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Research Article

On Properties of Graded Rings with respect to Group Homomorphisms

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Let G be a group and R be a G-graded ring with non-zero unity. The goal of our article is reconsidering some well-known concepts on graded rings using a group homomorphism $\alpha: G \longrightarrow G$. Next is to examine the new concepts compared to the known concepts. For example, it is known that (R, G) is weak if whenever $g \in G$ such that $R_g = 0$, then $R_{g^{-1}} = 0$. In this article, we also introduce the concept of α-weakly graded rings, where (R, G) is said to be α-weak whenever $g \in G$ such that $R_g = 0$, and $R_{\alpha(g)} = 0$. Note that if G is abelian, then the concepts of weakly and α-weakly graded rings coincide with respect to the group homomorphism $\alpha(g) = g^{-1}$. We introduce an example of non-weakly graded ring that is α-weak for some α. Similarly, we establish and examine the concepts of α-non-degenerate, α-regular, α-strongly, α-first strongly graded rings, and α-weakly crossed product.

1. Introduction

Let G be a group and R be a ring with unity 1. If additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ exist for all $g, h \in G$, then R is said to be G-graded. A G-graded ring R is denoted by (R, G). The elements of R_g are called homogeneous of degree g and R_e. The identity component of R is a subring of R with $1 \in R_e$. For $x \in R$, it can be written uniquely as $\sum_{g\in G} x_g$ where x_g is the component of x in R_g and $x_g = 0$ except finitely many.

The set of all homogeneous elements of R is $\bigcup_{g\in G} R_g$ and is denoted by $h(R)$. The support of (R, G) is defined by $\text{supp}(R, G) = \{g \in G: R_g \neq 0\}$. Let R be a graded ring and let I be an ideal of R. If $I = \bigoplus_{g\in G} (I \cap R_g)$, i.e., if whenever $x \in I$, then $x_g \in I$ for all $g \in G$. Then I is said to be a graded ideal. An ideal of a graded ring is not necessarily graded (see [1, 2]).

The concepts of faithful, non-degenerate, regular, and strongly graded rings have been introduced and examined in [2]; (R, G) is said to be faithful if for all $g, h \in G$, $a_g \in R_g - \{0\}$, we have $a_g R_h \neq \{0\}$ and $R_h a_g \neq \{0\}$. It is clear if (R, G) is faithful, then $\text{supp}(R, G) = G$. (R, G) is said to be non-degenerate if for all $g \in G$, $a_g \in R_g - \{0\}$, we have $a_g R_{g^{-1}} \neq \{0\}$ and $R_{g^{-1}} a_g \neq \{0\}$. Otherwise, (R, G) is said to be degenerate. (R, G) is said to be regular if for all $g \in G$, $a_g \in R_g$, we have $a_g \in a_g R_{g^{-1}} a_g$. Evidently, every regular graded ring is considered non-degenerate graded, and every faithful graded ring is said to be non-degenerate. (R, G) is said to be strong if $R_g R_h = R_{gh}$ for all $g, h \in G$. Undoubtedly, every strongly graded ring is faithful. Firstly, strongly graded rings have been introduced and investigated in [3]; (R, G) is said to be first strongly if $1 \in R_g R_{g^{-1}}$ for all $g \in \text{supp}(R, G)$. Therefore, every strongly graded ring is first strong. We can certainly say that if (R, G) is strong, then $\text{supp}(R, G) = G$. On the other hand, if (R, G) is first strong, then $\text{supp}(R, G)$ is a subgroup of G. Surely, (R, G) is first strong if and only if $\text{supp}(R, G)$ is a subgroup of G and $R_g R_h = R_{gh}$ for all $g, h \in \text{supp}(R, G)$. Additionally, in [3], the concept of second strongly graded rings was introduced; (R, G) is second strong if $\text{supp}(R, G)$ is a monoid in G and $R_g R_h = R_{gh}$ for all $g, h \in \text{supp}(R, G)$. Apparently, every strongly graded ring is second strong. Furthermore, every first strongly graded ring is second strong. In fact, if (R, G) is second strong and $\mathrm{supp}(R, G)$ is a subgroup of G, then (R, G) is first strong. Moreover, (R, G) is first strong if and only if (R, G) is second strong and non-degenerate. Furthermore, (R, G) is strong if and only if (R, G) is second strong and faithful. In [4], the concept of weakly graded rings was proposed and investigated; (R, G) is said to be weak if $g \in G$ with $R_g = \{0\}$, and $R_{g^{-1}} = \{0\}$. Every non-degenerate graded ring is weak, and if $\mathrm{supp}(R, G)$ is a subgroup of G, then (R, G) is weak. Therefore, we say that every first strongly graded ring is weak, and every strongly graded ring is weak. On the other hand, a second strongly graded ring is not necessarily weak, and a weakly graded ring is not necessarily second strong (see [4]). Following [2], (R, G) is said to be a crossed product if R_g contains a unit for all $g \in G$. On the other hand, the concept of weakly crossed product (crossed product over the support) has been proposed in [5] and was investigated in [4] where (R, G) is said to be a weakly crossed product if R_g contains a unit for all $g \in \mathrm{supp}(R, G)$. It is clear that if (R, G) is a crossed product, then (R, G) is a weakly crossed product. However, the converse is not necessarily true. If $|G| \geq 2$, then the trivial graduation of R ($R_e = R$ and $R_g = 0$ otherwise) is a weakly crossed product, but it is not a crossed product since for $g \neq e$, R_g does not contain units. For more terminology, see [6–9].

In this article, we also reconsider some of the aforementioned concepts using a group homomorphism $\alpha\colon G \longrightarrow G$ and examine the new concepts compared to the aforementioned concepts. In Section 2, we introduce the concepts of α-non-degenerate and α-regular graded rings and examine them in comparison to non-degenerate and regular graded rings. (R, G) is said to be α-non-degenerate. If for all $g \in G$, $a_g \in R_g - \{0\}$, we have $a_g R_{\alpha(g)} \neq \{0\}$ and $R_{\alpha(g)} a_g \neq \{0\}$. Otherwise, (R, G) is said to be α-degenerate. Clearly, if G is abelian, then the concepts of non-degenerate and α-non-degenerate graded rings coincide regarding the group homomorphism $\alpha(g) = g^{-1}$. However, Example 1 introduces a degenerate graded ring that is α-non-degenerate for some α. In Section 3, we establish the concepts of α-strongly and α-first strongly graded rings and study them in comparison to strongly and first strongly graded rings. If $R_g R_{\alpha(g)} = R_{g\alpha(g)}$, for all $g \in G$, then (R, G) is said to be α-strong. It is clear that every strongly graded ring is α-strong, for every α. However, if (R, G) is α-strong, then (R, G) is not necessarily strong, as we see in Example 3 where (R, G) is said to be α-first strong if $R_g R_{\alpha(g)} = R_{g\alpha(g)}$, for all $g \in \mathrm{supp}(R, G)$. Clearly, every α-strongly graded ring is α-first strong. However, the converse is not necessarily true, as we see in Example 4.

Additionally, if (R, G) is first strong and $\alpha(\mathrm{supp}(R, G)) \subseteq \mathrm{supp}(R, G)$, then (R, G) is α-first strong. However, if (R, G) is α-first strong, then (R, G) is not necessarily first strong, even if $\alpha(\mathrm{supp}(R, G)) \subseteq \mathrm{supp}(R, G)$, as mentioned in Example 3; (R, G) is α-first strong since it is α-strong, and $\alpha(\mathrm{supp}(R, G)) \subseteq \mathrm{supp}(R, G)$, but (R, G) is not first strong since $\mathrm{supp}(R, G)$ is not a subgroup of G. In Section 4, we present the concepts of α-weakly graded rings and α-weakly crossed products and study them in comparison to weakly graded rings and weakly crossed products. (R, G) is said to be α-weak if whenever $g \in G$ such that $R_g = \{0\}$, then $R_{\alpha(g)} = \{0\}$. It is clear that if G is abelian, then the concepts of weakly and α-weakly graded rings coincide regarding the group homomorphism $\alpha(g) = g^{-1}$. However, Example 5 is a non-weakly graded ring that is α-weak for some α. (R, G) is said to be α-weakly crossed product if $R_{\alpha(g)}$ contains a unit for all $g \in \mathrm{supp}(R, G)$. The concepts of weakly crossed product and α-weakly crossed product coincide with respect to the identity homomorphism. However, Example 8 is on non-weakly crossed product that is α-weakly crossed product for some α.

2. α-Non-Degenerate and α-Regular Graded Rings

In this section, we introduce the concepts of α-non-degenerate and α-regular graded rings and examine them compared to non-degenerate and regular graded rings.

Definition 1. Let R be a G-graded ring and $\alpha\colon G \longrightarrow G$ be a group homomorphism. Then,

(1) (R, G) is added to be α-non-degenerate if for all $g \in G$, $a_g \in R_g - \{0\}$, we have $a_g R_{\alpha(g)} \neq \{0\}$ and $R_{\alpha(g)} a_g \neq \{0\}$. Otherwise, R is said to be α-degenerate.

(2) (R, G) is expressed to be α-regular if for all $g \in G$, $a_g \in R_g$, we have $a_g \in a_g R_{\alpha(g)} a_g$.

Clearly, if G is abelian, then the concepts of non-degenerate and α-non-degenerate (regular and α-regular) graded rings coincide regarding the group homomorphism $\alpha(g) = g^{-1}$. However, the next example introduces a degenerate graded ring that is α-non-degenerate for some α.

Example 1. Let $R = K[X]$, where K is a field, and $G = \mathbb{Z}$. Then R is G-graded by $R_n = \mathrm{KX}^n$, $n \geq 0$ and $R_n = \{0\}$. Consider the group homomorphism $\alpha\colon G \longrightarrow G$ such that $\alpha(n) = 2n$. Let $n \in G$ and $a_n \in R_n - \{0\}$, $R_n \neq \{0\}$; then $R_n = \mathrm{KX}^n$, and $a_n = aX^n$, for some $a \in K - \{0\}$. Since $R_n \neq \{0\}$, $n \geq 0$, and $\alpha(n) = 2n$ is also a non-negative integer, $R_{\alpha(n)} = R_{2n} = \mathrm{KX}^{2\mathrm{n}}$, which implies that $a_n\ R_{\alpha(n)} = aX^n \mathrm{KX}^{2n} \neq \{0\}$ as $0 \neq aX^{3n} \in a_n\ R_{\alpha(n)}$. Similarly, $R_{\alpha(n)}\ a_n \neq \{0\}$. Hence, (R, G) is α-non-degenerate. On the other hand, (R, G) is degenerate since $X \in R_1 - \{0\}$ with $XR_{1^{-1}} = X\ R_{-1} = \{0\}$.

Remark 2. Clearly, every faithful graded ring is α-non-degenerate, for every α. However, if (R, G) is α-non-degenerate, then (R, G) is not necessarily faithful as in Example 1, where (R, G) is α-non-degenerate, but (R, G) is not faithful since $\mathrm{supp}(R, G) \neq G$.

Theorem 3. *Every α-regular graded ring is α-non-degenerate.*

Proof. Let (R, G) be α-regular. Suppose that $g \in G$ and $a_g \in R_g - \{0\}$. This gives us $0 \neq a_g \in a_g R_{\alpha(g)} a_g$ as (R, G) is α-regular, which implies that $a_g R_{\alpha(g)} \neq 0$ and $R_{\alpha(g)} a_g \neq 0$, and therefore (R, G) is α-non-degenerate. □

Remark 4. If (R,G) is α-non-degenerate, then (R,G) is not necessarily α-regular as in Example 1, (R,G) is α-non-degenerate, and (R,G) is not α-regular since $X \in R_1$ with $X \notin XR_{\alpha(1)}X = XR_2X = XKX^2X = KX^4$.

Theorem 5. *Let (R,G) be α-non-degenerate such that $\alpha^2 = I$. If $R_g = \{0\}$, for some $g \in G$, then $R_{\alpha(g)} = \{0\}$.*

Proof. Assume that $R_{\alpha(g)} \neq 0$. Then there is $a_{\alpha(g)} \in R_{\alpha(g)} - \{0\}$; therefore, since (R,G) is α-non-degenerate, $0 \neq a_{\alpha(g)} R_{\alpha(\alpha(g))} = a_{\alpha(g)}R_g$, which implies that $R_g \neq \{0\}$, which is a contradiction. Hence, $R_{\alpha(g)} = \{0\}$. □

Theorem 6. *Let (R,G) be α-non-degenerate such that $\alpha^2 = \alpha$. Suppose that $H = \{g \in G\colon g^2 \in \operatorname{Ker}(\alpha)\}$. If I will be a graded right (left) ideal of R with $I_e = \{0\}$, then $I_{\alpha(g)} = \{0\}$, for all $g \in H$.*

Proof. Let $g \in H$. Then $I_{\alpha(g)}R_{\alpha(g)} = (I \cap R_{\alpha(g)})\, R_{\alpha(g)} \subseteq I\, R_{\alpha(g)} \cap R_{\alpha(g)}R_{\alpha(g)} \subseteq I \cap R_{\alpha(g)\alpha(g)} = I \cap R_{\alpha(g^2)} = I \cap R_e = I_e = \{0\}$. So, $I_{\alpha(g)}\ R_{\alpha(g)} = \{0\}$. Let $a_{\alpha(g)} \in I_{\alpha(g)}$. Then $a_{\alpha(g)}\ R_{\alpha(g)} = \{0\}$. If $a_{\alpha(g)} \neq 0$, then $a_{\alpha(g)} \in R_{\alpha(g)} - \{0\}$, and since (R,G) is α-non-degenerate, $0 \neq a_{\alpha(g)}R_{\alpha(\alpha(g))} = a_{\alpha(g)}R_{\alpha(g)}$, which is a contradiction. So, $a_{\alpha(g)} = 0$, and therefore $I_{\alpha(g)} = \{0\}$. □

Corollary 7. *Let (R,G) be α-non-degenerate such that $\alpha^2 = \alpha$. Suppose that $g^2 = e$, for all $g \in G$. If I is a graded right (left) ideal of R with $I_e = \{0\}$, then $I_{\alpha(g)} = \{0\}$, for all $g \in G$.*

Proof. Let $g \in G$. Then $g^2 = e$, and so $g^2 \in \operatorname{Ker}(\alpha)$, which implies that $g \in H$, where $H = \{g \in G\colon g^2 \in \operatorname{Ker}(\alpha)\}$, so $H = G$, and hence the result holds by Theorem 6. □

Corollary 8. *Let (R,G) be α-non-degenerate, where α is the identity homomorphism. Suppose that $g^2 = e$, for all $g \in G$. If I will be a graded right (left) ideal of R with $I_e = \{0\}$, then $I = \{0\}$.*

Proof. By Corollary 7, $I_g = \{0\}$, for all $g \in G$, which implies that $I = \{0\}$.

The next example shows that the condition "(R,G) is α-non-degenerate regarding the identity homomorphism" in Corollary 8 is necessary. □

Example 2. Consider $R = \left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\colon a,b,c \in K\right\}$, where K is a field, and $G = \mathbb{Z}_2$ that satisfies $g^2 = e$, for all $g \in G$.

Then R is G-graded by $R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$ and $R_1 = \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}$.

Now, $I = R\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}$ is a graded left ideal of R that satisfies

$$I_e = I_0 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}, \text{but } I \neq \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}. \tag{1}$$

Note that (R,G) is α-degenerate with respect to the identity homomorphism since $A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in R_1 - \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ with $A_1R_{\alpha(1)} = A_1R_1 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$.

Theorem 9. *Let (R,G) be α-non-degenerate, where α is the identity homomorphism. Suppose that $g^2 = e$, for all $g \in G$. If R_e is a field, then R is graded simple.*

Proof. Let I be a graded ideal of R. If $I_e = \{0\}$, then by Corollary 8, $I = \{0\}$. Suppose that $I_e \neq \{0\}$. Then I_e is a non-zero ideal of R_e, and then $I_e = R_e$. Assume that $g \in G$. Then $R_g = R_gR_e = R_gI_e = R_g(I \cap R_e) \subseteq R_gI \cap R_gR_e \subseteq I \cap R_g \subseteq I$. So, $R_g \subseteq I$, for all $g \in G$, which implies that $R \subseteq I$, and hence $I = R$. Thus, R is graded simple. □

Corollary 10. *Let (R,G) be α-non-degenerate, where α is the identity homomorphism. Suppose that $g^2 = e$, for all $g \in G$. If R_e is a field, then R is graded Artinian and graded Noetherian.*

Theorem 11. *Consider $R = K[X]$, where K is a field. If R is G-graded such that $R_e \subseteq K$, then either (R,G) is not α-non-degenerate regarding the identity homomorphism or there is $g \in G$ such that $g^2 \neq e$.*

Proof. Since $R_e \subseteq K$, R_e is a commutative ring with unity. Let $0 \neq r \in R_e$. Then $0 \neq r \in K$, and then r is unit in K, and hence $r^{-1} \in R_{e^{-1}} = R_e$ by ([4], Lemma 3.4). Hence, R_e is a subfield of K. Let f be a non-constant homogeneous element of R. Then $I_a = \langle r f^{n+a}\colon r \in R, n \in \mathbb{N}\rangle$ $(a \in \mathbb{N})$ is a graded ideal of R with $I_1 \supset I_2 \supset \ldots$, but $f^n \notin I_n$ for all $n \in \mathbb{N}$; let $a \in \mathbb{N}$ and $g \in I_{a+1}$. Then $g = r f^{n+(a+1)}$ for some $r \in R$ and $n \in \mathbb{N}$ and then $g = r f^{(n+1)+a} \in I_a$. Suppose that $f^n \in I_n$ for some $n \in \mathbb{N}$. Then $f^n = r f^{t+n}$ for some $t \in \mathbb{N}$ and then, since R is an integral domain, $1 = r f^t$, a contradiction since $t \in \mathbb{N}$ and f is non-constant. Therefore, there is no $s \in \mathbb{N}$ with $I_n = I_s$ for all $n \geq s$, i.e., R is not Artinian. Hence, by Corollary 10, either (R,G) is not α-non-degenerate regarding the identity homomorphism or there exists $g \in G$ such that $g^2 \neq e$. □

Remark 12. In Example 1, $R = K[X]$ is graded where $R_e = K$ is a field, and one can prove that (R,G) is α-non-degenerate regarding the identity homomorphism. Absolutely, $g^2 \neq e$, for all $g \in G$.

Theorem 13. *Let R be a graded ring such that R has no zero divisors. Suppose that $\alpha\colon G \longrightarrow G$ is a group homomorphism and $H = \{g \in G\colon g^2 \in \operatorname{Ker}(\alpha)\}$. If I will be a graded right (or left) ideal of R with $I_e = \{0\}$, then $I_{\alpha(g)} = \{0\}$, for all $g \in H$.*

Proof. Let $g \in H$. Then $I_{\alpha(g)}R_{\alpha(g)} = (I \cap R_{\alpha(g)})R_{\alpha(g)} \subseteq IR_{\alpha(g)} \cap R_{\alpha(g)}R_{\alpha(g)} \subseteq I \cap R_{\alpha(g)\alpha(g)} = I \cap R_{\alpha(g^2)} = I \cap R_e = I_e = \{0\}$. So, $I_{\alpha(g)}R_{\alpha(g)} = \{0\}$, and then either $I_{\alpha(g)} = \{0\}$ or $R_{\alpha(g)} = \{0\}$. If $R_{\alpha(g)} = \{0\}$, then $I_{\alpha(g)} = I \cap R_{\alpha(g)} = \{0\}$. Hence, $I_{\alpha(g)} = \{0\}$, for all $g \in H$. □

Proposition 14. *If* (R, G) *is* α*-non-degenerate, then* $\alpha(supp(R, G)) \subseteq supp(R, G)$.

Proof. Let $h \in \alpha(\text{supp}(R, G))$. So there exists $g \in \text{supp}(R, G)$ such that $\alpha(g) = h$, and then there exists $a_g \in R_g - \{0\}$, and so $a_g R_{\alpha(g)} \neq 0$, which implies that $R_{\alpha(g)} \neq 0$, and hence $h = \alpha(g) \in \text{supp}(R, G)$. □

Remark 15. Example 2 shows that the converse of Proposition 14 is not necessarily true as $\alpha(\text{supp}(R, G)) = \text{supp}(R, G)$, but (R, G) is α-degenerate.

Theorem 16. *Let R be a graded ring such that R has no zero divisors and* $\alpha: G \longrightarrow G$ *be a group homomorphism. Thus,* $\alpha(supp(R, G)) \subseteq supp(R, G)$ *if and only if* (R, G) *is* α*-non-degenerate.*

Proof. Suppose that $\alpha(\text{supp}(R, G)) \subseteq \text{supp}(R, G)$. Let $g \in G$ and $a_g \in R_g - \{0\}$. Then $g \in \text{supp}(R, G)$, and so $\alpha(g) \in \text{supp}(R, G)$, which implies that $a_g R_{\alpha(g)} \neq \{0\}$ and $R_{\alpha(g)} a_g \neq \{0\}$. Hence, (R, G) is α-non-degenerate. The converse holds by Proposition 14.

Let R and S be two G-graded rings. Then a ring homomorphism $f: R \longrightarrow S$ is said to be a graded homomorphism if $f(R_g) \subseteq S_g$, for all $g \in G$ [2]. Note that if $f: R \longrightarrow S$ is a ring homomorphism and $g \notin \text{supp}(R, G)$, then $f(R_g) = f(\{0\}) = \{0\} \subseteq S_g$. So, f being graded homomorphism is equivalent to $f(R_g) \subseteq S_g$ for all $g \in \text{supp}(R, G)$. We have the following. □

Lemma 17. *If* $f: R \longrightarrow S$ *is a graded epimorphism, then* $f(R_g) = S_g$ *for all* $g \in G$.

Proof. Let $g \in G$. Then $f(R_g) \subseteq S_g$ as f is a graded homomorphism. Let $a_g \in S_g$. If $a_g = 0$, then $a_g = f(0) \in f(R_g)$. Suppose that $a_g \neq 0$. Since f is onto, there exists $m \in R - \{0\}$ such that $f(m) = a_g$. Suppose that $m = \sum_{i=1}^{n} m_{g_i}$, where $m_{g_i} \in R_{g_i}$, $g_i \neq g_j$ for $i \neq j$. Then $a_g = f(m) = \sum_{i=1}^{n} f(m_{g_i}) = \sum_{i=1}^{k} f(m_{g_{t_i}})$, where $1 \leq t_i \leq n$ and $f(m_{g_{t_i}}) \neq 0$ for all $1 \leq i \leq k$. Since $f(m_{g_{t_i}}) \in S_{g_{t_i}}$, $a_g \in S_g \cap \sum_{i=1}^{k} S_{g_{t_i}}$. Thus, $g = g_{t_1} = \cdots = g_{t_n}$ and hence $k = 1$ and $f(m_{g_{t_i}}) = f(m_g) = a_g$. Thus, $S_g \subseteq f(A_g)$, and hence $f(R_g) = S_g$. □

Theorem 18. *Let* $f: R \longrightarrow S$ *be a graded isomorphism. Then* (R, G) *is* α*-non-degenerate if and only if* (S, G) *is* α*-non-degenerate.*

Proof. Suppose that (R, G) is α-non-degenerate. Let $g \in G$ and $s_g \in S_g - \{0\}$. By Lemma 17, $f(R_g) = S_g$. So, there exists $a_g \in R_g - \{0\}$ such that $f(a_g) = s_g$. Since R is α-non-degenerate, $a_g R_{\alpha(g)} \neq 0$, and then $0 \neq f(a_g R_{\alpha(g)}) = s_g S_{\alpha(g)}$. Similarly, $S_{\alpha(g)} s_g \neq 0$, and hence (S, G) is α-non-degenerate. Conversely, let $g \in G$ and $a_g \in R_g - \{0\}$. Assume that $f(a_g) = s_g$. Then $s_g \in S_g - \{0\}$, and then $s_g S_{\alpha(g)} \neq 0$, which implies that $0 \neq f^{-1}(s_g S_{\alpha(g)}) = a_g R_{\alpha(g)}$. Similarly, $R_{\alpha(g)} a_g \neq 0$, and hence (R, G) is α-non-degenerate. □

3. α-Strongly and α-First Strongly Graded Rings

In this section, we establish the concepts of α-strongly and α-first strongly graded rings and study them in comparison to strongly and first strongly graded rings.

Definition 19. Let R be a G-graded ring and $\alpha: G \longrightarrow G$ be a group homomorphism. Therefore, (R, G) is said to be α-strong if $R_g R_{\alpha(g)} = R_{g\alpha(g)}$, for all $g \in G$.

It is clear that every strongly graded ring is α-strong, for every α. However, if (R, G) is α-strong, then (R, G) is not necessarily strong, as we see in the following example.

Example 3. Let $R = K[X]$, where K is a field, and $G = \mathbb{Z}$. Then R is G-graded by $R_n = KX^n$, $n \geq 0$ and $R_n = \{0\}$ otherwise. Consider the group homomorphism $\alpha: G \longrightarrow G$ such that $\alpha(n) = 2n$. Let n be a non-negative integer. Then $R_n R_{\alpha(n)} = R_n R_{2n} = KX^n KX^{2n} = KX^{3n} = R_{3n} = R_{n+2n} = R_{n+\alpha(n)}$. Let n be a negative integer. Then $R_n R_{\alpha(n)} = \{0\} = R_{3n} = R_{n+\alpha(n)}$. So, $R_n R_{\alpha(n)} = R_{n+\alpha(n)}$, for all $n \in \mathbb{Z}$, and hence (R, G) is α-strong. On the other hand, (R, G) is not considered strong since $R_{-1} R_1 = \{0\} \neq R_0$.

Proposition 20. *Let R be a graded ring. Then* (R, G) *is strongly graded if and only if* $R_g R_{g^{-1}} = R_e$*, for all* $g \in G$.

Proof. Suppose that $R_g R_{g^{-1}} = R_e$, for all $g \in G$. Let $g, h \in G$. Then $R_{gh} = R_{gh} R_e = R_{gh} R_{h^{-1}} R_h \subseteq R_{ghh^{-1}} R_h \subseteq R_g R_h \subseteq R_{gh}$, and $R_g R_h = R_{gh}$. Hence, (R, G) is strong. The converse is clear.

The next result is a consequence of Proposition 20. □

Corollary 21. *If G is abelian, then strongly graded and* α*-strongly graded rings coincide regarding the group homomorphism* $\alpha(g) = g^{-1}$.

Theorem 22. *Let R be a G-graded ring and* $\alpha: G \longrightarrow G$ *be a group homomorphism. If* $1 \in R_g R_{\alpha(g)}$*, for all* $g \in G$*, then* (R, G) *is strong, and hence* (R, G) *is* α*-strong. Moreover, G is abelian.*

Proof. Let $g \in G$. Then $0 \neq 1 \in R_g R_{\alpha(g)} \cap R_e \subseteq R_{g\alpha(g)} \cap R_e$, and then $g\alpha(g) = e$, which implies that $\alpha(g) = g^{-1}$. So, $1 \in R_g R_{g^{-1}}$. Now, let $x \in R_e$. Then $x = 1.x \in R_g R_{g^{-1}} R_e \subseteq R_g R_{g^{-1}}$. Thus, $R_e \subseteq R_g R_{g^{-1}} \subseteq R_e$, which implies that $R_g R_{g^{-1}} = R_e$, and thus (R, G) is strong by Proposition 20, and hence (R, G) is α-strong. Moreover, since $\alpha(g) = g^{-1}$, for all $g \in G$ and α is homomorphism, then G is abelian. □

Definition 23. Let R be a G-graded ring and $\alpha: G \longrightarrow G$ be a group homomorphism. Then (R, G) is said to be α-first strong if $R_g R_{\alpha(g)} = R_{g\alpha(g)}$, for all $g \in \text{supp}(R, G)$.

Clearly, every α-strongly graded ring is α-first strong. However, the converse is not necessarily true, as we see in the following example.

Example 4. Let $R = M_2(K)$, where K is a field, and $G = \mathbb{Z}_4$ and consider the identity homomorphism α on G. Then R is G-graded by

$$\begin{aligned} R_0 &= \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \\ R_2 &= \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}, \\ R_1 &= R_3 = \{0\}, \text{and, so } \operatorname{supp}(R,G) = \{0,2\}. \end{aligned} \tag{2}$$

By easy calculations, we find that $R_0R_{\alpha(0)} = R_0 = R_{0+\alpha(0)}$ and $R_2R_{\alpha(2)} = R_0 = R_{2+\alpha(2)}$. So, (R,G) is α-first strong. On the other hand, (R,G) is not α-strong since $R_1R_{\alpha(1)} = \{0\} \neq R_2 = R_{1+\alpha(1)}$.

Remark 24. Clearly, if (R,G) is a first strong and $\alpha\colon G \longrightarrow G$ is a group homomorphism with $\alpha(\operatorname{supp}(R,G)) \subseteq \operatorname{supp}(R,G)$, then (R,G) is α-first strong. However, if (R,G) is α-first strong, then (R,G) is not necessarily first strong, even if $\alpha(\operatorname{supp}(R,G)) \subseteq \operatorname{supp}(R,G)$, as in Example 2. (R,G) is α-first strong since it is α-strong, and $\alpha(\operatorname{supp}(R,G)) \subseteq \operatorname{supp}(R,G)$, but (R,G) is not first strong since $\operatorname{supp}(R,G)$ is not a subgroup of G.

Proposition 25. *Let (R,G) be first strong. If (R,G) is α-non-degenerate, then (R,G) is α-first strong.*

Proof. Apply Proposition 14 and Remark 24. ☐

Proposition 26. *Let R be a graded ring. Then (R,G) is first strong if and only if $R_gR_{g^{-1}} = R_e$, for all $g \in \operatorname{supp}(R,G)$.*

Proof. Suppose that $R_gR_{g^{-1}} = R_e$, for all $g \in \operatorname{supp}(R,G)$. Then as $1 \in R_e$, $1 \in R_gR_{g^{-1}}$, for all $g \in \operatorname{supp}(R,G)$, and hence (R,G) is first strong. The converse is clear.

The next result is a consequence of Proposition 26. ☐

Corollary 27. *If G is abelian, then the first strongly graded and α-first strongly graded rings coincide regarding the group homomorphism $\alpha(g) = g^{-1}$.*

Theorem 28. *Let R be a G-graded ring and $\alpha\colon G \longrightarrow G$ be a group homomorphism. If $1 \in R_gR_{\alpha(g)}$, for all $g \in \operatorname{supp}(R,G)$, then (R,G) is first strong, and hence (R,G) is α-first strong. Furthermore, $\operatorname{supp}(R,G)$ is an abelian subgroup of G.*

Proof. Let $g \in \operatorname{supp}(R,G)$. Then $0 \neq 1 \in R_gR_{\alpha(g)} \cap R_e \subseteq R_{g\alpha(g)} \cap R_e$, and then $g\alpha(g) = e$, which suggests that $\alpha(g) = g^{-1}$. So, $1 \in R_gR_{g^{-1}}$, and thus (R,G) is first strong. So, $\operatorname{supp}(R,G)$ is a subgroup of G, and then $\alpha(\operatorname{supp}(R,G)) = (\operatorname{supp}(R,G))^{-1} \subseteq \operatorname{supp}(R,G)$, which implies that (R,G) is α-first strong by Remark 24. Since $\alpha(g) = g^{-1}$, for all $g \in \operatorname{supp}(R,G)$ and α is homomorphism, then $\operatorname{supp}(R,G)$ will be an abelian subgroup of G. ☐

Theorem 29. *Let R be a graded ring such that R is an integral domain. Then $gh = hg$, for all $g,h \in \operatorname{supp}(R,G)$.*

Proof. Let $g,h \in \operatorname{supp}(R,G)$. Then $\{0\} \neq R_gR_h \subseteq R_{gh}$, and then, since R is commutative, $\{0\} \neq R_gR_h = R_hR_g \subseteq R_{hg}$. So, $\{0\} \neq R_gR_h \subseteq R_{gh} \cap R_{hg}$, which implies that gh = hg. ☐

Theorem 30. *Let R be a graded ring over a finite group G. If (R,G) is second strong, then (R,G) is α-first strong, for every α.*

Proof. Let $\alpha\colon G \longrightarrow G$ be a group homomorphism. Assume that $g \in \operatorname{supp}(R,G)$. Since G is finite, $g^{-1} = g^n$, for some positive integer n, and then $g^{-1} = g^n \in \operatorname{supp}(R,G)$ since $\operatorname{supp}(R,G)$ is a monoid, and hence $\operatorname{supp}(R,G)$ is a subgroup of G, and then since (R,G) is second strong, (R,G) is first strong. Since G is finite, $\alpha(g) = g^m$, for some positive integer m, and then $\alpha(g) \in \operatorname{supp}(R,G)$.

Hence, $\alpha(\operatorname{supp}(R,G)) \subseteq \operatorname{supp}(R,G)$, which implies that (R,G) is α-first strong by Remark 24. ☐

4. α-Weakly Graded Rings and α-Weakly Crossed Products

In this section, we present the concepts of α-weakly graded rings and α-weakly crossed products and study them in comparison to weakly graded rings and weakly crossed products.

Definition 31. Let R be a G-graded ring and $\alpha\colon G \longrightarrow G$ be a group homomorphism. Then (R,G) is said to be α-weak if whenever $g \in G$ such that $R_g = \{0\}$, then $R_{\alpha(g)} = \{0\}$.

Clearly, if G is abelian, then the concepts of weakly and α-weakly graded rings coincide regarding the group homomorphism $\alpha(g) = g^{-1}$. However, the following example is on non-weakly graded ring that is α-weakly for some α.

Example 5. Let $R = K[X]$ and $G = \mathbb{Z}$. Then R is G-graded by $R_n = \mathrm{K}\mathrm{X}^n$, $n \geq 0$ and $R_n = \{0\}$. Otherwise, consider the group homomorphism $\alpha\colon G \longrightarrow G$ such that $\alpha(n) = 2n$. Let $n \in G$ be such that $R_n = \{0\}$. Then n is a negative number, and then $\alpha(n) = 2n$ is also a negative number, which implies that $R_{\alpha(n)} = \{0\}$. Hence, (R,G) is α-weak. On the other hand, (R,G) is not weak since $R_{-1} = \{0\}$ but $R_{-1^{-1}} = R_1 \neq \{0\}$.

Theorem 32. *Let (R,G) be α-non-degenerate such that $\alpha^2 = I$. So (R,G) is α-weak.*

Proof. Apply Theorem 5.

The later result is a consequence of Theorem 32. ☐

Corollary 33. *Let R be a graded ring and $\alpha\colon G \longrightarrow G$ be a group homomorphism such that $\alpha^2 = I$. If (R,G) is faithful, then (R,G) is α-weak.*

The following result is a consequence of Theorems 3 and 32.

Corollary 34. *Let R be a graded ring and $\alpha\colon G \longrightarrow G$ be a group homomorphism such that $\alpha^2 = I$. If (R,G) is α-regular, (R,G) is α-weak.*

Theorem 35. *Let R be a graded such that R has no zero divisors and α: $G \longrightarrow G$ be a group homomorphism such that $\alpha^2 = I$. Then (R, G) is α-non-degenerate if and only if (R, G) is α-weak.*

Proof. Suppose that (R, G) is α-weak. Let $g \in G$ and $a_g \in R_g - \{0\}$. Then $R_g \neq \{0\}$. If $R_{\alpha(g)} = \{0\}$, then as R is α-weak, $\{0\} = R_{\alpha(\alpha(g))} = R_g$, which is a contradiction. So, $R_{\alpha(g)} \neq \{0\}$, and thus $a_g R_{\alpha(g)} \neq \{0\}$ and $R_{\alpha(g)} a_g \neq \{0\}$, and hence (R, G) is α-non-degenerate. The converse holds from Theorem 32. □

Theorem 36. *Let R be a graded ring and α: $G \longrightarrow G$ be a group homomorphism such that $\alpha^2 = I$. Then $\alpha(supp(R, G)) \subseteq supp(R, G)$ if and only if (R, G) is α-weak.*

Proof. Suppose that $\alpha(\mathrm{supp}(R, G)) \subseteq \mathrm{supp}(R, G)$. Let $g \in G$ such that $R_g = \{0\}$. If $R_{\alpha(g)} \neq \{0\}$, $\alpha(g) \in \mathrm{supp}(R, G)$, and then $g = \alpha(\alpha(g)) \in \alpha(\mathrm{supp}(R, G)) \subseteq \mathrm{supp}(R, G)$, which is a contradiction. So, $R_{\alpha(g)} = \{0\}$, and therefore (R, G) is α-weak. Conversely, let $h \in \alpha(\mathrm{supp}(R, G))$. Then there is $g \in \mathrm{supp}(R, G)$ such that $h = \alpha(g)$. If $R_{\alpha(g)} = \{0\}$, then $R_g = R_{\alpha(\alpha(g))} = \{0\}$, which is a contradiction. So, $R_{\alpha(g)} \neq \{0\}$, and then $h \in \mathrm{supp}(R, G)$, and hence $\alpha(\mathrm{supp}(R, G)) \subseteq \mathrm{supp}(R, G)$. □

Proposition 37. *Every strongly graded ring is α-weak, for every α.*

Proof. Let (R, G) be strong and α: $G \longrightarrow G$ be a group homomorphism. Then $\mathrm{supp}(R, G) = G$, and hence (R, G) is α-weak.

Similarly, one can prove the following. □

Proposition 38. *Every faithful graded ring is α-weak, for every α.*

The next example shows that an α-weakly graded ring is not necessarily strongly graded or faithful graded or even first strongly graded.

Example 6. Let $R = M_3(K)$ and $G = \mathbb{Z}_7$. Then R is G-graded by

$$R_0 = \begin{pmatrix} K & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}, R_1 = \begin{pmatrix} 0 & K & 0 \\ 0 & 0 & K \\ 0 & 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 0 & K \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$R_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K & 0 & 0 \end{pmatrix}, R_6 = \begin{pmatrix} 0 & 0 & 0 \\ K & 0 & 0 \\ 0 & K & 0 \end{pmatrix},$$
$$R_3 = R_4 = \{0\}. \tag{3}$$

Clearly, (R, G) is α-weak regarding the identity homomorphism, but (R, G) is not strong and not faithful since $\mathrm{supp}(R, G) \neq G$. Also, (R, G) is not first strong since $\mathrm{supp}(R, G)$ is not a subgroup of G.

Also, the next example is on a non-strongly graded ring that is α-weakly graded, for every α.

Example 7. Let $R = K[X]$ and $G = \mathbb{Z}_3$. Then R is G-graded by $R_0 = \langle 1, X^3, X^6, \ldots. \rangle$, $R_1 = \langle X, X^4, X^7, \ldots. \rangle$ and $R_2 = \langle X^2, X^5, X^8, \ldots. \rangle$. Since $\mathrm{supp}(R, G) = G$, (R, G) is α-weak, for every α. On the other hand, (R, G) is not strong since $R_1 R_2 \neq R_0$ as $1 \in R_0$ such that $1 \notin R_1 R_2$.

Definition 39. Let R be a G-graded ring and α: $G \longrightarrow G$ be a group homomorphism. Then (R, G) is said to be α-weakly crossed product if $R_{\alpha(g)}$ contains a unit for all $g \in \mathrm{supp}(R, G)$.

Clearly, the concepts of weakly crossed product and α-weakly crossed product coincide regarding the identity homomorphism. However, the following example is a non-weakly crossed product that is α-weakly crossed product for some α.

Example 8. Let $R = K[X]$ and $G = \mathbb{Z}$. Then R is G-graded by $R_n = \mathrm{K}X^n$, $n \geq 0$ and $R_n = \{0\}$ otherwise. Consider the group homomorphism α: $G \longrightarrow G$ such that $\alpha(n) = 0$, for all $n \in G$. Let $n \in \mathrm{supp}(R, G)$. Then $R_{\alpha(n)} = R_0$ contains a unit. Hence, (R, G) is α-weakly crossed product. On the other hand, (R, G) is not weakly crossed product since $1 \in \mathrm{supp}(R, G)$ with R_1 does not contain any unit.

Proposition 40. *Let R be a G-graded ring and α: $G \longrightarrow G$ be the group homomorphism $\alpha(g) = e$, for all $g \in G$. Then (R, G) is α-weakly crossed product.*

Proof. Let $g \in \mathrm{supp}(R, G)$. Then $R_{\alpha(g)} = R_e$ contains a unit. Hence, (R, G) is α-weakly crossed product. □

Proposition 41. *Let (R, G) be α-weakly crossed product. Then $\alpha(supp(R, G)) \subseteq supp(R, G)$.*

Proof. Let $h \in \alpha(\mathrm{supp}(R, G))$. Then there is $g \in \mathrm{supp}(R, G)$ such that $\alpha(g) = h$. Since (R, G) is α-weakly crossed product, $R_{\alpha(g)}$ contains a unit, and then $R_{\alpha(g)} \neq 0$, which implies that $h = \alpha(g) \in \mathrm{supp}(R, G)$. Thus, $\alpha(\mathrm{supp}(R, G)) \subseteq \mathrm{supp}(R, G)$. □

Remark 42. The converse of Proposition 41 is not necessarily true; as in Example 6, (R, G) is not α-weakly crossed product regarding the identity homomorphism since $2 \in \mathrm{supp}(R, G)$ with $R_{\alpha(2)} = R_2$ does not contain any unit, but $\alpha(\mathrm{supp}(R, G)) = \mathrm{supp}(R, G)$.

Theorem 43. *Let R be a graded ring such that R has no zero divisors. If (R, G) is α-weakly crossed product, then (R, G) is α-non-degenerate.*

Proof. Let $g \in G$ and $a_g \in R_g - \{0\}$. Then $g \in \mathrm{supp}(R, G)$, and then $R_{\alpha(g)}$ contains a unit, which implies that $R_{\alpha(g)} \neq 0$. So, $a_g R_{\alpha(g)} \neq 0$ and $R_{\alpha(g)} a_g \neq 0$. Hence, (R, G) is α-non-degenerate.

The next example introduces a α-non-degenerate graded ring that is not a α-weakly crossed product. □

Example 9. Let $R = M_2(K)$, where K is a field, and $G = \mathbb{Z}$. Then R is G-graded by

$$R_0 = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, R_1 = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}, R_{-1} = \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}, \quad R_n = \{0\}. \tag{4}$$

Otherwise, one can prove that (R, G) is α-non-degenerate regarding the group homomorphism $\alpha(n) = -n$, but (R, G) is not α-weakly crossed product since $-1 \in \mathrm{supp}(R, G)$ with $R_{\alpha(-1)} = R_1$ does not contain any unit.

Theorem 44. *If (R, G) is α-weakly crossed product and $\alpha^2 = I$, then (R, G) is α-weak.*

Proof. Let $g \in G$ such that $R_g = \{0\}$. If $R_{\alpha(g)} \neq \{0\}$, then $\alpha(g) \in \mathrm{supp}(R, G)$, and then $R_g = R_{\alpha(\alpha(g))}$ contains a unit, which is a contradiction. So, $R_{\alpha(g)} = \{0\}$, and hence (R, G) is α-weak.

The next example is on α-weakly graded ring that is not α-weakly crossed product. □

Example 10. Let $R = M_3(K)$, where K is a field, and $G = \mathbb{Z}_2$. Then R is G-graded by

$$R_0 = \begin{pmatrix} K & 0 & K \\ 0 & K & 0 \\ K & 0 & K \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & K & 0 \\ K & 0 & K \\ 0 & K & 0 \end{pmatrix}. \tag{5}$$

Clearly, (R, G) is α-weak, for every α, but (R, G) is not α-weakly crossed product regarding the identity homomorphism since $1 \in \mathrm{supp}(R, G)$ with $R_{\alpha(1)} = R_1$ does not contain any unit.

Proposition 45. *If (R, G) is α-weakly crossed product, then for all $g \in supp(R, G)$, there exists a unit $r \in R_{\alpha(g)}$ such that $R_{\alpha(g)} = R_e r$ (i.e., $R_{\alpha(g)}$ is a cyclic R_e-module).*

Proof. Let $\in \mathrm{supp}(R, G)$. Then $R_{\alpha(g)}$ contains a unit, say r. Let $x \in R_{\alpha(g)}$. Then $x = x.1 = x.r^{-1}.r \in R_{\alpha(g)} R_{(\alpha(g))^{-1}} r = R_{\alpha(g)} R_{\alpha(g^{-1})} r \subseteq R_{\alpha(g)\alpha(g^{-1})} r = R_{\alpha(gg^{-1})} r = R_{\alpha(e)} r = R_e r$. So, $R_{\alpha(g)} \subseteq R_e r \subseteq R_e R_{\alpha(g)} \subseteq R_{\alpha(g)}$. Hence, $R_{\alpha(g)} = R_e r$.

Similarly, one can prove the following. □

Proposition 46. *If (R, G) is α-weakly crossed product, then for all $\in supp(R, G)$, there exists a unit $r \in R_{\alpha(g)}$ such that $R_{\alpha(g)} = r R_e$.*

Theorem 47. *If (R, G) is α-weakly crossed product, then $R_{\alpha(g)}$ is isomorphic to R_e, as an R_e-module, for all $g \in supp(R, G)$.*

Proof. Let $g \in \mathrm{supp}(R, G)$. Then by Proposition 46, $R_{\alpha(g)} = r R_e$, for some unit $r \in R_{\alpha(g)}$, and then $f: R_e \longrightarrow R_{\alpha(g)}$ such that $f(x) = \mathrm{r}x$ is an R_e-isomorphism. □

5. Conclusion

Let G be a group, $\alpha: G \longrightarrow G$ be a group homomorphism, and R be a G-graded ring. In this article, we established and investigated the perceptions of α-weakly, α-non-degenerate, α-regular, α-strongly, α-first strongly graded rings, and α-weakly crossed product. We have indeed examined these notions comparable to weakly, non-degenerate, regular, strongly, first strongly graded rings, and weakly crossed product.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] R. Hazrat, *Graded Rings and Graded Grothendieck Groups*, Cambridge University press, Cambridge, UK, 2016.

[2] C. Nastasescu and F. Oystaeyen, *Methods of Graded Rings*, Springer-Verlag, Berlin, Germany, 2004.

[3] M. Refai, “Various types of strongly graded rings,” *Abhath Al-Yarmouk Journal (Pure Sciences and Engineering Series)*, vol. 4, no. 2, pp. 9–19, 1995.

[4] M. Refa`i and R. Abu-Dawwas, “On properties of graded rings and graded modules,” *Proyecciones (Antofagasta)*, vol. 41, no. 6, pp. 1413–1437, 2022.

[5] R. Abu-Dawwas, “More on crossed product over the support of graded rings,” *International Mathematical Forum*, vol. 5, no. 36, pp. 3121–3126, 2010.

[6] Y. Fan and B. Kulshammer, “Group-graded rings and finite block theory,” *Pacific Journal of Mathematics*, vol. 196, no. 1, pp. 177–186, 2000.

[7] S. Goto and K. Watanabe, “On graded rings I,” *Journal of the Mathematical Society of Japan*, vol. 30, no. 2, pp. 179–213, 1978.

[8] C. Nastasescu and N. Rodino, “Group graded rings and smash products,” *Rendiconti del Seminario Matematico della Università di Padova*, vol. 74, pp. 129–137, 1985.

[9] Pratibha, R. K. Mishra, and R. Mohan, “A report on graded rings and graded modules,” *Global Journal of Pure and Applied Mathematics*, vol. 13, no. 9, pp. 6827–6853, 2017.