline{u}, \mathsf{F}^m\overline{v}) \leq \sum_{i=0}^{p-1} \tau(\mathsf{F}^m\beta_i, \mathsf{F}^m\beta_{i+1}). \quad (14)$$

Letting the limit $m \longrightarrow +\infty$, we get $\tau(\overline{u}, \overline{v}) = 0$, which is a contradiction. Hence, the fixed point is unique. □

Example 1. Let $Y = U \cup V$, where $U = \{(u(\eta), v(\eta)): u(\eta) = 1 - \eta; v(\eta) = 0; 0 \leq \eta \leq 1\}$ and $V = \{(u(s), v(s)): u(s) = 0; v(s) = 1 + s; 0 \leq s \leq 1\}$. The metric space $\mathbb{R}^2$ with a usual metric τ has Y as a complete subspace. Also, Y is ν-chainable for any $\nu > 1$.

Case I: Consider $M(1-\eta_1, 0)$, $N(1-\eta_2, 0) \in U$ such that $\tau(M, N) = |\eta_1 - \eta_2| \leq 1$. Hence, $M = \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n = N$, and $\tau(\alpha_i, \alpha_{i+1}) \leq 1 < \nu$ for all $\alpha_i \in U$, where $i = 0, 1, 2, \ldots, n$.

Case II: Consider $M(0, 1+s_1), N(0, 1+s_2) \in V$ such that $\tau(M, N) = |s_1 - s_2| \leq 1$. Hence, $M = \alpha_0, \alpha_1, \ldots, \alpha_n = N$, and $\tau(\alpha_i, \alpha_{i+1}) \leq 1 < \nu$ for all $\alpha_i \in V$ where $i = 0, 1, 2, \cdots, n$.

Case III: Consider $M(1-\eta, 0) \in U, N(0, 1+s) \in V$ such that $\tau(M, N) \leq \tau((1-\eta, 0), (0,0)) + \tau((0,0), (0,1)) + \tau((0,1), (0, 1+s))$.

Hence, $M = \alpha_0, \alpha_1, \ldots, \alpha_k = (0,0), \alpha_{k+1} = (0,1), \alpha_{k+2}, \alpha_{k+3}, \ldots, \alpha_n = N$. Then, $\tau(\alpha_i, \alpha_{i+1}) \leq 1 < \nu$ for all $\alpha_i \in U$ where $i = 0, 1, 2, \ldots, k-1$, $\tau(\alpha_k, \alpha_{k+1}) = 1 < \nu$ and $\tau(\alpha_i, \alpha_{i+1}) \leq 1 < \nu$ for all $\alpha_i \in V$, for $i = k+1, \ldots, n-1$.

Hence, $\tau(\alpha_i, \alpha_{i+1}) \leq 1 < \eta$ for $i = 0, 1, 2, 3, \ldots, n$.

Therefore, for any $M, N \in Y$, there is a ν-chain from M to N, that is, there are finite number of points $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ in Y with $M = \alpha_0; N = \alpha_n$ such that $d(\alpha_i, \alpha_{i+1}) < \nu \forall i = 0, 1, \ldots, n-1$. Hence, Y is ν-chainable for any $\nu > 1$.

Define ρ: $[0, +\infty) \longrightarrow [0, +\infty)$ by $\rho(\lambda) = \lambda^2/2$. Clearly, ρ is a continuous, nondecreasing function such that $\rho(\eta) > 0$ for $\eta > 0$ and $\rho(0) = 0$.

Define F: $Y \longrightarrow Y$ by $\mathsf{F}(u(\lambda), v(\lambda)) = (1 - \lambda + \lambda^2/2, 0)$.

First, we demonstrate that F is not a uniform local (Banach) contraction. On the contrary, suppose that F is a uniform local contraction. So, there exists $\epsilon > 0, 0 \leq \gamma < 1$ such that for all $\lambda, \mu \in Y$, $\tau(\lambda, \mu) < \epsilon$,

$$\tau(\mathsf{F}(\lambda), \mathsf{F}(\mu)) < \gamma\,\tau(\lambda, \mu). \quad (15)$$

Choose $\eta = \min\{\epsilon, 1 - \gamma/2\}$.

Now, choose the points $\lambda(1,0)$ and $\mu(1-\eta,0)$ such that $\tau(\lambda,\mu) = \eta$ and $\tau(\mathsf{F}\lambda, \mathsf{F}\mu) = \tau((1,0),(1-\eta+\eta^2/2,0)) = \eta - \eta^2/2$. Since $\tau(\lambda,\mu) = \eta < \epsilon$, relation (15) is satisfied. Thus, we obtain $\eta - \eta^2/2 < \gamma\eta$ or $\eta - \eta^2/2 < (1-2\eta)\eta$ [because $\eta \le 1 - \gamma/2 \Longrightarrow \gamma \le 1 - 2\eta$] or $-1/2\eta^2 < -2\eta^2$, which is absurd. As a result, there is a contradiction and thus F is not a uniform local contraction.

We now demonstrate that F is not a ρ-weak contraction. Choose a pair of points $\lambda(1,0)$ and $\mu(0,2)$ of Y corresponding to $\eta = 0$ and $s = 1$, respectively, such that $\tau(\lambda,\mu) = \sqrt{1^2+2^2} = \sqrt{5}$; $\tau(\lambda,\mu) - \rho(\tau(\lambda,\mu)) = \sqrt{5} - 5/2 < 0$.

Again, $\mathsf{F}(1,0) = (1,0)$, $\mathsf{F}(0,2) = (1-1+1/2,0) = (1/2,0)$. $\tau(\mathsf{F}\lambda,\mathsf{F}\mu) = 1/2$, which shows that it is not a ρ-weak contraction.

Now, we show that F is a $(\epsilon-\rho-\sigma)$-uniformly local weak contraction map, for some $\nu > 0$.

Let us now consider the following cases.

Case-I: Choose two points $M(u(\eta),v(\eta)) \in U$ and $N(u(s),v(s)) \in V$ where $0 \le \eta \le 1$ and $0 \le s \le 1$ such that

$$\begin{aligned}
\tau(M,N) &= \sqrt{(1-\eta)^2+(1+s)^2} \ge 1+s \ge 1,\\
\tau(M,\mathsf{F}M) &= \tau\left((1-\eta,0),\left(1-\eta+\frac{\eta^2}{2},0\right)\right) = \frac{\eta^2}{2} \le \frac{1}{2},\\
\tau(N,\mathsf{F}N) &= \tau\left((0,1+s),\left(1-s+\frac{s^2}{2},0\right)\right) = \sqrt{(1+s)^2+\left(1-s+\frac{s^2}{2}\right)^2} \ge 1+s \ge 1.
\end{aligned} \tag{16}$$

Consider

$$\begin{aligned}
\frac{\tau(M,\mathsf{F}N)+\tau(N,\mathsf{F}M)}{2} &= \frac{1}{2}\left[\tau\left((1-\eta,0),\left(1-s+\frac{s^2}{2},0\right)\right)+\tau\left((0,s+1),\left(1-\eta+\frac{\eta^2}{2},0\right)\right)\right]\\
&= \frac{1}{2}\left[\left|\eta - s + \frac{s^2}{2}\right| + \sqrt{(s+1)^2+\left(1-\eta+\frac{\eta^2}{2}\right)^2}\right]\\
&\ge \frac{1}{2}(s+1)\\
&\ge \frac{1}{2}.
\end{aligned} \tag{17}$$

So, $\sigma(M,N) = \min\{\tau(M,N), \tau(M,\mathsf{F}M), \tau(N,\mathsf{F}N), (\tau(M,\mathsf{F}N)+\tau(N,\mathsf{F}M))/2\}$, which implies $\sigma(M,N) = \tau(M,\mathsf{F}M)$ and $\sigma(M,N) - \rho(\sigma(M,N)) = \eta^2/2 - \eta^4/8$.

Now, $\tau(\mathsf{F}M,\mathsf{F}N) = \tau((1-\eta+\eta^2/2,0),(1-s+s^2/2,0)) = |\eta - s - 1/2(\eta^2 - s^2)|$.

Define a function $T\colon \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ by $T(\eta,s) = \eta^2/2 - \eta^4/8 - |\eta - s - 1/2(\eta^2-s^2)|$. At $(1,1/4)$, T is continuous and $T(1,1/4) = 1/2 - 1/8 - |1 - 1/4 - 1/2(1-1/16)| = 3/32 > 0$. Thus, by the neighbourhood property of the continuous functions, there is a δ-neighbourhood of the point $(1,1/4)$ where the function assumes only positive values. Thus, for $\eta = 1$ and $s = 1/4$, the corresponding pair of points $(0,0) \in U$ and $(0,5/4) \in V$ satisfies (4) and there is some δ with $0.1 > \delta > 0$ such that for all pair of points $\mathfrak{R}(u(\eta),v(\eta)) \in U$ and $S(u(s),v(s)) \in V$, where $\eta \in [1-\delta,1]$ and $s \in [0,\delta)$, (4) remains satisfied.

Furthermore, consider $\omega \ge 0$ such that $\omega \le \delta$. The points corresponding to parametric values $\eta = 1-\omega$ and $s = \omega + 1/4$ are $P(\omega,0)$ and $Q(0,5/4+\omega)$, respectively, and the distance between them is $\tau(M,N) = \tau((\omega,0),(0,5/4+\omega)) = \sqrt{\omega^2+(5/4+\omega)^2}$. $\sigma(M,N) - \rho(\sigma(M,N)) = (1-\omega)^2/2 - (1-\omega)^4/8$, $0 \le \omega < \delta < 0.1$. Consider

$$\begin{aligned}\tau(\digamma M,\digamma N) &= \tau\left(\left(\omega+\frac{(1-\omega)^2}{2},0\right),\left(1-\omega-\frac{1}{4}+\frac{(\omega+1/4)^2}{2},0\right)\right)\\ &= 1-\omega-\frac{1}{4}+\frac{1}{2}\left(\omega^2+\frac{1}{16}+\frac{\omega}{2}\right)-\omega-\frac{1}{2}\left(\omega^2+1-2\omega\right)\\ &= 1-\frac{1}{4}+\frac{1}{32}-\frac{1}{2}+\omega\left(-1+\frac{1}{4}-1+1\right)\\ &= \frac{9}{32}-\frac{3.\omega}{4}\\ &\le \frac{9}{32}.\end{aligned} \tag{18}$$

Now, $\sigma(M,N)-\rho(\sigma(M,N)) = (1-\omega)^2/2-(1-\omega)^4/8 \ge 0.3$ *for* $\omega \in [0,0.1]$. $d(\digamma M,\digamma N) \le \sigma(M,N)-\rho(\sigma(M,N))$ *holds for* $\omega \in [0,0.1]$.

Thus, in this case, we see that a pair of point (M,N) (where $M \in U, N \in V$) whose distance is less than $\nu = \sqrt{(0.01)^2+(5/4+0.01)^2} \approx 1.27$. (for $\omega = 0.01$) satisfies (4).

Case II: Take $M(u(s_1),v(s_1)), N(u(s_2),v(s_2)) \in V$, where $0 \le s_1 \le s_2 \le 1$. Consider

$$\begin{aligned}\tau(\digamma M,\digamma N) &= 1-s_1+\frac{s_1^2}{2}-\left(1-s_2+\frac{s_2^2}{2}\right)\\ &= (s_2-s_1)-\frac{1}{2}\left(s_2^2-s_1^2\right),\\ \tau(M,N) &= s_2-s_1 \le 1,\\ \tau(M,\digamma M) &= d\left((0,1+s_1),\left(1-s_1+\frac{s_1^2}{2},0\right)\right)\\ &= \sqrt{(1+s_1)^2+\left(1-s_1+\frac{s_1^2}{2}\right)^2} \ge 1,\\ \tau(N,\digamma N) &= \sqrt{(1+s_2)^2+\left(1-s_2+\frac{s_2^2}{2}\right)^2} \ge 1,\\ \frac{\tau(M,\digamma N)+\tau(N,\digamma M)}{2} &= \frac{1}{2}\left[\sqrt{(1+s_1)^2+\left(1-s_1+\frac{s_1^2}{2}\right)^2}+\sqrt{(1+s_2)^2+\left(1-s_2+\frac{s_2^2}{2}\right)^2}\right] \ge 1.\end{aligned} \tag{19}$$

Therefore, $\sigma(M,N) = \tau(M,N)$ *and* $\sigma(M,N)-\rho(\sigma(M,N)) = \tau(M,N)-1/2[\tau(M,N)]^2 = s_2-s_1-1/2(s_2-s_1)^2$. *Since* $0 < s_1 \le s_2$, *we note that* $s_2^2-s_1^2 = (s_2-s_1)(s_2+s_1) \ge (s_2-s_1)^2$. *Thus we have,* $(s_2-s_1)-1/2(s_2^2-s_1^2) \le (s_2-s_1)-1/2(s_2-s_1)^2$.

Thus, (4) *is satisfied. In particular,* $\tau(P,Q) < \eta$ *and* (4) *also holds.*

Case-III: Take $M(u(\eta_1),v(\eta_1)), N(u(\eta_2),v(\eta_2)) \in U$, $0 \le \eta_1 \le \eta_2 \le 1$. Consider

$$
\begin{aligned}
\tau(\digamma M, \digamma N) &= 1-\eta_1+\frac{\eta_1^2}{2}-\left(1-\eta_2+\frac{\eta_2^2}{2}\right)\\
&= (\eta_2-\eta_1)-\frac{1}{2}(\eta_2^2-\eta_1^2),\\
\tau(M,N) &= \eta_2-\eta_1,\\
\tau(M, \digamma M) &= 1-\eta_1+\frac{\eta_1^2}{2}-(1-\eta_1)=\frac{\eta_1^2}{2},\\
\tau(N, \digamma N) &= \left(1-\eta_2+\frac{\eta_2^2}{2}\right)-(1-\eta_2)=\frac{\eta_2^2}{2},\\
\frac{\tau(M, \digamma N)+\tau(N, \digamma M)}{2} &= \frac{1}{2}\left[\frac{\eta_2^2}{2}+\frac{\eta_1^2}{2}\right]\geq \frac{\eta_1^2}{2}.
\end{aligned}
\tag{20}
$$

Then, we see that $\tau(M, \digamma M)\leq \tau(N, \digamma N)$ and $\tau(M, \digamma M)\leq (\tau(M, \digamma N)+\tau(N, \digamma M))/2$. So, either $\sigma(M,N) = \tau(M,N)$ or $\tau(M, \digamma M)$.

Subcase-I: If $\sigma(M,N)=\tau(M,N)$, then $\sigma(M,N)-\rho(\sigma(M,N)) = (\eta_2-\eta_1)-1/2(\eta_2-\eta_1)^2$. Since $0\leq\eta_1\leq\eta_2$, $\eta_2^2-\eta_1^2=(\eta_2+\eta_1)(\eta_2-\eta_1)\geq(\eta_2-\eta_1)^2$. Thus, we have, $(\eta_2-\eta_1)-1/2(\eta_2^2-\eta_1^2)\leq(\eta_2-\eta_1)-1/2(\eta_2-\eta_1)^2$. Thus, (4) is satisfied. In particular, if $\tau(M,N)<\nu$, then (4) also holds.

Subcase-II: If $\sigma(M,N)=\tau(M, \digamma M)$, then $\sigma(M,N)-\rho(\sigma(M,N)) = \eta_1^2/2-\eta_1^4/8$. Since $0\leq\eta_1\leq\eta_2\leq 1$, we get $(\eta_2-\eta_1)-1/2(\eta_2^2-\eta_1^2)\leq\eta_1^2/2-\eta_1^4/8$.

Hence, $\tau(\digamma M, \digamma N)\leq\sigma(M,N)-\rho(\sigma(M,N))$ holds for any subcases. In particular, if $\tau(M,N)<\nu$, then Theorem 4 is satisfied. Hence, from the abovementioned cases, we can conclude that the function $\digamma$ is an $(\epsilon-\rho-\sigma)$- uniformly locally weak contraction for $\nu=1.27$ and the metric space (Y,τ) is 1.27-chainable. Thus, all the conditions of Theorem 4 are satisfied and point (1, 0) is a fixed point of $\digamma$.

Remark 5. In our present theorem, the contraction $(\epsilon-\rho-\sigma)$ uniformly locally weak contraction which is not trivial from $(\epsilon-\rho)$- uniformly local weak contraction, that is, if for each $u, v\in Y$, $\tau(u,v)<\epsilon$, then $\tau(\digamma u, \digamma v)\leq\tau(u,v)-\rho(\tau(u,v))$. It does not imply in general that $\tau(\digamma u, \digamma v)\leq\sigma(u,v)-\rho(\sigma(u,v))\leq\tau(u,v)-\rho(\tau(u,v))$. Hence, $\sigma(u,v)-\rho(\sigma(u,v))$ and $\tau(u,v)-\rho(\tau(u,v))$ are not comparable in general.

Now, we give the result for the following Ćirić type contraction.

Theorem 6. *Suppose that* $\digamma: Y\longrightarrow Y$ *is self mapping on a complete,* ϵ*-chainable metric space* (Y,τ) *satisfying*

$$
\tau(\digamma u, \digamma v)\leq\alpha\max\left\{\tau(u,v), \tau(u, \digamma u), \tau(v, \digamma v), \frac{\tau(u, \digamma v)+\tau(v, \digamma u)}{2}\right\},
\tag{21}
$$

where $0\leq\alpha\leq(q/(q+1))(<1)$, $0<q<1$ *and* $u, v\in Y$. *Then,* $\digamma$ *has unique fixed point* $\overline{u}$ *in* Y.

Proof. Let $u\in Y$ be an arbitrary element. We construct a sequence $\{u_n\}$ such that $u_0=u$, $u_1=\digamma u_0$, and $u_2=\digamma u_1,\ldots,u_i=\digamma^i u$, for all $i\in N$. As Y is ϵ-chainable, let $u=\alpha_0,\alpha_1,\ldots,\alpha_n=\digamma u$ be an $\epsilon-$ chain from u to $\digamma u$, where $d(\alpha_i,\alpha_{i+1})<\epsilon$, for all $i=0,1,2,\ldots,n-1$. Since $\tau(\alpha_i,\alpha_{i+1})<\epsilon$, for all $i=0,1,2,\ldots,n-1$, (21) is also satisfied for every pair of consecutive elements of the chain. Consider

$$
\begin{aligned}
\tau(\digamma\alpha_i, \digamma\alpha_{i+1}) &\leq \alpha\max\left\{\tau(\alpha_i,\alpha_{i+1}), \tau(\alpha_i, \digamma\alpha_i), \tau(\alpha_{i+1}, \digamma\alpha_{i+1}), \frac{\tau(\alpha_i, \digamma\alpha_{i+1})+\tau(\alpha_{i+1}, \digamma\alpha_i)}{2}\right\}\\
&= \alpha\max\left\{\tau(\alpha_i,\alpha_{i+1}), \tau(\alpha_i,\alpha_{i+1}), \tau(\alpha_{i+1},\alpha_{i+2}), \frac{\tau(\alpha_i,\alpha_{i+2})}{2}\right\}\\
&\leq \alpha\max\left\{\tau(\alpha_i,\alpha_{i+1}), \tau(\alpha_{i+1},\alpha_{i+2}), \frac{\tau(\alpha_i,\alpha_{i+1})+\tau(\alpha_{i+1},\alpha_{i+2})}{2}\right\}\\
&< \alpha\epsilon.
\end{aligned}
\tag{22}
$$

Hence, $\tau(\digamma\alpha_i, \digamma\alpha_{i+1})<\epsilon$, $0\leq\alpha\leq(q/(q+1))<1$. Inductively, we obtain $\tau(\digamma^m\alpha_i, \digamma^m\alpha_{i+1})<\epsilon$ for any $m\in N$.

Let $\mathfrak{R}_m^i=\tau(\digamma^m\alpha_i, \digamma^m\alpha_{i+1})$. Consider

$$\begin{aligned}
\mathfrak{R}^i_{m+1} &= \tau\left(\mathsf{F}^{m+1}\alpha_i, \mathsf{F}^{m+1}\alpha_{i+1}\right)\\
&\leq \alpha \max\Big\{\tau\left(\mathsf{F}^m\alpha_i, \mathsf{F}^m\alpha_{i+1}\right), \tau\left(\mathsf{F}^m\alpha_i, \mathsf{F}^{m+1}\alpha_i\right), \tau\left(\mathsf{F}^m\alpha_{i+1}, \mathsf{F}^{m+1}\alpha_{i+1}\right),\\
&\quad \frac{\tau\left(\mathsf{F}^m\alpha_i, \mathsf{F}^{m+1}\alpha_{i+1}\right) + \tau\left(\mathsf{F}^m\alpha_{i+1}, \mathsf{F}^{m+1}\alpha_i\right)}{2}\Big\}\\
&= \alpha \max\Big\{\tau\left(\mathsf{F}^m\alpha_i, \mathsf{F}^m\alpha_{i+1}\right), \tau\left(\mathsf{F}^m\alpha_i, \mathsf{F}^m\alpha_{i+1}\right), \tau\left(\mathsf{F}^m\alpha_{i+1}, \mathsf{F}^m\alpha_{i+2}\right),\\
&\quad \frac{\tau\left(\mathsf{F}^m\alpha_i, \mathsf{F}^m\alpha_{i+2}\right) + \tau\left(\mathsf{F}^m\alpha_{i+1}, \mathsf{F}^m\alpha_{i+1}\right)}{2}\Big\}\\
&\leq \alpha \max\left\{\tau\left(\mathsf{F}^m\alpha_i, \mathsf{F}^m\alpha_{i+1}\right), \tau\left(\mathsf{F}^m\alpha_i, \mathsf{F}^m\alpha_{i+1}\right), \frac{\tau\left(\mathsf{F}^m\alpha_i, \mathsf{F}^m\alpha_{i+1}\right) + \tau\left(\mathsf{F}^m\alpha_{i+1}, \mathsf{F}^m\alpha_{i+2}\right)}{2}\right\}\\
&\leq \alpha\tau\left(\mathsf{F}^m\alpha_i, \mathsf{F}^m\alpha_{i+1}\right).
\end{aligned} \tag{23}$$

Therefore, using $0 \leq \alpha \leq q/(q+1) < 1$, $\mathfrak{R}^i_{m+1} < \tau(\mathsf{F}^m\alpha_i, \mathsf{F}^m\alpha_{i+1}) = \mathfrak{R}^i_m$.

Thus, $\{\mathfrak{R}^i_m\}$ is a nonincreasing sequence and being bounded below (0 is a lower bound), it must be convergent. Suppose $\lim_{m\longrightarrow+\infty}\mathfrak{R}^i_m = \mathfrak{R}^i \geq 0$ for each $i = 0, 1, 2, \ldots, n-1$.

Again, taking $m \longrightarrow \infty$ in (23), we get $\lim_{m\longrightarrow+\infty}\mathfrak{R}^i_{m+1} \leq \lim_{m\longrightarrow+\infty}\alpha\,\mathfrak{R}^i_m$, and it implies $(1-\alpha)\mathfrak{R}^i \leq 0$. As $0 \leq \alpha \leq (q/(q+1)) < 1$, $1-\alpha > 0$, $\mathfrak{R}^i \geq 0$, we get $\mathfrak{R}^i = 0$. Hence it is easy to see that,

$$\lim_{m\longrightarrow+\infty} \tau(u_m, u_{m+1}) = 0. \tag{24}$$

Now, we will show that $\{u_n\}$ is a Cauchy sequence. Suppose that for a given $\epsilon > 0$, there exists a natural number k such that

$$\tau(u_m, u_{m+1}) < \frac{\epsilon}{2q}, \tag{25}$$

for all $m \geq k$. Now, using the triangle inequality in any of the cases, we have

$$\begin{aligned}
\tau(\mathsf{F}u_m, \mathsf{F}u_n) &\leq \alpha \max\left\{\tau(u_m, u_n), \tau(u_m, \mathsf{F}u_m), \tau(u_n, \mathsf{F}u_n), \frac{\tau(u_m, \mathsf{F}u_n) + \tau(u_n, \mathsf{F}u_m)}{2}\right\}\\
&\leq \alpha.\left[\tau(u_m, \mathsf{F}u_m) + \tau(\mathsf{F}u_m, \mathsf{F}u_n) + \tau(\mathsf{F}u_n, u_n)\right].
\end{aligned} \tag{26}$$

After rewriting, we get

$$\tau(\mathsf{F}u_m, \mathsf{F}u_n) \leq \frac{\alpha}{1-\alpha}\left[\tau(u_m, \mathsf{F}u_m) + \tau(u_n, \mathsf{F}u_n)\right]. \tag{27}$$

Using (25) and assumption of α, we have, for all $n, m \geq k$

$$\tau(\mathsf{F}u_m, \mathsf{F}u_n) < \frac{\alpha}{1-\alpha}\left(\frac{\epsilon}{2q} + \frac{\epsilon}{2q}\right) < q.\frac{\epsilon}{q} = \epsilon. \tag{28}$$

Therefore, the sequence $\{u_n\}$ is a Cauchy sequence. As Y is complete, there exists $\overline{u} \in Y$ such that $u_n \longrightarrow \overline{u}$, as $n \longrightarrow +\infty$. Take $u_n = \overline{u}$ in (27), we get for all $m \geq 0$

$$\begin{aligned}
\tau(u_{m+1}, \mathsf{F}\overline{u}) &= \tau(\mathsf{F}u_m, \mathsf{F}\overline{u})\\
&\leq \frac{\alpha}{1-\alpha}\left[\tau(u_m, \mathsf{F}u_m) + \tau(\overline{u}, \mathsf{F}\overline{u})\right)\\
&\leq q\left[\tau(u_m, u_{m+1}) + \tau(\overline{u}, \mathsf{F}\overline{u})\right].
\end{aligned} \tag{29}$$

Taking the limit $m \longrightarrow +\infty$, we have, $\tau(\overline{u}, \mathsf{F}\overline{u}) \leq q[\tau(\overline{u}, \overline{u}) + \tau(\overline{u}, \mathsf{F}\overline{u})]$, that is, $(1-q)\tau(\overline{u}, \mathsf{F}\overline{u}) \leq 0$. As $(1-q) > 0$, it follows that $\tau(\overline{u}, \mathsf{F}\overline{u}) = 0$.

If possible, suppose that $\overline{v}$ is another fixed point of Y. From (27), we get

$$\tau(\overline{u}, \overline{v}) = \tau(\mathsf{F}\overline{u}, \mathsf{F}\overline{v}) \leq q[\tau(\overline{u}, \mathsf{F}\overline{u}) + \tau(\overline{v}, \mathsf{F}\overline{v})] = 0. \tag{30}$$

Therefore, $\overline{u} = \overline{v}$, which is a contradiction. Hence, the result. □

Example 2. Let $Y = [0, 1]$ with a usual metric d. Here, $Y = [0, 1]$ is a connected, complete $\epsilon-$ chainable metric space. Define F_q: $Y \longrightarrow Y$ by

$$\mathsf{F}_q(u) = \begin{cases} \dfrac{q}{2(q+1)} & \text{if} \quad 0 \leq u < 1,\\ 0, & \text{if} \quad u = 1, \end{cases} \tag{31}$$

for $0 < q < 1$. Then, $\mathsf{F}_q(u)$ has a unique fixed point.

Solution. *Choose* $u, v \in [0, 1)$. *Consider* $\tau(\mathsf{F}_q(u), \mathsf{F}_q(v)) = |\mathsf{F}_q(u) - \mathsf{F}_q(v)| = |q/2(q+1) - q/2(q+1)| = 0$. *Hence,* $0 \leq \alpha \max\{\tau(u,v), \tau(u, \mathsf{F}_q u), \tau(v, \mathsf{F}_q v), (\tau(u, \mathsf{F}_q v) + \tau(v, \mathsf{F}_q u))/2\}$.

If $u, v = 1$, *we get* $\tau(\mathsf{F}_q(1), \mathsf{F}_q(1)) = 0 \leq \alpha \max\{\tau(1,1), \tau(1, \mathsf{F}_q(1)), \tau(1, \mathsf{F}_q(1)), (\tau(1, \mathsf{F}_q(1)) + \tau(1, \mathsf{F}_q(1)))/2\}$.

Now, choose $u \in [0,1)$ *and* $v = 1$. *Then,* $\tau(\mathsf{F}_q(u), \mathsf{F}_q(1)) = |q/2(q+1) - 0| = q/2(q+1)$.

Now, $\max\{\tau(u,1), \tau(u, \mathsf{F}_q(u)), \tau(1, \mathsf{F}_q(1)), (\tau(u, \mathsf{F}_q(1)) + \tau(1, \mathsf{F}_q(u)))/2\} = \tau(1, \mathsf{F}_q(1)) = 1$.

So, $\tau(\mathsf{F}_q(u), \mathsf{F}_q(1)) \leq \alpha \max\{\tau(u,1), \tau(u, \mathsf{F}_q(u)), \tau(1, \mathsf{F}_q(1)), (\tau(u, \mathsf{F}_q(1)) + \tau(1, \mathsf{F}_q(u)))/2\}$, for $0 \leq \alpha \leq q/(q+1)$.

Therefore, $\mathsf{F}_q(u)$ satisfies all the conditions of Theorem 6 and it has a unique fixed point at $q/2(q+1)$.

Here, it is interesting to note that F_q is an uncountable function for $0 < q < 1$ defined on $Y = [0, 1]$. In particular, our function has a fixed point $\mathsf{F}_{q_0}(u) = q_0/2(q_0+1)$ for each $q_0 \in (0,1), u \in [0,1]$, for example, for $q_0 = 1/2$, $\mathsf{F}_{1/2}(1/6) = 1/6$.

Now, we provide the following example in a non-chainable metric space where inequality (21) does not hold.

Example 3. Suppose that $Y = \{0\} \cup \{1/n\colon n \in \mathbb{N}\}$, where $\mathbb{N}$ is a set of natural numbers, $\tau(u,v) = |u - v|$, for all $u, v \in Y$. Here, Y is complete but not chainable for $\epsilon < 1/2$.

Define a mapping $\mathsf{F}\colon Y \longrightarrow Y$ as

$$\mathsf{F}(u) = \begin{cases} 1, & u = 0, \\ \dfrac{1}{n+1}, u = \dfrac{1}{n}, & n \in \mathbb{N}. \end{cases} \tag{32}$$

Choose $u = (1/3), v = (1/2) \in Y$. Then, $\mathsf{F}(1/3) = 1/4$; $\mathsf{F}(1/2) = 1/3$. So,

$$\begin{aligned} \tau\left(\mathsf{F}\left(\frac{1}{3}\right), \mathsf{F}\left(\frac{1}{2}\right)\right) &= \left|\frac{1}{4} - \frac{1}{3}\right| = \frac{1}{12}, \\ \tau\left(\frac{1}{3}, \mathsf{F}\left(\frac{1}{3}\right)\right) &= \left|\frac{1}{3} - \frac{1}{4}\right| = \frac{1}{12}, \\ \tau\left(\frac{1}{2}, \mathsf{F}\left(\frac{1}{2}\right)\right) &= \left|\frac{1}{2} - \frac{1}{3}\right| = \frac{1}{6}, \\ \tau\left(\frac{1}{3}, \frac{1}{2}\right) &= \left|\frac{1}{3} - \frac{1}{2}\right| = \frac{1}{6}, \\ \frac{\tau(1/2, \mathsf{F}(1/3)) + \tau(1/3, \mathsf{F}(1/2))}{2} &= \frac{|1/2 - 1/4| + |1/3 - 1/3|}{2} = \frac{1}{8}. \end{aligned} \tag{33}$$

Therefore, $\max\{\tau(\mathsf{F}(1/3), \mathsf{F}(1/2)), \tau(1/3, \mathsf{F}(1/3)), \tau(1/3, 1/2), (\tau(1/2, \mathsf{F}(1/3)) + \tau(1/3, \mathsf{F}(1/2)))/2\} = 1/6$. Hence, using Theorem 6, we get $1/12 \leq \alpha.1/6$, $1/2 \leq \alpha \leq (q/(q+1))$. It implies $q \geq 1$, which is a contradiction. Hence, in a non-chainable metric space, inequality (21) does not hold.

Remark 8. In Theorems 4 and 6, we use chainable metric space which is more general as compared to metric space. Here, in Example 1, we show that the operator is not a uniform local (Banach) contraction as well as not a ρ- weak contraction, but it satisfies the inequality (4) in a chainable metric space for $\epsilon = 1.27$ and we get a unique fixed point. But in Example 1, we cannot say the existence of a fixed point in metric space.

3. Conclusion

In our present paper, we use $(\epsilon - \rho - \sigma)$ uniformly local weak contraction in ϵ-chainable metric spaces. We get a unique fixed point for such contraction. We claim that our results generalized some existence fixed point results in metric space and generalized metric space because we prove our results within ϵ-chainable restrictions. Actually, ϵ-chainable metric space is a subclass of metric space. Some examples show that results are satisfied with ϵ-chainable metric spaces but it does not validate in the setting of metric spaces and generalized metric spaces. So, in future, many fixed point results may be established in Hilbert space, Banach space, etc. Also, it may be possible for applying in the integral equation and differential equation. Our study is more significant for the researchers in the fixed point theory. We can say that our results may indicate the new direction of possible future research.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally.

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