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Research Article

The Fuzzy Prime Spectrum of Partially Ordered Sets

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We study the space of prime fuzzy ideals (and the space of maximal fuzzy ideals as a subspace) equipped with the hull-kernel topology in partially ordered sets. Mainly, we investigate the conditions for which the fuzzy prime spectrum of a poset is compact, Hausdorff, and normal, respectively.

1. Introduction

Partially ordered sets (posets) play a fundamental role in various branches of mathematics, providing a rich framework for studying order relations and their associated structures. In 1995, Halaś [1] introduced the general theory of ideals in a partially ordered set as an abstraction of ideals in lattices. Other scholars have also been proposed different definitions for ideals in a poset. For instance, closed or normal ideals of a poset were introduced by Birkhoff in [2]. Later in 1954, Frink [3] proposed the definition of ideals in a poset using lower and upper cones. Venkatanarasimhan [4] has also developed the theory of semiideals in a poset which are known by the name down sets. In 1979, Enré [5] has come up with a new definition of ideals in a poset called m-ideals, generalizing almost all definitions proposed by other scholars. In 1995, Halaś and Rachunek [6] have studied primness in the class of ideals of a poset. Further investigations have been conducted by Erné [7] providing several characterizing theorems for prime and maximal ideals. The theory of prime ideals has been also extended to the class of n-normal posets by Halaš et al. [8] and to the class of 0-distributive lattices by Joshi and Mundlik [9].

In 2016, Mundlik et al. [10] studied the hull-kernel topology on prime ideals in posets. The hull-kernel topology is a well-known topology that provides a powerful tool for analyzing the convergence, compactness, and connectedness properties in the class of ideals.

In recent years, the study of fuzzy sets and their applications in algebraic structures, particularly in lattices and ordered algebras, has gained significant attention, leading to the exploration of fuzzy ideals and their properties. Fuzzy sets allow for the representation of partial membership, capturing the idea that elements can belong to a set to varying degrees. This concept has proven to be a powerful tool in modeling uncertainty and vagueness, making it particularly well-suited for analyzing posets where precise membership may not always be applicable. In this context, the notion of fuzzy ideals and filters in posets has been studied by Assaye Alaba et al. [11, 12] as a natural extension of fuzzy ideals in lattices. Fuzzy ideals provide a flexible framework for studying the behavior of fuzzy subsets that possess certain desirable properties within a poset. They offer a more comprehensive understanding of the structure and relationships between elements, allowing for a deeper exploration of the underlying order relations. They further investigate the primness and maximality conditions in the class of fuzzy ideals in a general poset in [13]. In addition, they obtained sufficient conditions for the existence of fuzzy prime ideals in a poset as in the lattice of its fuzzy ideals.

The space of fuzzy prime ideals, called the hull-kernel topology for fuzzy prime ideals, has been studied by many authors in different classes of general algebraic structures such as rings (see [14]), modules (see [15]), hemirings (see [16]), semirings and Γ-Semirings (see [17]), distributive lattices (see [18]), C-algebras (see [19]), etc. This topology provides a powerful tool to explore and hence to utilize the important topological properties that the class of fuzzy prime ideals possesses inherently.

The motivation behind this study lies in the desire to identify the natural properties of the fuzzy prime spectrum of a poset and to understand the conditions under which the fuzzy prime spectrum of a poset exhibits the well-known topological properties such as compactness, Hausdorffness, and normality. Compactness, for example, is a fundamental property in topology that has implications in various areas of mathematics, including measure theory and mathematical analysis. By characterizing the conditions under which the fuzzy prime spectrum is compact, we can identify posets where the space of fuzzy prime ideals behaves in a particularly well-behaved manner. Hausdorffness, another crucial topological property, ensures the separation of points in a space, allowing for distinctness and uniqueness. Investigating the conditions for Hausdorffness in the hull-kernel topology for fuzzy prime ideals provides insights into the relationships between different elements within the poset and sheds light on the structure of fuzzy prime ideals. Furthermore, the study of normality in the fuzzy prime spectrum of a poset is motivated by its fundamental role in topology. Normal spaces possess certain desirable properties, enabling the existence of certain types of continuous functions and preserving important topological properties. By examining the conditions for which the fuzzy prime spectrum becomes a normal space, we contribute to the understanding of the broader topological structure of fuzzy ideals in partially ordered sets.

It is known that the unit interval $[0,1]$ has so many important properties: lattice theoretic property as well as topological properties. For instance, it forms a complete residuated lattice with three standard product and residuum operations (Lukasiewicz operations, Godel operations and Goguen (product) operations). By abstracting its lattice properties, Goguen [20] was first to define the concept of L-fuzzy sets by replacing the unit interval $[0,1]$ by a general complete lattice L in the definition of fuzzy subsets. Swamy and Swamy [21] mentioned that complete Brouwerian lattices are the most appropriate candidates to have the truth values of general fuzzy statements. Taking this into consideration, we prefer complete Brouwerian lattices to be the set of truth degrees for our fuzzy statements.

2. Preliminaries

A partially ordered set (or shortly a poset) is a system $(Z, \leq)$ consisting of a nonempty set Z together with a partial ordering "$\leq$" on Z, where by a partial ordering on Z, we mean a subset $\leq$ of $Z \times Z$ which is transitive $(\forall x, y, z \in Z, [(x, y) \in \leq \text{and } (y, z) \in \leq] \Longrightarrow (x, z) \in \leq)$, antisymmetric $(\forall x, y \in Z, [(x, y) \in \leq \text{and } (y, x) \in \leq] \Longrightarrow x = y)$, and reflexive $(\forall x \in Z, (x, x) \in \leq)$. If $\leq$ is a partial ordering on Z and $y, z \in Z$, then we write $y \leq z$ instead of $(y, z) \in \leq$. For $H \subseteq Z$, an element p in Z is said to be an upper bound (respectively, a lower bound) of H if $h \leq p$ (respectively, $p \leq h$) for all $h \in H$. We will denote by H^u (respectively, H^l) the set of all upper bounds (respectively, lower bounds) of H. We write H^{ul} (respectively, H^{lu}), to denote $\{H^u\}^l$ (respectively, $\{H^l\}^u$). For any $p, q \in Z$, we will use standard notations p^l and $(p, q)^l$ to denote the sets $\{p\}^l$ and $\{p, q\}^l$. In a dual manner, $\{p\}^u$ and $\{p, q\}^u$ will be denoted by p^u and $(p, q)^u$, respectively. For sets S and T in a poset Z, one can easily verify that $S \subseteq S^{ul}$ and $S \subseteq S^{lu}$. Moreover, $S \subseteq T \Longrightarrow [S^u \supseteq T^u \text{ and } S^l \supseteq T^l]$. It is also obvious that $p^{ul} = p^l$ and $p^{lu} = p^u$ for all $p \in Z$. For $S \subseteq Z$, its infimum (or greatest lower bound) is a lower bound p_0 of S such that $x \leq p_0$ for all lower bounds x of S. Similarly, the supremum (least upper bound) of S is an upper bound q_0 of S such that $q_0 \leq y$ for all upper bounds y of S. We write $p_0 = \inf S$ (respectively, $p_0 = \sup S$) to say that p_0 is the infimum (respectively, the supremum) of S. For $p, q \in Z$, if the infimum (respectively, the supremum) of the set $\{p, q\}$ exists, then it will be denoted by $p \wedge q$ (respectively, $p \vee q$). A poset Z is said to have the least (respectively, the greatest) element if there is an element in Z denoted by 0 (respectively, 1) such that $0 \leq x$ (respectively, $x \leq 1$) for all $x \in Z$. By a bounded poset, we mean a poset having both the least and the greatest elements.

An element p in a bounded poset Z is said to be complemented if there is $q \in Z$ such that $(p, q)^{lu} = (p, q)^{ul} = Z$. If every element of Z is complemented, then we say that Z is complemented. By a Boolean poset, we mean a distributive complemented poset. More details about Boolean posets can be found in [22].

An element p^* in a poset Z with 0 is said to be the pseudocomplement of $p \in Z$, if $(p, p^*)^l = \{0\}$, and for $x \in Z$, $(p, x)^l = \{0\}$ implies $x \leq p^*$. A poset Z with least element 0 is said to be pseudocomplemented if for all $p \in z$, there is $p^* \in Z$ such that $(p, p^*) = \{0\}$ and for $x \in Z$, $(p, x) = \{0\}$ implies $x \leq p^*$. In this case, p^* is unique and is called the pseudocomplement of p in Z [23]. Given a poset Z, by a u-ideal of Z, we mean an ideal I of Z such that $(p, q)^u \cap I \neq \varnothing$ for all $p, q \in I$ [6]. In a dual manner, a filter F of Z is called an l-filter provided that $(p, q)^l \cap F \neq \varnothing$ for all $p, q \in F$ [1]. A proper ideal (respectively, a proper filter P) of Z is called prime if for all $p, q \in Z$, $(p, q)^l \subseteq P$ implies $p \in P$ or $q \in P$ [6].

3. Fuzzy Prime Spectrum of a Poset

The present paper is the continuation of Assaye Alaba et al.'s work [11–13], and so we will follow the same notations, definitions, and results on fuzzy ideals, u-fuzzy ideals, fuzzy filters, l-fuzzy filters, prime fuzzy ideals, prime fuzzy filters, maximal fuzzy ideals, and maximal fuzzy filters in a poset.

A fuzzy subset μ of $Z \in \text{Pos}^0$ (respectively, Pos^1) is a fuzzy ideal (respectively, a fuzzy filter) if $\mu(0) = 1$ (respectively, $\mu(1) = 1$) and for any $p, q \in Z$, $\mu(z) \geq \mu(p) \wedge \mu(q)$, for all $z \in (p, q)^{ul}$ (respectively, $z \in (p, q)^{lu}$).

A fuzzy ideal μ of $Z \in \text{Pos}^0$ (respectively, Pos^1) is a u-fuzzy ideal (respectively, an l-fuzzy filter) if an for any

$p, q \in Z$, there exists $z \in (p,q)^u$ (respectively, $z \in (p,q)^l$) such that $\mu(z) = \mu(p) \wedge \mu(q)$.

For our work, we collect some useful notations from [10–13]:

(1) **Br** denotes the family of all nontrivial complete Brouwerian lattices.

(2) For $L \in \mathbf{Br}$, we denote by $p(L)$ the set of all prime elements of L.

(3) **Pos** denotes the collection of all posets.

(4) $\mathbf{Pos}^0$ denotes the collection of all posets with the least element 0.

(5) $\mathbf{Pos}^1$ denotes the collection of all posets with the top element 1.

(6) $\mathbf{Pos}^B$ denotes the collection of all bounded posets.

(7) For $L \in \mathbf{Br}$ and $Z \in \mathbf{Pos}$, we write $\mu \in L^Z$ to say that μ is an L-fuzzy subset of Z (or simply a fuzzy subset if there is no confusion on L); that is, $\mu: Z \longrightarrow L$ is a mapping.

(8) For $Z \in \mathbf{Pos}$, $\mathcal{NF}(Z)$ is the set of normalized fuzzy subsets of Z; that is, $\mathcal{NF}(Z) = \{\mu \in L^Z: \mu(p) = 1 \text{ for some } p \in Z\}$.

(9) For $Z \in \mathbf{Pos}$, $\mathcal{FI}(Z)$ is the set of all fuzzy ideals of Z.

(10) For $Z \in \mathbf{Pos}$, $\mathcal{FI}^u(Z)$ is the set of all u-fuzzy ideals of Z.

(11) For $Z \in \mathbf{Pos}$, $\mathcal{FF}(Z)$ is the set of all fuzzy filters of Z.

(12) For $Z \in \mathbf{Pos}$, $\mathcal{FF}^l(Z)$ is the set of all l-fuzzy filters of Z.

(13) For $Z \in \mathbf{Pos}$, $\mathcal{PFI}(Z)$ is the set of all prime fuzzy ideals of Z.

(14) For $Z \in \mathbf{Pos}$, $\mathcal{PFF}(Z)$ denotes the set of all prime fuzzy filters of Z.

(15) For $Z \in \mathbf{Pos}$, $\mathcal{MFI}(Z)$ is the set of all maximal fuzzy ideals of Z.

(16) For $Z \in \mathbf{Pos}$, $\mathcal{MFF}(Z)$ denotes the set of all maximal fuzzy filters of Z.

(17) $\mathbb{P}_{\text{MFP}} = \{Z \in \mathbf{Pos}^0$: every maximal fuzzy filter of Z is prime$\}$

(18) $\mathbb{P}^l_{\text{MFP}} = \{Z \in \mathbb{P}_{\text{MFP}}$: every maximal u − fuzzy filter of Z is maximal$\}$

(19) $\mathbb{P}_{\text{MIP}} = \{Z \in \mathbf{Pos}^1$: every maximal fuzzy ideal of Z is prime$\}$

(20) $\mathbb{P}^u_{\text{MIP}} = \{Z \in \mathbb{P}_{\text{MIP}}$: every maximal u − fuzzy ideal of Z is maximal$\}$.

Definition 1. For $\mu \in \mathcal{NF}(Z)$, its generalized complement denoted by μ^c is a fuzzy subset of Z defined as follows:

$$\mu^c(z) = \begin{cases} \bigwedge_{y \in Z} \mu(y), & \text{if } \mu(z) = 1, \\ 1, & \text{otherwise}, \end{cases} \tag{1}$$

for all $z \in Z$.

Lemma 2. *Let $Z \in \mathbf{Pos}$ and $\mu \in \mathcal{FI}(Z)$ with $Im(\mu) = \{1, \alpha\}$, for some $\alpha \in p(L)$. Then, $\mu \in \mathcal{PFI}(Z)$ if and only if $\mu^c \in \mathcal{FF}^l(Z)$.*

Proof. Suppose that $\mu \in \mathcal{PFI}(Z)$. Then, it was proved in [13] that the level set $\mu_* = \{p \in Z: \mu(p) = 1\}$ is a prime ideal of Z. In this case, μ^c is of the form

$$\mu^c(z) = \begin{cases} 1, & if\ z \notin \mu_*, \\ \alpha, & \text{otherwise}, \end{cases} \tag{2}$$

for all $z \in Z$. Now, we fist show that $\mu^c \in \mathcal{FF}(Z)$. Clearly, we have $\mu^c(1) = 1$. Let $p, q \in Z$ and $x \in (p,q)^{\text{lu}}$. If $p \in \mu_*$ or $q \in \mu_*$, then it follows that $\mu^c(x) \geq \alpha = \mu^c(p) \wedge \mu^c(q)$. Otherwise, if $p \notin \mu_*$ and $q \notin \mu_*$, then being μ_* a prime ideal, we get $(p,q)^l \nsubseteq \mu_*$. So, there exists $y \in (p,q)^l$ such that $y \notin \mu_*$. Since $x \in (p,q)^{\text{lu}} \subseteq y^u$, we have $y \leq x$, and hence, $x \notin \mu_*$. Thus,

$$\mu^c(x) = 1 = \mu^c(p) \wedge \mu^c(q). \tag{3}$$

Therefore, $\mu^c \in \mathcal{FF}(Z)$. To show $\mu^c \in \mathcal{FF}^l(Z)$, let $p, q \in Z$. If $p \notin \mu_*$ and $q \notin \mu_*$, then $\mu^c(p) \wedge \mu^c(q) = 1$. Again, since μ_* is a prime ideal, we have $(a,b)^l \nsubseteq \mu_*$. So, there is $x \in (p,q)^l$ such that $x \notin \mu_*$ and so

$$\mu^c(x) = 1 = \mu^c(p) \wedge \mu^c(q). \tag{4}$$

Let $p \in \mu_*$ or $q \in \mu_*$. Then, since $0 \in (p,q)^l \subseteq p^l, q^l \subseteq \mu_*$, we have $\varnothing \neq (p,q)^l \subseteq \mu_*$, and hence, $\mu^c(x) = \alpha = \mu^c(p) \wedge \mu^c(q)$, for all $x \in (p,q)^l$. Hence, in either cases, there exists $x \in (p,q)^l$ such that $\mu^c(x) = \mu^c(p) \wedge \mu^c(q)$. Thus, μ^c is an l-fuzzy filter, and hence, $\mu^c \in \mathcal{FF}^l(Z)$.

Conversely, assume that $\mu^c \in \mathcal{FF}^l(Z)$. Then, by Theorem 3.3 of [13], it is enough to check that the set $\mu_* = \{x \in Z: \mu(x) = 1\}$ is a prime ideal of Z. It is clear that μ_* is a proper ideal of Z. Let $p, q \in Z$ such that $(p,q)^l \subseteq \mu_*$. This implies that $\mu^c(x) = \alpha$ for all $x \in (p,q)^l$. Since μ^c is an l-fuzzy filter, there exists $y \in (p,q)^l$ such that

$$\mu^c(y) = \mu^c(p) \wedge \mu^c(q). \tag{5}$$

So that $\mu^c(p) \wedge \mu^c(q) = \alpha$. Since α is prime and hence irreducible in L, we have either $\mu^c(p) = \alpha$ or $\mu^c(q) = \alpha$, i.e., either $p \in \mu_*$ or $q \in \mu_*$. So, μ_* is a prime ideal. Hence, it is proved. □

Lemma 3. *Let $Z \in \mathbf{Pos}$ and $\mu \in \mathcal{FF}(Z)$ with $Im(\mu) = \{1, \alpha\}$, for some $\alpha \in p(L)$. Then, $\mu \in \mathcal{PFF}(Z)$ if and only if $\mu^c \in \mathcal{FI}^u(Z)$.*

Lemma 4. *For any $p \in Z$ and $\alpha \in L$, $[p_\alpha)$ (the fuzzy filter generated by p_α) where p_α is a fuzzy subset of Z called a fuzzy point of Z given by*

$$p_\alpha(z) = \begin{cases} \alpha, & \text{if } z = p, \\ 0, & \text{if otherwise}, \end{cases} \tag{6}$$

which belongs to the class $\mathcal{FF}^u(Z)$.

The next lemma proves existence of prime fuzzy ideals in posets from $\mathbb{P}^l_{\text{MFP}}$.

Lemma 5. *Let $Z \in \mathbb{P}^l_{MFP}$. If L has dual atoms, then the class $\mathscr{PFI}(Z)$ is nonempty.*

Proof. Let $0 \neq p \in Z$ and α a dual atom in L. Put

$$\mathfrak{F} = \{\sigma \in l - \text{FF}(Z)\colon\ [p_\alpha) \subseteq \sigma, \sigma(0) = \alpha\}, \tag{7}$$

where $[p_\alpha)$ is a fuzzy filter of Z generated by the fuzzy point p_α which is characterized by

$$[p_\alpha)(z) = \begin{cases} 1, & \text{if } z = 1, \\ \alpha, & \text{if } z \in [p) - \{1\}, \\ 0, & \text{if } z \notin [p), \end{cases} \tag{8}$$

for all $z \in Z$. Now, since $0 \neq p \in Z$ and $(0] \cap [p) = \varnothing$, by Zorn's lemma, there exists a maximal l-filter $[F \supseteq p)$ such that $(0] \cap F = \varnothing$. Define a fuzzy subset of Z by

$$\mu_F(z) = \begin{cases} 1, & \text{if } z \in F, \\ \alpha, & \text{otherwise}, \end{cases} \tag{9}$$

for all $z \in Z$. Then, it is clear that μ_F is an l-fuzzy filter containing $[p_\alpha)$ and $\mu_F(0) = \alpha$, and hence, $\mu_F \in \mathfrak{F}$. Thus, $\mathfrak{F}$ is nonempty and hence form a poset together with the inclusion ordering of fuzzy sets. Moreover, it can be easily verified that it satisfies the hypothesis of Zorn's lemma. So, by applying Zorn's lemma, we can find a maximal member, let say η in $\mathfrak{F}$. Now, we show that η is maximal in the set of all l-fuzzy filters. Let θ be a proper element in l-FF(Z) such that $\eta \subseteq \theta$. Then, $\alpha = \eta(0) \leq \theta(0)$. Since α is a dual atom, we have either $\theta(0) = \alpha$ or $\theta(0) = 1$. If $\theta(0) = 1$, then $1 = \theta(0) \leq \theta(z)$, for all $z \in Z$. Hence, $\theta(z) = 1$, for all $z \in Z$, which contradicts to the fact that θ is proper. Therefore, $(0) = \alpha$, and also, it is clear that $[p_\alpha) \subseteq \theta$ and so $\theta \in \mathfrak{F}$. Thus, by maximality of η in $\mathfrak{F}$, we get $\eta = \theta$. This proves that η is maximal in l-FF(Z).

Due to the assumption that every maximal l-fuzzy filter is maximal among all fuzzy filters, we have that η is a maximal fuzzy filter. Further, as $Z \in \mathbb{P}^l_{\text{MFP}}$, so η is a prime fuzzy filter. Thus, η would be a two-valued fuzzy set and can be characterized as

$$\eta(z) = \begin{cases} 1, & if\ z \in \eta_*, \\ \beta, & \text{otherwise}, \end{cases} \tag{10}$$

for all $z \in Z$, where $\beta \in p(L)$. It further yields that η^c is a u-fuzzy ideal. Moreover, η^c is a prime fuzzy ideal and is described as

$$\eta^c(z) = \begin{cases} 1, & \text{if } z \in Z - \eta_*, \\ \beta, & \text{otherwise}, \end{cases} \tag{11}$$

for all $z \in Z$. Therefore, the set $\mathscr{PFI}(Z)$ of all prime fuzzy ideals of Z is nonempty.

Let $Z \in \textbf{Pos}$, and θ be any fuzzy subset of Z. Define sets by:

$$\begin{aligned} \Upsilon(\theta) &= \{\mu \in \mathscr{PFI}(Z)\colon\ \theta \subseteq \mu\} \text{ and} \\ \sum(\theta) &= \{\mu \in \mathscr{PFI}(Z)\colon\ \theta \nsubseteq \mu\} = \mathscr{PFI}(Z) - \Upsilon(\theta). \end{aligned} \tag{12}$$

□

Lemma 6. *Let $Z \in$ **Pos**. Then, for any $\theta, \sigma \in FI(Z)$:*

$$\eta \subseteq \theta \Rightarrow \sum(\eta) \subseteq \sum(\theta) \text{ and} \sum(\eta) = \sum((\eta]). \tag{13}$$

Proof. Let $\mu \in \Sigma(\eta)$. This implies that $\eta \nsubseteq \mu$. Since $\eta \subseteq \theta$, it is clear that $\theta \nsubseteq \mu$, and hence, $\mu \in \sum(\theta)$. Therefore, $\sum(\eta) \subseteq \sum(\theta)$ and again as $\eta \subseteq (\eta]$, we clearly have $\sum(\eta) \subseteq \sum((\eta])$. To show the other inclusion, let $\mu \in \sum((\eta])$. Then, $(\eta \nsubseteq \mu]$. Now, we claim that $\eta \nsubseteq \mu$. Suppose that $\eta \subseteq \mu$. This implies that $(\eta] \subseteq (\mu] = \mu$, which is a contradiction. Hence, the claim is true. Therefore, $\mu \in \Sigma(\eta)$, and hence, $\Sigma((\eta) \subseteq \Sigma(\eta)$. Therefore, $\Sigma(\eta) = \Sigma((\eta])$. □

Theorem 7. *The collection $\mathscr{C} = \{\Sigma(\theta)\colon\ \theta \in FI(Z)\}$ forms a topology on $\mathscr{PFI}(Z)$.*

Proof. Since $\Sigma(\chi_{\{0\}}) = \varnothing$ and $\Sigma(\overline{1}) = \mathscr{PFI}(Z)$, $\mathscr{C}$ contains both $\varnothing$ and $\mathscr{PFI}(Z)$. Let $\Sigma(\theta_1), \Sigma(\theta_2) \in \mathscr{C}$.

Then, since $\Sigma(\theta_1) \cap \Sigma(\theta_2) = \Sigma(\theta_1 \cap \theta_2) \in \mathscr{C}$, $\mathscr{C}$ is closed under finite intersection. Further, let $\{\Sigma(\theta_i)\colon\ i \in \Delta\} \subseteq \mathscr{C}$.

Then $\Sigma(\cup_{i\in\Delta}\theta_i) = \Sigma((\cup_{i\in\Delta}\theta_i]) \in \mathscr{C}$. Now we claim that $\cup_{i\in\Delta}\Sigma(\theta_i) = \Sigma(\cup_{i\in\Delta}\theta_i)$. Let $\mu \in \cup_{i\in\Delta}\Sigma(\theta_i)$. Then, $\mu \in \Sigma(\theta_i)$ for some $i \in \Delta$; that is, $\theta_i \nsubseteq \mu$ for some $i \in \Delta$. Since $\theta_i \subseteq (\cup_{i\in\Delta}\theta_i]$, we get $(\cup_{i\in\Delta}\theta_i] \nsubseteq \mu$ so that $\mu \in \Sigma((\cup_{i\in\Delta}\theta_i) = \Sigma(\cup_{i\in\Delta}\theta_i)$. Hence, $\cup_{i\in\Delta}\Sigma(\theta_i) \subseteq \Sigma(\cup_{i\in\Delta}\theta_i)$. To prove the other inclusion, let $\mu \in \Sigma(\cup_{i\in\Delta}\theta_i)$. Then, $\cup_{i\in\Delta}\theta_i \nsubseteq \mu$. This implies that $\theta_i \nsubseteq \mu$ for some $i \in \Delta$ so that $\mu \in \Sigma(\theta_i) \subseteq \cup_{i\in\Delta}\Sigma(\theta_i)$, and hence, $\cup_{i\in\Delta}\Sigma(\theta_i) \subseteq \Sigma(\cup_{i\in\Delta}\theta_i)$. Therefore, $\cup_{i\in\Delta}\Sigma(\theta_i) = \Sigma(\cup_{i\in\Delta}\theta_i) = \Sigma((\cup_{i\in\Delta}\theta_i]) \in \mathscr{C}$. Hence, $\mathscr{C}$ is closed under arbitrary unions. Therefore, $(\mathscr{PFI}(Z), \mathscr{C})$ is a topological space.

The topological space $(\mathscr{PFI}(Z), \mathscr{C})$ is called the spectrum of prime fuzzy ideals of the poset Z, and it is denoted by Spec(Z). □

Remark 8. For any fuzzy ideal θ of a poset Z, the sets $\sum(\theta)$ and $\Upsilon(\theta)$, respectively, are open sets and closed sets in the topological space Spec(Z).

Lemma 9. *Let $p, q \in Z$ and $\alpha, \beta \in L - \{0\}$. Then, there exists $z \in (p, q)^l$ such that*

$$\Sigma(p_\alpha) \cap \Sigma(q_\beta) = \Sigma(z_{\alpha\wedge\beta}). \tag{14}$$

Proof. Suppose that $\mu \in \Sigma(p_\alpha) \cap \Sigma(q_\beta)$. Then, $\mu \in \Sigma(p_\alpha)$ and $\mu \in \Sigma(q_\beta)$. This implies that $p_\alpha \nsubseteq \mu$ and $q_\beta \nsubseteq \mu$. Thus, we have $\alpha \nleq \mu(p)$ and $\beta \nleq \mu(q)$. Since $\text{Im}(\mu) = \{1, \gamma\}$, where γ is a prime element, $\mu(p) = \mu(q) = \gamma$. This implies that $p \notin \mu_*$ and $\notin \mu_*$. But as μ_* is a prime ideal, we have $(p, q)^l \nsubseteq \mu_*$, and hence, there exists $z \in (x, y)^l$ such that $z \notin \mu_*$. Thus, $\mu(z) = \gamma$ and $\alpha \nleq \mu(z)$ and $\beta \nleq \mu(z)$. By primness of γ, we have $\alpha \wedge \beta \nleq \mu(z)$, and hence, $z_{\alpha\wedge\beta} \nsubseteq \mu$, that is, $\mu \in \Sigma(z_{\alpha\wedge\beta})$, so there exists $z \in (p, q)^l$ such that

$$\Sigma(p_\alpha) \cap \Sigma(q_\beta) \subseteq \Sigma(z_{\alpha\wedge\beta}). \quad (15)$$

To show the other inclusion, let $\mu \in \Sigma(z_{\alpha\wedge\beta})$ for some $z \in (p, q)^l$. Then, $z_{\alpha\wedge\beta} \nsubseteq \mu$, that is, $\alpha \wedge \beta \nleq \mu(z)$. This implies that $\alpha \nleq \mu(z)$ and $\beta \nleq \mu(z)$, otherwise if $\alpha \leq \mu(z)$ or $\beta \leq \mu(z)$, we have $\alpha \wedge \beta \leq \mu(z)$, which is a contradiction. Again, since $z \in (p, q)^l$, we clearly have $\mu(p) \leq \mu(z)$ and $\mu(q) \leq \mu(z)$, and hence, $\alpha \nleq \mu(p)$ and $\beta \nleq \mu(q)$. Thus, $p_\alpha \nsubseteq \mu$ and $q_\beta \nsubseteq \mu$, and so we have $\mu \in \Sigma(p_\alpha) \cap \Sigma(q_\beta)$ and hence

$$\Sigma(z_{\alpha\wedge\beta}) \subseteq \Sigma(p_\alpha) \cap \Sigma(q_\beta). \quad (16)$$

This proves the lemma. □

Corollary 10. *Let Z be a meet semilattice. Then, for any $p, q \in Z$ and, $\beta \in L - \{0\}$, we have*

$$\Sigma(p_\alpha) \cap \Sigma(q_\beta) = \sum((p \wedge q)_{\alpha\wedge\beta}). \quad (17)$$

Lemma 11. *Let $Z \in \mathbb{P}^l_{MFP}$. The collection*

$$\mathscr{B} = \{\Sigma(p_\alpha) : p \in Z, \alpha \in L - \{0\}, \quad (18)$$

forms a basis for the open sets of $\mathscr{PFI}(Z)$.

Proof. Let $\Sigma(\theta)$ be any open set in $(\mathscr{PFI}(Z), \mathscr{C})$ and $\mu \in \Sigma(\theta)$. Then, $\theta \nsubseteq \mu$. Then, there exists $p \in Z$ such that $\theta(p) \nleq \mu(p)$. If we put $\theta(p) = \alpha$, we have $p_\alpha \subseteq \theta$ and $p_\alpha \nsubseteq \mu$ so that $\mu \in \Sigma(p_\alpha) \subseteq \Sigma(\theta)$. Thus, the collection $\{\Sigma(p_\alpha) : p \in Z, \alpha \in L - \{0\}\}$ forms a basis for the open sets of X. □

For any subset Y of $\mathscr{PFI}(Z)$, denote $\overline{Y}$, the closure of Y in $\mathscr{PFI}(Z)$, the smallest closed set containing Y.

Lemma 12. *Let $Z \in$ **Pos**. For any $\emptyset \neq Y \subseteq \mathscr{PFI}(Z)$, we have*

$$\overline{Y} = \Upsilon\left(\bigcap_{\mu\in Y} \mu\right). \quad (19)$$

Proof. Clearly, $\Upsilon(\cap_{\mu\in Y}\mu)$ is a closed set of $\mathscr{PFI}(Z)$ and $Y \subseteq \Upsilon(\cap_{\mu\in Y}\mu)$. We show that any closed set F containing Y also contains $\Upsilon(\cap_{\mu\in Y}\mu)$. Since F is closed, by Remark 8, $F = \Upsilon(\theta)$ for some fuzzy ideal θ of Z. As $Y \subseteq \Upsilon(\theta)$, we have $\theta \subseteq \mu$ for every $\mu \in Y$. Therefore, $\theta \subseteq \cap_{\mu\in Y}\mu$, and thus, $\Upsilon(\cap_{\mu\in Y}\mu) \subseteq \Upsilon(\theta) = F$. Hence, $\Upsilon(\cap_{\mu\in Y}\mu)$ is the smallest closed set containing Y. Therefore, $\overline{Y} = \Upsilon(\cap_{\mu\in Y}\mu)$. □

Definition 13. A topological space X is defined to be

(1) A T_0-space if for any $x \neq y \in X$, there exists an open set H in X such that either $x \in H$ and $y \notin H$ or $y \in H$ and $x \notin H$

(2) A T_1-space if for any $x \neq y \in X$, there exist two open sets G and H in X such that either $x \in G - H$ and $y \in H - G$ or $y \in G - H$ and $x \in H - G$

Theorem 14. *$\mathscr{PFI}(Z)$ is a T_0-space for all $Z \in$ **Pos**^{0}.*

Proof. Let θ, μ be any two distinct points in $\mathscr{PFI}(Z)$. Then, either $\theta \nsubseteq \mu$ or $\nsubseteq \theta$. Assume without loss of generality $\theta \nsubseteq \mu$. This implies that $\mu \in \Sigma(\theta)$ and $\theta \notin \Sigma(\theta)$. Thus, we get an open set $\Sigma(\theta)$ containing μ but not θ. Therefore, $\mathscr{PFI}(Z)$ is a T_0-space. □

Lemma 15. *Let μ and σ be any two points in $\mathscr{PFI}(Z)$. Then, $\sigma \in \overline{\{\mu\}}$ if and only if $\mu \subseteq \sigma$.*

Proof. Let $\mu, \sigma \in \mathscr{PFI}(Z)$ and $\sigma \in \overline{\{\mu\}}$. Then, $\mu \in \Sigma(\theta)$ for each neighborhood $\Sigma(\theta)$ of σ. Thus, $\theta \nsubseteq \mu$ for all $\theta \nsubseteq \sigma$. Suppose that $\mu \nsubseteq \sigma$. Then, $\mu \nsubseteq \mu$, which is a contradiction. Therefore, $\mu \subseteq \sigma$.

Conversely, suppose that $\mu \subseteq \sigma$ and $\Sigma(\theta)$ be any neighborhood of σ. Then, $\theta \nsubseteq \sigma$. Since $\mu \subseteq \sigma$, we get $\theta \nsubseteq \mu$, which gives that $\mu \in \Sigma(\theta)$; that is, $\{\mu\} \cap \Sigma(\theta) \neq \emptyset$. Thus $\sigma \in \overline{\{\mu\}}$.

In the following lemma, we obtain a set of equivalent conditions for which $\mathscr{PFI}(Z)$ becomes a T_1-space. Recall that a topological space X is said to be a T_1-space if $\{x\}$ is a closed set for any $x \in X$. □

Lemma 16. *For any $Z \in \mathbb{P}^l_{MFP}$, the following conditions are equivalent:*

(1) $\mathscr{PFI}(Z)$ is a T_1-space

(2) $\{\mu\} = \Upsilon(\mu)$ for all $\mu \in \mathscr{PFI}(Z)$

(3) $(\mathscr{PFI}(Z), \subseteq)$ is antichain

Proof

(1) ⇒ (2): Let $\mu \in \mathscr{PFI}(Z)$. Then, since $\mathscr{PFI}(Z)$ is T_1, we have $\{\mu\} = \overline{\{\mu\}} = \Upsilon(\mu)$, by Lemma 12.

(2) ⇒ (3): Suppose that $\mu, \sigma \in \mathscr{PFI}(Z)$ such that $\mu \subseteq \sigma$. Then, $\sigma \in \overline{\{\mu\}} = \Upsilon(\mu) = \{\mu\}$, and hence, $\mu = \sigma$. Therefore, $(\mathscr{PFI}(Z), \subseteq)$ is antichain.

(3) ⇒ (1): Let $\sigma \in \overline{\{\mu\}}$ then by Lemma 15 $\mu \subseteq \sigma$. Since $(\mathscr{PFI}(Z), \subseteq)$ is an antichain, we have $\sigma = \mu$. therefore $\Upsilon(\mu) = \overline{\{\mu\}} = \{\mu\}$ for each $\mu \in \mathscr{PFI}(Z)$. Thus, $\mathscr{PFI}(Z)$ is a T_1-space. □

Corollary 17. *For a pseudocomplemented meet-semilattice Z, the following conditions are equivalent:*

(1) $\mathscr{PFI}(Z)$ is a T_1-space

(2) $\{\mu\} = \Upsilon(\mu)$ for all $\mu \in \mathscr{PFI}(Z)$

(3) $(\mathscr{PFI}(Z), \subseteq)$ is antichain

Definition 18. We say a topological space X reducible if there are two closed sets U and W such that $X = U \cup W$ and $U \neq X \neq W$. X is irreducible if it is not reducible. In particular, a closed subset V of X is irreducible if for any closed sets U and W in X:

$$V = U \cup W \Rightarrow [\text{either } V = U \text{ or } V = W]. \quad (20)$$

In the following lemma, we describe what irreducible sets look like in $\mathscr{PFI}(Z)$.

Lemma 19. *Let $Z \in \mathbb{P}^l_{MFP}$ and $\varnothing \neq V$ be closed in $\mathscr{PFI}(Z)$. Then, $\cap_{\mu\in V}\mu$ being a prime fuzzy ideal is a necessary and sufficient condition for V to be irreducible.*

Proof. Suppose that V is irreducible. Put $\sigma = \cap_{\mu\in V}\mu$. Since $V \neq \varnothing$, it is clear that σ is proper. Let p_α and q_β be fuzzy points in Z such that $(p_\alpha] \cap (q_\beta] \subseteq \sigma$. Then, $(p_\alpha] \cap (q_\beta] \subseteq \mu$ for all $\mu \in V$. Thus, for all $\mu \in V$, either $p_\alpha \subseteq \mu$ or $q_\beta \subseteq \mu$. Hence, $V = (V \cap \Upsilon(p_\alpha)) \cup (V \cap \Upsilon(q_\beta))$. Since V is irreducible and $(V \cap \Upsilon(p_\alpha))$ and $(V \cap \Upsilon(q_\beta))$ are closed, we have either $V = (V \cap \Upsilon(p_\alpha))$ or $V = (V \cap \Upsilon(q_\beta))$ and hence $V \subseteq \Upsilon(p_\alpha)$ or $V \subseteq \Upsilon(q_\beta)$. Hence, $p_\alpha \subseteq \sigma$ or $q_\beta \subseteq \sigma$. Therefore, $\sigma = \cap_{\mu\in V}\mu$ is a prime fuzzy ideal of Z.

Conversely suppose that $\sigma = \cap_{\mu\in W}\mu$ is a prime fuzzy ideal of Z. Let $W = W_1 \cup W_2$, where W_1 and W_2 are closed sets in X. Put $\theta = \cap_{\mu\in W_1}\mu$ and $\eta = \cap_{\mu\in W_2}\mu$. Thus, $\sigma \subseteq \theta$ and $\sigma \subseteq \eta$. Also, observe that $\theta \cap \eta = \sigma$ and since $\sigma = \cap_{\mu\in W}\mu$ is a prime fuzzy ideal of Z, we have either $\theta \subseteq \sigma$ or $\eta \subseteq \sigma$. Therefore, $\sigma = \theta$ and $\sigma = \eta$. Then, by Lemma 12, we clearly have $W = \overline{W} = \Upsilon(\cap_{\mu\in W}\mu) = \Upsilon(\sigma) = \Upsilon(\cap_{\mu\in W_1}\mu) = \overline{W_1} = W_1$ or $W = \overline{W} = \Upsilon(\cap_{\mu\in W}\mu) = \Upsilon(\eta) = \Upsilon(\cap_{\mu\in W_2}\mu) = \overline{W_2} = W_2$. □

Lemma 20. *Let Z be a complemented poset. For $\alpha \in L - \{1\}$, if we define $X_\alpha = \{\mu \in \mathscr{PFI}(Z)\colon Im(\mu) = \{1,\alpha\}\}$, then X_α is an antichain.*

Proof. Let $\mu, \sigma \in X_\alpha$ such that $\mu \subseteq \sigma$. Suppose that $\mu \neq \sigma$. Then, there exists $p \in Z$ such that $\mu(p) \neq 1$ and $\sigma(p) = 1$. This implies that $p \notin \mu_*$ and $p \in \sigma_*$. By complementedness of Z, there exists p' such that $(p, p')^{ul} = Z$. Since μ_* is a prime ideal, $(p, p')^l = \{0\} \subseteq \mu_*$, we have $p' \in \mu_* \subsetneq \sigma_*$. Thus, we have $Z = (p, p')^{ul} \subseteq \sigma_*$ and hence $\sigma = \overline{1}$, which is a contradiction. Therefore, $\mu = \sigma$, and hence, X_α is antichain. □

Recall that X_α can be made a subspace of X by the relativized topology $\mathscr{C}_\alpha$ where

$$\mathscr{C}_\alpha = \{\Sigma(\theta) \cap X_\alpha\colon \theta \in \mathscr{PFI}(Z)\}. \tag{21}$$

If we put $S = \{\beta \in L\colon \beta \nleq \alpha\}$, then $\varnothing \neq S \subseteq L$. It is evident that the family

$$\mathscr{B}_\alpha = \{\Sigma(p_\beta) \cap X_\alpha\colon p \in Z \text{ and } \beta \in S\}, \tag{22}$$

constitutes a base for $\mathscr{C}_\alpha$.

Corollary 21. *Let Z be a complemented poset. If $p(L)$ is an antichain, then $\mathscr{PFI}(L)$ is antichain.*

Definition 22. A topological space X is Hausdorff (or T_2-space) if any two distinct points can be separated by means of disjoint open sets. That is, for any $x \neq y \in X$, there are two disjoint open sets G and H in X such that $x \in G$ and $y \in H$.

Theorem 23. *Let $Z \in \mathbb{P}^l_{MFP}$. If Z is complemented, then*

(1) $\Sigma(p_\beta) \cap X_\alpha$ is both open and closed, for all $\beta \in L$ such that $\beta \nleq \alpha$

(2) X_α is a Hausdorff space

Proof

(i) We now claim that $\Sigma(p_\beta) \cap X_\alpha = \Upsilon(p'_\beta) \cap X_\alpha$, where p' is the complement of p in Z. Let $\mu \in \Sigma(p_\beta) \cap X_\alpha$. Then, $\beta \nleq \mu(p)$ and $\mathrm{Im}(\mu) = \{1,\alpha\}$ so that $\mu(p) = \alpha$. Hence, $\beta \nleq \alpha$ and $p \notin \mu_*$. Since μ_* is a prime ideal, $(p, p')^l = \{0\} \subseteq \mu_*$, we have $p' \in \mu_*$. Consequently, $\mu(p') = 1$ and $p'_\beta \subseteq \mu$, and thus, $\mu \in \Upsilon(p'_\beta) \cap X_\alpha$. On the other hand, let $\mu \in \Upsilon(p'_\beta) \cap X_\alpha$. Then, $\beta \leq \mu(p')$ and $\mathrm{Im}(\mu) = \{1,\alpha\}$. As $\alpha \nleq \beta$, we have $\mu(p') = 1$. It follows that $p' \in \mu_*$, and hence, $p \notin \mu_*$. Otherwise, if $p' \in \mu_*$, we have $Z = (p, p')^{ul} \subseteq \mu_*$, which is a contradiction. Therefore, $p_\beta \nsubseteq \mu$, and thus, $\mu \in \Sigma(p_\beta) \cap X_\alpha$. Hence, this is the claim.

(ii) Let $\mu \neq \sigma \in X_\alpha$. Without loss of generality, we can assume that $\mu \nsubseteq \sigma$. Since $\mathrm{Im}(\mu) = \mathrm{Im}(\sigma) = \{1,\alpha\}$, there exists $p \in Z$ such that $\mu(p) \neq 1$ and $\sigma(p) = 1$. Thus, $p \notin \mu_*$ and $p \in \sigma_*$, and hence, $p' \in \mu_*$ and $p' \notin \sigma_*$. Otherwise, if $p' \notin \mu_*$ and $p' \in \sigma_*$, we have $\{0\} = (p, p')^l \nsubseteq \mu_*$ and $Z = (p, p')^{ul} \subseteq \sigma_*$, which is a contradiction. Whence $\mu(p) = \sigma(p') = \alpha$ and $\mu(p') = \sigma(p) = 1$. Let $\beta \in L$ such that $\beta \nleq \alpha$. Then, $p_\beta \nsubseteq \mu$ and $p'_\beta \nsubseteq \sigma$. Hence, $\mu \in \Sigma(p_\beta)$ and $\sigma \in \Sigma(p'_\beta)$ and $\Sigma(p_\beta) \cap \Sigma(p'_\beta) = \Sigma(z_\beta)$, for some $z \in (p, p')^l$. Since $(p, p')^l = \{0\}$, we have $\Sigma(p_\beta) \cap \Sigma(p'_\beta) = \Sigma(0_\beta) = \varnothing$. So, X_α is Hausdorff.

In a topological space X, by a clopen set, we mean a set which is closed and open. We say that X is totally disconnected if any two distinct points in X can be separated by means of a clopen set, That is, for any $u \neq v \in X$, we can find a clopen set O such that $u \in O$ and $v \notin O$. □

Corollary 24. *Let $Z \in \mathbb{P}^l_{MFP}$. If Z is complemented, then X_α is totally disconnected.*

4. Space of Maximal Fuzzy Ideals

This section is devoted to the study of the space of maximal fuzzy ideals of a poset Z as a subspace of the fuzzy prime spectrum $\mathscr{PFI}(Z)$ under the assumption that every maximal fuzzy ideal is prime. Moreover, we derive equivalent conditions for the space of maximal fuzzy ideals of a poset to be a normal space.

Remark 25. Note that if $Z \in \mathbf{Pos}^1$ and L has dual atoms, then by applying Zorn's Lemma, we can find a maximal fuzzy ideal in Z. Therefore, $\mathscr{MFI}(Z) \neq \varnothing$ and so to study $\mathscr{MFI}(Z)$ as a subspace of $\mathscr{PFI}(Z)$, first we have to make sure that the inclusion $\mathscr{MFI}(Z) \subseteq \mathscr{PFI}(Z)$ holds. Note that if $Z \in \mathbb{P}^u_{\mathrm{MIP}}$, then $\mathscr{MFI}(Z) \subseteq \mathscr{PFI}(Z)$. Thus, it follows from the assumption $Z \in \mathbb{P}^u_{\mathrm{MIP}}$ that $\mathscr{MFI}(Z)$ forms a subspace of $\mathscr{PFI}(Z)$. Any open set of $\mathscr{MFI}(Z)$ is of the form $\Sigma(\theta) \cap \mathscr{MFI}(Z)$. For every fuzzy ideal θ of Z if we consider

$$\Upsilon_M(\theta) = \mathscr{MFI}(Z) \cap \Upsilon(\theta) \text{ and } \Sigma_M(\theta) = \mathscr{MFI}(Z) \cap \Sigma(\theta), \tag{23}$$

then we have the following:

$$\begin{aligned}\Upsilon_M(\theta) &= \mathscr{MFI}(Z) - \Sigma_M(\theta) \\ &= \mathscr{MFI}(Z) - \Sigma(\theta) \text{ and} \\ \Sigma_M(\theta) &= \mathscr{MFI}(Z) - \Upsilon_M(\theta) \\ &= \mathscr{MFI}(Z) - \Upsilon(\theta).\end{aligned} \tag{24}$$

Lemma 26. *Let $Z \in \mathbb{P}^u_{MIP}$. Then, the family $\mathscr{B} = \{\Sigma_M((p_\alpha]) : p \in Z, \alpha \in L\}$ forms a basis for $\mathscr{MFI}(Z)$ as a subspace of $\mathscr{PFI}(Z)$.*

Lemma 27. *Let $Z \in$ **Pos**^{1}, $p \in Z$ and $\alpha \in L$. Then, $(p_\alpha]$ (the fuzzy ideal of Z generated by the fuzzy point p_α) is a u-fuzzy ideal of Z.*

Lemma 28. *Let $Z \in$ **Pos**^{1}, α in $L - \{1\}$ and $\mu \in \mathscr{FI}^u(Z)$ such that $Im(\mu) = \{1, \alpha\}$. If every maximal element in $\mathscr{FI}^u(Z)$ is maximal in $\mathscr{FI}(Z)$, then for every μ, μ is maximal in $\mathscr{FI}^u(Z)$ if and only if for any $p \in Z$ such that $\mu(p) \neq 1$, there exists $q \in Z$ such that $\mu(q) = 1$ and $(p, q)^u = \{1\}$.*

Proof. Let μ be a maximal u-fuzzy ideal of Z and $\mu(p) \neq 1$.

Consider the fuzzy subset σ defined by, for any $q \in Z$,

$$\sigma(q) = \sup\left\{\bigwedge_{i=1}^{n} \mu(a_i) : a_1, a_2 \cdots a_n \in Z, q \in (a_1, a_2 \cdots a_n, p)^{\text{ul}}\right\}. \tag{25}$$

We show that σ is a fuzzy ideal. Since $0 \in (0, p)^{\text{ul}}$, it is clear that $\sigma(0) = 1$ and let $r, s \in Z$ and $q \in (r, s)^{\text{ul}}$. Now,

$$\begin{aligned}\sigma(r) \wedge \sigma(s) = &\sup\left\{\bigwedge_{i=1}^{n} \mu(a_i) : a_1, a_2 \cdots, a_n \in Z, r \in (a_1, a_2 \cdots, a_n, p)^{\text{ul}}\right\} \\ &\wedge \sup\left\{\bigwedge_{j=1}^{m} \mu(b_j) : b_1, b_2 \cdots, b_m \in Z, s \in (b_1, b_2 \cdots, b_m, p)^{\text{ul}}\right\} \\ &= \sup\left\{\bigwedge_{i=1}^{n} \mu(a_i) \wedge \bigwedge_{j=1}^{m} \mu(b_j) : a_1, a_2, \ldots, a_n, b_1, b_2 \cdots, b_m \in Z, r \in (a_1, a_2 \cdots, a_n, p)^{\text{ul}}, s \in (b_1, b_2 \cdots, b_m, p)^{ul}\right\} \\ &\le \sup\left\{\bigwedge_{i=1}^{n} \mu(a_i) \wedge \bigwedge_{j=1}^{m} \mu(b_j) : a_1, a_2, \cdots a_n, b_1, b_2 \cdots, b_m \in Z,\right. \\ q \in (r, s)^{\text{ul}} &\subseteq (a_1, a_2 \cdots a_n, b_1, b_2 \cdots, b_m, p)^{\text{ul}} \\ &= \sigma(q).\end{aligned} \tag{26}$$

Thus, $\sigma \in \mathscr{FI}(Z)$. Since for any $q \in Z$, $q \in (q, p)^{\text{ul}}$, we clearly have $\sigma(q) \geq \mu(q)$, and hence, $\mu \subseteq \sigma$. Again since $p \in (0, p)^{\text{ul}}$, we clearly have $\sigma(p) \geq \mu(0) = 1$, and hence, $\sigma(p) = 1$. This shows that $\mu \subsetneq \sigma$. It follows that $\mu(p) < \sigma(p)$ for some $p \in L$; that is, $\mu(p) \neq 1$. Again, by the hypothesis "every maximal element of $\mathscr{FI}^u(Z)$ is maximal in $\mathscr{FI}(Z)$," μ would be maximal, and hence, $\sigma = \overline{1}$. Since $1 \in Z$, we have $(1) = 1$, and hence, there exist $a_1, a_2 \cdots, a_n \in Z$ such that $1 \in (a_1, a_2 \cdots, a_n, p)^{\text{ul}}$ and $\wedge_{i=1}^{n} \mu(a_i) = 1$. Then, $(a_1, a_2 \cdots, a_n, p)^u = \{1\}$. But, as $\mu \in \mathscr{FI}^u(Z)$, there should be some $q \in (a_1, a_2 \cdots, a_n)^u$ such that $\mu(q) = \wedge_{i=1}^{n} \mu(a_i)$, and hence, $\mu(q) = 1$. Therefore, $(p, q)^u = (a_1, a_2 \cdots, a_n, p)^u = \{1\}$.

Conversely, assume the condition to be true and let $\theta \in \mathscr{FI}^u$ such that $\mu \subsetneq \theta \subseteq \overline{1}$. Then, there exists $p \in Z$ such that $\mu(p) \neq 1$ and $\theta(p) = 1$ Then, by hypothesis, there exits $q \in Z$ such that $\mu(q) = 1$ and $(p, q)^u = \{1\}$. This implies that $(p, q)^{\text{ul}} = Z$ and since $p, q \in \theta_* = \{z \in Z : \theta(z) = 1\}$, we have $Z = (p, q)^{\text{ul}} \subseteq \theta_*$, and hence, $\theta = \overline{1}$. Hence, μ is a maximal u-fuzzy ideal.

Let us recall a dually dense element in a poset. Let $Z \in$ **Pos**^{1} and $p \in Z$. We say that p is dually dense whenever $p^\intercal = \{1\}$, where $p^\intercal = \{q \in Z : (q, p)^u\} = \{1\}$. We denote by D_1, the set of all dually dense elements in Z. □

Lemma 29. *Let $Z \in$ **Pos**^{B}. Assume that every maximal element of $\mathscr{FI}^u(Z)$ is maximal in the class $\mathscr{FI}(Z)$. Then,*

$$\cap\{\mu : \mu \text{ is maximal in } \mathscr{FI}^u(Z)\} = \chi_{D_1}. \tag{27}$$

Proof. Let $\mathscr{MF}^u(Z)$ be the collection of all maximal u-fuzzy ideals of Z. Suppose $\chi_{D_1}(p) = 1$. Then, $p \in D_1$, that is $p^\intercal = \{1\}$. Suppose on the contrary that $(\cap_{\eta \in \mathscr{MF}^u(Z)} \eta)(p) \neq 1$. Then, there exists a maximal u-fuzzy ideal μ such that $\mu(p) \neq 1$. By Lemma 28, there exists $q \in Z$ such that $\mu(q) = 1$ and $(p, q)^u = \{1\}$. This gives $q \in p^\intercal = \{1\}$, and hence, $\mu(q) = \mu(1) \neq 1$ which is a contradiction to the fact that $\mu(q) = 1$. On the other hand, let $(\cap_{\eta \in \mathscr{MF}^u(Z)} \eta)(p) = 1$. Then, $\eta(p) = 1$ for all maximal u-fuzzy ideals. Suppose $\chi_{D_1}(p) \neq 1$. Then, $\notin D_1$, and hence, there exists $q \in p^{\tau}$ such that $q \neq 1$. By Zorn's lemma, there is a maximal u-fuzzy ideal μ with $\chi_{(q]} \subseteq \mu$. As $\mu(p) = 1$ and $\mu(q) = 1$, we have , $q \in \mu_*$, and this implies that $Z = (p,q)^{ul} \subseteq \mu_*$ is a contradiction to the maximality of μ_*. □

Corollary 30. *Let $Z \in \mathbf{Pos}^B$ and assume that that every maximal element of $\mathscr{F}\mathscr{I}^u(Z)$ is maximal in the class $\mathscr{F}\mathscr{I}(Z)$. Then, χ_{D_1} is a fuzzy ideal of Z.*

Lemma 31. *Let $Z \in \mathbb{P}^u_{MIP}$. Then, the closure of the set $\mathscr{M}\mathscr{F}^u(Z)$ in $\mathscr{P}\mathscr{F}\mathscr{I}(Z)$ is $\Upsilon(\chi_{D_1})$.*

Proof. It follows from Lemma 12 that

$$\begin{aligned}\overline{\mathscr{M}\mathscr{F}^u(Z)} &= \Upsilon\left(\bigcap_{\mu \in \mathscr{M}\mathscr{F}^u(Z)} \mu\right)\\ &= \Upsilon\left(\chi_{D_1}\right).\end{aligned} \tag{28}$$

Hence, this is proved. □

Definition 32 (see [24]). By π_0-space, we mean a topological space for which each of its nonempty open set contains a nonempty closed set.

One can easily verify that every T_1-space can be viewed as a π_0-space. One of our first aims here is to provide an example of a poset for which its fuzzy prime spectrum $\mathscr{P}\mathscr{F}\mathscr{I}(Z)$ is not a π_0-space. For, consider the poset Z_6 whose Hasse diagram is depicted in Figure 1 and $L = \{0, \alpha, 1\}$ where $1 > \alpha > 0$. Here, $\mathscr{P}\mathscr{F}\mathscr{I}(Z_6) = \left\{\alpha_{(a]}, \alpha_{(b]}, \alpha_{(c]}, \alpha_{(d]}\right\}$. The collection of open sets is

$$\left\{\varnothing, \Sigma\left(\alpha_{(a]}\right), \Sigma\left(\alpha_{(b]}\right), \Sigma\left(\alpha_{(c]}\right), \Sigma\left(\alpha_{(d]}\right), \Sigma\left(\alpha_{\{0,a,b\}}\right), \mathscr{P}\mathscr{F}\mathscr{I}(Z_6)\right\}. \tag{29}$$

Now, $\Sigma(\alpha_{(a]}) = \mathscr{P}\mathscr{F}\mathscr{I}(Z) - \Upsilon(\alpha_{(a]}) = \left\{\alpha_{(b]}\right\}$ and so that $\left\{\alpha_{(b]}\right\}$ is a nonempty open set which does not contain a nonempty closed set. Thus, $\mathscr{P}\mathscr{F}\mathscr{I}(Z_6)$ is not π_0, and hence, it is not a T_1-space, as every T_1-space is a π_0-space.

Lemma 33. *Let $Z \in \mathbb{P}^u_{MIP}$ with the least element 0. If*

$$\cap\left\{\eta\colon \eta \in \mathscr{M}\mathscr{F}^u(Z)\right\} = \chi_{\{0\}}, \tag{30}$$

then $\mathscr{P}\mathscr{F}\mathscr{I}(Z)$ is a π_0-space.

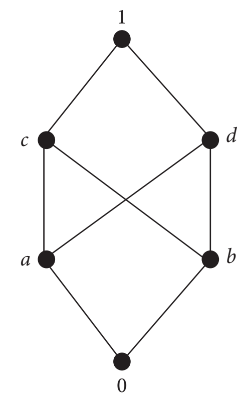

Figure 1: Hasse diagram for the poset Z_6.

Proof. Suppose that $\cap_{\eta \in \mathscr{M}\mathscr{F}^u(Z)} \eta = \chi_{\{0\}}$. By Lemma 29, $\chi_{D_1} = \chi_{\{0\}}$. Let $\Sigma(\mu)$ be any nonempty proper open set in $\mathscr{P}\mathscr{F}\mathscr{I}(Z)$. Clearly, $\chi_{D_1} = \chi_{\{0\}} \subsetneq \mu$. Then, there exists a nonzero $p \in Z$ such that $\mu(p) > 0$. Let say $\beta = \mu(p)$. Since p is nonzero, we have $\chi_{D_1}(p) = 0$. By Lemma 29, $(\cap_{\eta \in \mathscr{M}\mathscr{F}^u(Z)} \eta)(p) = 0$. If $\eta(p) \geq \beta$ for all $\eta \in \mathscr{M}\mathscr{F}^u(Z)$, then $(\cap_{\eta \in \mathscr{M}\mathscr{F}^u(Z)} \eta)(p) \geq \beta > 0$ which is a contradiction. Thus, there is a maximal u-fuzzy ideal θ such that $\theta(p) \ngeq \beta$, and thus, $\mu \nsubseteq \theta$. Since $Z \in \mathbb{P}^u_{\mathrm{MIP}}$, it follows that θ is a prime fuzzy ideal. Further, $\Upsilon(\theta) = \{\theta\}$ yields, and hence, $\{\theta\}$ is a closed set. Therefore, $\{\theta\}$ is the desired nonempty closed set such that $\{\theta\} \subseteq \Sigma(\mu)$, and consequently, $\mathscr{P}\mathscr{F}\mathscr{I}(Z)$ is a π_0 space. □

Remark 34. But, the converse of Lemma 33 need not necessarily be true in general. Consider the poset Z_8 whose Hasse diagram is given in Figure 2 and let $L = \{0, \alpha, 1\}$, where $1 > \alpha > 0$. Here, $\mathscr{P}\mathscr{F}\mathscr{I}(Z_8) = \alpha_{(e]}, \alpha_{(f]}$. By Lemma 16, $\mathscr{P}\mathscr{F}\mathscr{I}(Z_8)$ is a T_1-space, and hence, it is π_0, but the intersection of all maximal fuzzy ideals of Z_8 is nonzero.

Definition 35. Given two subsets G and H of a topological space X, we say that G is weakly separable from H if $G \cap \overline{H} = \varnothing$.

Theorem 36. *Let $Z \in \mathbb{P}^u_{MIP}$. Let us put*

$$\mathfrak{S} = \left\{Y\colon Y \text{ is a subspace of } \mathscr{P}\mathscr{F}\mathscr{I}\ (Z) \text{ such that } Y \text{ is not weakly separable with } \{\eta\} \text{ for all } \eta \in \mathscr{P}\mathscr{F}\mathscr{I}(Z) - Y\right\}. \tag{31}$$

Then, $\mathscr{M}\mathscr{F}\mathscr{I}(Z)$ is the least member of $\mathfrak{S}$.

Proof. We first show that $\mathscr{M}\mathscr{F}\mathscr{I}(Z) \in \mathfrak{S}$. For, let $\sigma \in \mathscr{P}\mathscr{F}\mathscr{I}(Z)$ be any element outside of $\mathscr{M}\mathscr{F}\mathscr{I}(Z)$; that is, $\sigma \notin \mathscr{M}\mathscr{F}\mathscr{I}(Z)$. By Lemma 12, $\overline{\{\sigma\}} = \Upsilon(\sigma)$. As $\sigma \notin \mathscr{M}\mathscr{F}\mathscr{I}(Z)$, there exists a maximal ideal $\theta \supseteq \sigma$. By the hypothesis, θ is a prime fuzzy ideal. Thus, $\theta \in \Upsilon(\sigma)$, and hence, $M \in \overline{\{\sigma\}}$. This implies $\mathscr{M}\mathscr{F}\mathscr{I}(Z) \cap \overline{\{\sigma\}} \neq \varnothing$, and therefore, $\mathscr{M}\mathscr{F}\mathscr{I}(Z)$ is not weakly separable from any element outside it. Next, we show that $\mathscr{M}\mathscr{F}\mathscr{I}(Z) \subseteq Y$ for all $Y \in \mathfrak{S}$. Let $Y \in \mathfrak{S}$ and η be a prime fuzzy ideal of Z such that $\eta \notin Y$. Then, $Y \cap \overline{\{\eta\}} \neq \varnothing$. So, by Lemma 12 there exists $\sigma \in Y$ such that $\eta \subseteq \sigma$. As $\eta \notin Y$, it should be the case that $\eta \subsetneq \sigma$ which implies that η is not maximal; i.e., $\eta \notin \mathscr{M}\mathscr{F}\mathscr{I}(Z)$ leads to a conclusion $\mathscr{M}\mathscr{F}\mathscr{I}(Z) \subseteq Y$. Therefore, $\mathscr{M}\mathscr{F}\mathscr{I}(Z)$ is the least member in $\mathfrak{S}$. □

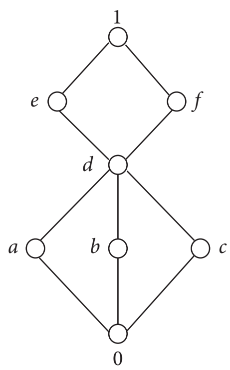

Figure 2: Hasse diagram for the poset Z_8.

Lemma 37. *Let Z be ideal distributive and $L \in \mathbf{Br}$ be such that for any $\alpha \in L-\{1\}$, there exists $\beta \in p(L)$ such that $\alpha \le \beta < 1$. For $S \subseteq L-\{1\}$ and $\{p_i\}_{i\in\Delta} \subseteq Z$, if $\mathscr{PFI}(Z) = \cup\{\Sigma((p_i)_\alpha): i \in\Delta, \alpha \in S\}$ for some index set Δ, then* $\sup\{t: t \in S\} = 1$.

Proof. Put $\sup\{t: t \in S\} = \alpha$. Assume on contrary that $\alpha \ne 1$. Then, by our assumption, there exists a prime element $\beta \in L$ such that $\alpha \le \beta < 1$. As Z is ideal distributive, Z need to necessarily have prime ideals and let P be a prime ideal in Z. Define $\mu: Z \longrightarrow L$ as follows:

$$\mu(p) = \begin{cases} 1, & \text{if } p \in P, \\ \beta, & \text{otherwise}, \end{cases} \quad (32)$$

for all $p \in Z$. Then, it is clear that $\mu \in \mathscr{PFI}(Z)$, and hence, $\mu \in \Sigma(p_t)$ for some $p \in \{p_i\}_{i\in\Delta}$ and $t \in S$. This implies that $p_t \nsubseteq \mu$. However, $\mu(p) > \beta \ge \alpha \ge t = p_t(p)$. Thus, $p_t \subseteq \mu$ is a contradiction. Therefore, $\alpha = 1$. □

Corollary 38. *Let $L \in \mathbf{Br}$ be such that for any $\alpha - \{1\}$, there exists a dual atom $\beta \in L$ such that $\alpha \le \beta < 1$. Then, for any $S \subseteq L-\{1\}$ and $\{p_i\}_{i\Delta} \subseteq Z$ and any subspace Y of $\mathscr{PFI}(Z)$ containing $\mathscr{MFI}(Z)$; if $Y \subseteq \cup\{\Sigma((p_i)_\alpha): i \in \Delta, \alpha \in S\}$ for some index set Δ, then* $\sup\{t: t \in S\} = 1$.

Lemma 39. *Let Z be a poset, $\{p_i\}_{i\in\Delta} \subseteq Z$, $S \subseteq L-\{0\}$ and $\beta = \sup\{t: t \in S\}$. Then,*

$$\bigcup\{\Sigma((p_i)_t): i \in \Delta, t \in S\} = \cup\left\{\Sigma\left((p_i)_\beta\right): i \in \Delta\right\}. \quad (33)$$

Proof. The proof is similar to that of Proposition 16 in [25].

We now turn our attention and go to study some compactness properties of the space of prime (respectively, maximal) fuzzy ideals in a poset. □

Lemma 40. *Let $L \in \mathbf{Br}$ be such that for any $\alpha \in L-\{1\}$, there exists $\beta \in p(L)$ such that $\alpha \le \beta < 1$. Let $Z \in \mathbb{P}^u_{MIP}$ and Y be any subspace of $\mathscr{PFI}(Z)$ containing $\mathscr{MFI}(Z)$. Then, Y is compact.*

Proof. Let Y be any subspace of $\mathscr{PFI}(Z)$ containing $\mathscr{MFI}(Z)$ and let $\{\Sigma((p_i)_t): i \in \Delta, t \in S \subseteq L-\{1\}\}$ be a basic open cover for Y and let $\beta = \sup\{t: t \in S\}$. Then, by Corollary 38 and Lemma 39, we clearly have

$$Y \subseteq \cup\{\Sigma((p_i)_t): i \in \Delta, t \in S \subseteq L-\{1\}\} = \cup\{\Sigma((p_i)_1): i \in \Delta\}. \quad (34)$$

Then, we have $Y \subseteq \Sigma((\bigcup_{i\in\Delta}(p_i)_1]) = \mathscr{PFI}(Z) - \Upsilon((\bigcup_{i\in\Delta}(p_i)_1])$. This implies that $Y \cap \Upsilon((\bigcup_{i\in\Delta}(p_i)_1]) = \varnothing$. We claim that $((\bigcup_{i\in\Delta}(p_i)_1] = \overline{1}.)$ Suppose that $((\bigcup_{i\in\Delta}(p_i)_1] \ne \overline{1}.)$ Then, by Zorn's lemma, there exists a maximal fuzzy ideal μ in $\mathscr{MFI}(Z)$ such that $((\bigcup_{i\in\Delta}(p_i)_1] \subseteq \mu.)$ Thus, $\mu \in \Upsilon((\bigcup_{i\in\Delta}(p_i)_1])$, a contradiction to the fact that $\mathscr{MFI}(Z) \subseteq Y \subseteq \Sigma((\bigcup_{i\in\Delta}(p_i)_1]) = \mathscr{PFI}(Z) - \Upsilon((\bigcup_{i\in\Delta}(p_i)_1])$. Hence, the claim holds true. This implies that the set $(\{p_i: i \in\Delta\}] = Z = (1]$. It was proved in [1] that the class $\mathrm{Id}(Z)$ of all ideals of Z forms an algebraic lattice with respect to the inclusion order and every principal ideal is a compact element in $\mathrm{Id}(Z)$. Then, it follows from this fact that there exist $i_1, i_2, \ldots, i_n \in\Delta$ such that $(p_{i_1}] \vee \ldots \vee (p_{i_n}] = (1] = Z$. Now, we show that $Y \cap \Upsilon((\bigcup_{i=1}^n (p_i)_1]) = \varnothing$. Let $\mu \in Y \cap \Upsilon((\bigcup_{i=1}^n (p_i)_1])$. Then, $\bigcup_{i=1}^n (p_i)_1 \subseteq \mu$. So, for each $= 1, 2, \ldots, n$, $(p_i)_1 \subseteq \mu$, and hence, $\mu(p_i) = 1$, i.e., $p_i \in \mu_*$ for all $i = 1, 2, \ldots, n$. This implies that

$$(1] = (p_1] \vee \ldots \vee (p_n] \subseteq \mu_*, \quad (35)$$

which is a contradiction. Thus, $Y \cap \Upsilon((\bigcup_{i=1}^n (x_i)_1]) = \varnothing$. So,

$$\begin{aligned} Y \subseteq \mathscr{PFI}(Z) - \Upsilon\left(\left(\bigcup_{i=1}^{n} (p_i)_1\right]\right) &= \Sigma\left(\left(\bigcup_{i=1}^{n} (p_i)_1\right]\right) \\ &= \Sigma\left(\bigcup_{i=1}^{n} (p_i)_1\right). \end{aligned} \quad (36)$$

□

Corollary 41. *For any $Z \in \mathbb{P}^u_{MIP}$, both $\mathscr{MFI}(Z)$ and $\mathscr{PFI}(Z)$ are compact.*

Corollary 42. *If Z is a bounded pseudocomplemented semilattice, then $\mathscr{PFI}(Z)$ is compact. In particular, $\mathscr{MFI}(Z)$ is compact.*

Theorem 43. *Let $Z \in \mathbb{P}^u_{MIP}$. Then, $\mathscr{MFI}(Z)$ is Hausdorff if and only if for any distinct elements $\mu', \mu'' \in \mathscr{MFI}(Z)$, we can find $a, b \in Z$ and $\alpha, \beta \in L$ such that $\alpha \nleq \mu'(a), \beta \nleq \mu''(b)$ and $(a_\alpha] \cap (b_\beta] \subseteq \cap_{\mu\in\mathscr{MFI}(Z)}\mu$*

Proof. Suppose that $\mathscr{MFI}(Z)$ is Hausdorff. Let μ', μ'' be any two distinct elements of $\mathscr{MFI}(Z)$. As $\mathscr{MFI}(Z)$ is Hausdorff, we get two fuzzy ideals θ and η of Z such that $\mathscr{MFI}(Z) - \Upsilon(\theta)$ and $\mathscr{MFI}(Z) - \Upsilon(\eta)$ are disjoint open neighborhoods of μ' and μ'', respectively. This gives that $\theta \nsubseteq \mu'$ and $\eta \nsubseteq \mu''$ so that there are two points $a, b \in Z$ with $\theta(a) \nleq \mu'(a)$ and $\eta(b) \nleq \mu''(b)$. Put $\alpha = \theta(a)$ and $\beta = \eta(b)$. Also, from

$$[\mathscr{MFI}(Z) - \Upsilon(\theta)] \cap [\mathscr{MFI}(Z) - \Upsilon(\eta)] = \varnothing, \quad (37)$$

we get $\mathcal{MFI}(Z) - \Upsilon(\theta \cap \eta) = \varnothing$. Therefore, $\Upsilon(\theta \cap \eta) = \mathcal{MFI}(Z)$. This gives $\theta \cap \eta \subseteq \mu$, for all $\mu \in \mathcal{MFI}(Z)$, and hence, $\theta \cap \eta \subseteq \cap_{\mu \in \mathcal{MFI}(Z)} \mu$. Now, let $p \in Z$, if $\notin (a,b)^l$, then it holds that

$$\left(\left(a_\alpha\right] \cap \left(b_\beta\right]\right)(p) = 0 \leq \left(\bigcap_{\mu \in \mathcal{MFI}(Z)} \mu\right)(p), \tag{38}$$

If $p \in (a,b)^l$, and $p \neq 0$, then

$$\begin{aligned}\left(\left(a_\alpha\right] \cap \left(b_\beta\right]\right)(p) &= \alpha \wedge \beta \\ &= \theta(a) \wedge \eta(b) \\ &\leq \theta(p) \wedge \eta(p) \\ &= (\theta \cap \eta)(p) \\ &\leq \left(\bigcap_{\mu \in \mathcal{MFI}(Z)} \mu\right)(p).\end{aligned} \tag{39}$$

On the other hand, if we are assuming that $p = 0$, then it is clear that $((a_\alpha] \cap (b_\beta])(p) = 1 = (\cap_{\mu \in \mathcal{MFI}(Z)} \mu)(p)$.

Conversely, let μ' and μ'' be any two distinct maximal fuzzy ideals in Z. By our assumption, there exist $a, b \in Z$ and $\alpha, \beta \in L$ such that $\alpha \nleq \mu'(a), \beta \nleq \mu''(b)$ and $(a_\alpha] \cap (b_\beta] \subseteq \cap_{\mu \in \mathcal{MFI}(Z)} \mu$. Clearly, $\mathcal{MFI}(Z) - U((a_\alpha])$ and $\mathcal{MFI}(Z) - U((b_\beta])$ are the neighborhoods of μ' and μ'', respectively, in $\mathcal{MFI}(Z)$. As $(a_\alpha] \cap (b_\beta] \subseteq \cap_{\mu \in \mathcal{MFI}(Z)} \mu$, we have

$$\begin{aligned}&\mathcal{MFI}(Z) - U((a_\alpha]) \cap \mathcal{MFI}(Z) - U\left(\left(b_\beta\right]\right) \\ &\quad = \mathcal{MFI}(Z) - U\left(\left(a_\alpha\right] \cap \left(b_\beta\right]\right) \\ &\quad = \varnothing.\end{aligned} \tag{40}$$

This shows that $\mathcal{MFI}(Z)$ is Hausdorff. □ □

Definition 44. A bounded poset Z is said to have the Fpm-property if for any $\mu' \neq \mu'' \in \mathcal{MFI}(Z)$, there are $a, b \in Z$ and $\alpha, \beta \in L$ such that $\alpha \nleq \mu'(a), \beta \nleq \mu''(b)$ and $(a_\alpha] \cap (b_\beta] = \chi_{\{0\}}$.

Lemma 45. *Let $Z \in \mathbb{P}^u_{MIP}$. If Z has the Fpm-property, then for each $\theta \in \mathcal{PFI}(Z)$, there is a unique $\eta \in \mathcal{MFI}(Z)$ such that $\theta \subseteq \eta$.*

Proof. Let $\theta \in \mathcal{PFI}(Z)$. Since Z is given to be bounded and hence with top element 1, we can find $\eta \in \mathcal{MFI}(Z)$ containing θ. To prove the uniqueness part, we use contradiction. Suppose on the contrary that θ is contained in two distinct maximal fuzzy ideals μ' and μ''. Since Z has the Fpm-property, there exist $a, b \in Z$ and $\alpha, \beta \in L$ such that $\alpha \nleq \mu'(a), \beta \nleq \mu''(b)$ and $(a_\alpha] \cap (b_\beta] = \chi_{\{0\}}$. As $(a_\alpha] \cap (b_\beta] = \chi_{\{0\}} \subseteq \theta$, we have $(a_\alpha] \subseteq \theta$ or $(b_\beta] \subseteq \theta$. In either case, we get a contradiction. Therefore, every prime fuzzy ideal is contained in a unique maximal fuzzy ideal.

We say that topological space X is normal provided that any two disjoint closed sets in X can be separated by means of disjoint open sets. That is, for any disjoint closed sets E and F in X, there exist two disjoint open sets G and H in X such that $E \subseteq G$ and $F \subseteq H$. Observe that if X is compact and Hausdorff, then it is normal. □

Lemma 46. *Let $Z \in \mathbb{Z}^u_{MIP}$. If Z has the Fpm-property, then $\mathcal{MFI}(Z)$ is Hausdorff. Moreover, it is normal.*

Proof. Let μ' and μ'' be two distinct distinct elements of $\mathcal{MFI}(Z)$. Since Z is a Fpm-poset, there exist $a, b \in Z$ and $\alpha, \beta \in L$ such that $\alpha \nleq \mu'(a), \beta \nleq \mu''(b)$ and $(a_\alpha] \cap (b_\beta] = \chi_{\{0\}} \subseteq \cap_{\mu \in \mathcal{MFI}(Z)} \mu$. By Theorem 43, $\mathcal{MFI}(Z)$ is Hausdorff. By Corollary 41, $\mathcal{MFI}(Z)$ is a compact space. Thus, $\mathcal{MFI}(Z)$ is a normal space.

The converse of the above lemma is true if the intersection of all maximal fuzzy ideals of Z is $\chi_{\{0\}}$. □

Theorem 47. *Let $Z \in \mathbb{P}^u_{MIP}$ with the bottom element 0. Assume that $\chi_{\{0\}} = \cap\{\eta\colon \eta \in \mathcal{MFI}(Z)\}$. Then, $\mathcal{MFI}(Z)$ is Hausdorff if and only if Z has the Fpm-property.*

Proof. It follows from Theorem 43. If $\mathcal{MFI}(Z)$ is Hausdorff, there exist $a, b \in Z$ and $\alpha, \beta \in L$ such that $\alpha \nleq \mu'(a), \beta \nleq \mu''(b)$ and $(a_\alpha] \cap (b_\beta] \subseteq \cap_{\mu \in \mathcal{MFI}(Z)} \mu = \chi_{\{0\}}$, for any distinct maximal ideals μ' and μ''. Thus, Z is a Fpm-poset. The converse follows from Lemma 46.

We conclude this paper by the following theorem: □

Corollary 48. *Let $Z \in \mathbb{P}^u_{MIP}$ with the least element 0 and assume that $\cap\{\theta\colon \theta \in \mathcal{MFI}(Z)\} = \chi_{\{0\}}$. Then, the following conditions are equivalent:*

(1) Z has the Fpm-property

(2) For any $\mu' \neq \mu'' \in \mathcal{MFI}(Z)$, we can find two open neighborhoods U and V of μ' and μ'' in $\mathcal{PFI}(Z)$ such that $U \cap V = \varnothing$

(3) $\mathcal{MFI}(Z)$ is normal

In addition, if Z is given to be complemented and $p(L)$ is antichain, then the following statement is equivalent with any one (and hence all) of the above three statements

(4) $\mathcal{PFI}(Z)$ is a normal space

It is observed that if Z is a bounded distributive lattice, then the intersection of all prime fuzzy ideals of Z becomes $\chi_{\{0\}}$. Thus, we can deduce the following corollary from the proof of Corollary 48.

Corollary 49. *The following conditions are equivalent for any bounded distributive lattice Z:*

(1) Z has the Fpm-property

(2) *For any* $\mu' \neq \mu'' \in \mathscr{MFI}(Z)$*, we can find two open neighborhoods* U *and* V *of* μ' *and* μ'' *in* $\mathscr{PFI}(Z)$ *such that* $U \cap V = \varnothing$

(3) $\mathscr{MFI}(Z)$ *is normal*

In addition, if Z *is a Boolean algebra, then the following statement is equivalent with any one (and hence all) of the above three statements*

(4) $\mathscr{PFI}(Z)$ *is a normal space*

5. Conclusions

This manuscript presents results on fuzzy prime ideals of partially ordered sets, focusing on the study of their topological properties within the context of the hull-kernel topology. Our investigation has centered around the space of prime fuzzy ideals, along with the space of maximal fuzzy ideals as a subspace. We explored the conditions for which the space of fuzzy prime ideals in partially ordered sets is compact, Hausdorff, and normal through which it deepened our understanding of the interplay between order relations and fuzzy ideals. We believe that these findings not only contribute to the field of fuzzy set theory but also provide valuable insights into the broader study of partially ordered sets and their associated structures.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All the authors contributed equally to the writing of this manuscript. They have also read and approved the final manuscript.

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