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Research Article

Some Inequalities Involving the Derivative of Rational Functions

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In this paper, we establish some inequalities involving the modulus of the derivative of rational functions with prescribed poles and restricted zeros. The obtained results generalize some known inequalities for rational functions. Moreover, our results also contain certain known polynomial inequalities.

1. Introduction

The modulus of complex polynomials on a circle and the locations of zeros of these polynomials have been studied for several years. We start with a result due to Bernstein [1]. Let P_n be the class of polynomials of degree at most n. If $p(z)$ is a polynomial of degree n, then the famous result, known as Bernstein

$$\left|p'(z)\right| \leq n \max_{|z|=1} |p(z)|. \tag{1}$$

Equality holds in (1) if and only if $p(z)$ has all zeros at the origin.

Erdös conjectured in 1944 which Lax [2] proved by improving (1) that for polynomials $p(z)$ of degree n and having no zeros in $|z| < 1$, we have

$$\max_{|z|=1} \left|p'(z)\right| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{2}$$

Equality in (2) holds for $p(z) = \lambda z^n + \mu, |\lambda| = |\mu|$.

On the other hand, if $p(z)$ is a polynomial of degree n having all zeros in $|z| \leq 1$, then

$$\max_{|z|=1} \left|p'(z)\right| \geq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{3}$$

Inequality (3) was demonstrated by Turán [3] and equality in (3) holds for polynomials which have all its zeros on $|z| = 1$. In the literature [4–7], there are many improvements of inequalities (2) and (3). For the class of rational functions, we write

$$\begin{aligned} w(z) &:= (z - a_1)(z - a_2)\cdots(z - a_n), \\ B(z) &= \left(\frac{1 - \overline{a_1}z}{z - a_1}\right)\left(\frac{1 - \overline{a_2}z}{z - a_2}\right)\cdots\left(\frac{1 - \overline{a_n}z}{z - a_n}\right) \\ &= \frac{w^*(z)}{w(z)}, \end{aligned} \tag{4}$$

where $w^*(z) = z^n\overline{w(1/z)}$ and $a_1, a_2, \ldots, a_n$ are complex numbers. The product $B(z)$ is known as Blaschke product and $|B(z)| = 1$, when $|z| = 1$.

Let P_m be the class of all polynomials of degree at most m. Now we define $R_{m,n}$ by

$$R_{m,n} = R_{m,n}(a_1, a_2, \ldots, a_n) := \left\{\frac{p(z)}{w(z)}: p \in P_m \text{ and } m \leq n\right\}. \tag{5}$$

Thus, $R_{m,n}$ is the set of all rational functions with poles $a_1, a_2, \ldots, a_n$ at most and with finite limit at ∞.

From now on, we denote $T_k = \{z\colon |z| = k\}$, D_k^- is the set of all points inside T_k, and D_k^+ is the set of all points outside T_k.

In 1995, Li et al. [8] extended Bernstein-type inequalities to a rational function by replacing z^n by Blaschke product $B(z)$. They obtained the following results.

Theorem 1. *If $r(z) \in R_{n,n}$, then*

$$\left|r'(z)\right| \le \left|B'(z)\right|\left|r(z)\right|, \tag{6}$$

for $z \in T_1$. The inequality is sharp and equality holds for $r(z) = \alpha B(z)$ with $|\alpha| = 1$.

Li et al. [8] also proved the following results.

Theorem 2. *If $r(z) \in R_{n,n}$ and all the zeros of $r(z)$ lie in $T_1 \cup D_1^+$, then*

$$\left|r'(z)\right| \le \frac{\left|B'(z)\right|}{2} \max_{z\in T_1} |r(z)|, \tag{7}$$

for $z \in T_1$.

Theorem 3. *If $r(z) \in R_{n,n}$ and all the zeros of $r(z)$ lie in $T_1 \cup D_1^-$, then*

$$\left|r'(z)\right| \ge \frac{\left|B'(z)\right|}{2} \max_{z\in T_1} |r(z)|, \tag{8}$$

for $z \in T_1$.

Theorem 4. *Suppose $r(z) \in R_{m,n}$ where $r(z)$ has exactly n poles $a_1, a_2, \dots, a_n$ and all the zeros of $r(z)$ lie in $T_1 \cup D_1^-$. Then, for $z \in T_1$,*

$$\left|r'(z)\right| \ge \frac{1}{2}\left\{\left|B'(z)\right| - (n-m)\right\}|r(z)|, \tag{9}$$

where m is the number of zeros of $r(z)$.

An extension of Theorem 4 was shown by Wali and Shah [9].

Theorem 5. *Suppose $r(z) \in R_{m,n}$ where $r(z)$ has exactly n poles $a_1, a_2, \dots, a_n$ and all the zeros of $r(z)$ lie in $T_1 \cup D_1^-$. Then, for $z \in T_1$,*

$$\left|r'(z)\right| \ge \frac{1}{2}\left\{\left|B'(z)\right| - (n-m) + \frac{\left|c_m\right| - \left|c_0\right|}{\left|c_m\right| + \left|c_0\right|}\right\}|r(z)|, \tag{10}$$

where m is the number of zeros of $r(z)$.

2. Main Results

We use the following lemmas for proving our main results. The first lemma was shown by Dubinin [10] (see also [11]).

Lemma 6. *For polynomial $p(z) = c_n z^n + \cdots + c_0$ at each point z of the circle $|z| = 1$ at which $p(z) \ne 0$, the inequalities are valid:*

$$\operatorname{Re}\left(\frac{zp'(z)}{p(z)}\right) \ge \frac{n-1}{2} + \frac{\left|c_n\right|}{\left|c_n\right| + \left|c_0\right|}. \tag{11}$$

Lemma 7. *If $z \in T_1$, then*

$$\operatorname{Re}\left(\frac{zw'(z)}{w(z)}\right) = \frac{n - \left|B'(z)\right|}{2}. \tag{12}$$

Lemma 7 was proved by Aziz and Zargar [12]. The next lemma is due to Chan and Malik [13] and Li et al. [8].

Lemma 8. *If $p(z) = c_0 + \sum_{v=\mu}^{m} c_v z^v, 1 \le \mu \le m,$ is a polynomial of degree m having all the zeros in $T_k \cup D_k^+, k \ge 1$, then*

$$\left|p'(z)\right| \le \frac{m}{1+k^{\mu}} |p(z)|, \tag{13}$$

for $z \in T_1$.

Lemma 9. *If $r(z) \in R_{n,n}$ and $r^*(z) = B(z)\overline{r(1/\overline{z})}$, then*

$$\left|\left(r^*(z)\right)'\right| + \left|r'(z)\right| \le \left|B'(z)\right| |r(z)|, \tag{14}$$

for $z \in T_1$.

In this paper, firstly, we obtain an inequality for the modulus of the derivative of rational functions with prescribed poles and restricted zeros.

Theorem 10. *Suppose $r(z) = (p(z)/w(z)) \in R_{m,n}$ where $r(z)$ has exactly n poles $a_1, a_2, \dots, a_n$ and all the zeros of $r(z)$ lie in $T_k \cup D_k^+, k \ge 1$ except the zeros of order s lying in the origin. Then, for $z \in T_1$,*

$$\left|r'(z)\right| \le \frac{1}{2}\left[\left|B'(z)\right| + 2s - n + \frac{2(m-s)}{1+k^{\mu}}\right]|r(z)|, \tag{15}$$

where $p(z) = z^s\left(\sum_{v=\mu}^{m-s} c_v z^v\right), 1 \le \mu \le m-s$.

Proof. Let $r(z) = (p(z)/w(z))$ where $p(z) = z^s h(z)$ and $h(z) = c_0 + \sum_{v=\mu}^{m-s} c_v z^v, 1 \le \mu \le m-s$ has all its zeros in $T_k \cup D_k^+$.

It is easy to see that

$$\begin{aligned}\frac{r'(z)}{r(z)} &= \frac{w(z)p'(z) - p(z)w'(z)}{(w(z))^2} \cdot \frac{w(z)}{p(z)} \\ &= \frac{h'(z)}{h(z)} + \frac{s}{z} - \frac{w'(z)}{w(z)}.\end{aligned} \tag{16}$$

This implies that

$$\frac{zr'(z)}{r(z)} = \frac{zh'(z)}{h(z)} - \frac{zw'(z)}{w(z)} + s. \tag{17}$$

Hence,

$$\operatorname{Re}\left(\frac{zr'(z)}{r(z)}\right) = \operatorname{Re}\left(\frac{zh'(z)}{h(z)}\right) - \operatorname{Re}\left(\frac{zw'(z)}{w(z)}\right) + s. \tag{18}$$

Since $h(z)$ has all its zeros in $T_k \cup D_k^+, k \geq 1$, Lemma 8 implies that

$$\mathrm{Re}\left(\frac{zh'(z)}{h(z)}\right) \leq \left|\frac{zh'(z)}{h(z)}\right| \leq \frac{m-s}{1+k^{\mu}}. \tag{19}$$

Also, Lemma 7 yields

$$\mathrm{Re}\left(\frac{zw'(z)}{w(z)}\right) = \frac{n-\left|B'(z)\right|}{2}. \tag{20}$$

From (18), we get

$$\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) \leq \left(\frac{m-s}{1+k^{\mu}}\right) - \left(\frac{n-\left|B'(z)\right|}{2}\right) + s, \tag{21}$$

for $z \in T_1$. Hence, for $|z| = 1$ and using (18), we have

$$\begin{aligned}\left|\frac{z(r^*(z))'}{r(z)}\right|^2 &= \left|\left|B'(z)\right| - \frac{zr'(z)}{r(z)}\right|^2 \\ &= \left|B'(z)\right|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - 2\left|B'(z)\right|\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) \\ &\geq \left|B'(z)\right|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - 2\left|B'(z)\right|\left[\frac{m-s}{1+k^{\mu}} - \left(\frac{n-\left|B'(z)\right|}{2}\right) + s\right],\end{aligned} \tag{22}$$

for $z \in T_1$. That is,

$$\left[\left|r'(z)\right|^2 + \left(n - \frac{2(m-s)}{1+k^{\mu}} - 2s\right)\left|B'(z)\right|\left|r(z)\right|^2\right]^{(1/2)} \leq \left|(r^*(z))'\right|, \tag{23}$$

for $z \in T_1$. Combining this with Lemma 9, we get

$$\left|r'(z)\right|^2 + \left(n - \frac{2(m-s)}{1+k^{\mu}} - 2s\right)\left|B'(z)\right|\left|r(z)\right|^2 \leq \left(\left|B'(z)\right|\left|r(z)\right| - \left|r'(z)\right|\right)^2, \tag{24}$$

for $z \in T_1$. Therefore,

$$\left|r'(z)\right| \leq \frac{1}{2}\left[\left|B'(z)\right| + 2s - n + \frac{2(m-s)}{1+k^{\mu}}\right]\left|r(z)\right|, \tag{25}$$

for $z \in T_1$. This completes the proof. □

By taking $s = 0$ in Theorem 10, we get the next result.

Corollary 11. *Suppose $r(z) = (p(z)/w(z)) \in R_{m,n}$ where $r(z)$ has exactly n poles $a_1, a_2, \ldots, a_n$ and all the zeros of $r(z)$ lie in $T_k \cup D_k^+, k \geq 1$. Then, we have for $z \in T_1$,*

$$\left|r'(z)\right| \leq \frac{1}{2}\left[\left|B'(z)\right| - n + \frac{2m}{1+k^{\mu}}\right]\left|r(z)\right|, \tag{26}$$

where $p(z) = \sum_{v=\mu}^{m} c_v z^v, 1 \leq \mu \leq m$.

By taking $k = 1$ in Corollary 11, we get the following result.

Corollary 12. *Suppose $r(z) = (p(z)/w(z)) \in R_{m,n}$ where $r(z)$ has exactly n poles $a_1, a_2, \ldots, a_n$ and all the zeros of $r(z)$ lie in $T_1 \cup D_1^+$. Then, we have for $z \in T_1$,*

$$\left|r'(z)\right| \leq \frac{1}{2}\left[\left|B'(z)\right| - (n-m)\right]\left|r(z)\right|, \tag{27}$$

where m is the number of zeros of $r(z)$.

If $r(z) \in R_{m,m}$ in Corollary 12, it follows that Corollary 12 reduces to Theorem 2.

Remark 13. If $p \in P_n$, let us define $a_j = \alpha, |\alpha| \geq 1$, for $j = 1, 2, 3, \ldots, n$.

Then, $w(z) = (z-\alpha)^n$ and $r(z) = (p(z)/(z-\alpha)^n)$. We get

$$r'(z) = \frac{(z-\alpha)^n p'(z) - n(z-\alpha)^{n-1}p(z)}{(z-\alpha)^{2n}} = -\left[\frac{np(z)+(\alpha - z)p'(z)}{(z-\alpha)^{n+1}}\right] = \frac{-D_\alpha p(z)}{(z-\alpha)^{n+1}}, \tag{28}$$

where $D_\alpha p(z) = \mathrm{np}(z) + (\alpha - z)p'(z)$ is the polar derivative of polynomial $p(z)$ with respect to the point α. It generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\longrightarrow\infty} \frac{D_\alpha p(z)}{\alpha} = p'(z). \tag{29}$$

For $B(z) = (w^*(z)/w(z))$, we have $B'(z) = (n(|\alpha|^2-1)/(z-\alpha)^2)((1-\overline{\alpha}z)/(z-\alpha))^{n-1}$.

Hence, $|B'(z)| = (n(|\alpha|^2-1)/|z-\alpha|^2)$ for $|z|=1$.

Now, the next result is obtained for the polar derivative.

Corollary 14. *Suppose $p(z) = z^s h(z)$ where $h(z) = c_0 + \sum_{v=\mu}^{m-s} c_v z^v, 1 \le \mu \le m-s$ and $p(z)$ has all its zeros in $T_k \cup D_k^+, k \ge 1$, except the zeros of order s lying in the origin. Then,*

$$\left|D_\alpha p(z)\right| \le \frac{|\alpha|+1}{2}\left[2s + \frac{2(m-s)}{1+k^\mu}\right]|p(z)|, \tag{30}$$

for $z \in T_1$.

Dividing both sides of the inequality (30) by $|\alpha|$ and letting $|\alpha| \longrightarrow \infty$, we get the following result of Kumar and Lal [14].

Corollary 15. *Suppose $p(z) = z^s h(z)$ has all the zeros lying in $T_k \cup D_k^+, k \ge 1$ except the zeros of order s lying in the origin. Then, for $z \in T_1$,*

$$\left|p'(z)\right| \le \left[\frac{m + sk^\mu}{1+k^\mu}\right]|p(z)|, \tag{31}$$

where $h(z) = c_0 + \sum_{v=\mu}^{m-s} c_v z^v, 1 \le \mu \le m-s$.

Theorem 16. *Suppose $r(z) = (p(z)/w(z)) \in R_{m,n}$ where $p(z) = (z-z_0)^s h(z)$ has the zeros z_0 of order s with $|z_0| > 1$ and $h(z) = \sum_{j=0}^{m-s} c_{j+s} z^j$ has all its zeros lying in $T_1 \cup D_1^-$. Then,*

$$\left|r'(z)\right| \ge \frac{1}{2}\left[\left|B'(z)\right| - (n-m+s) + \frac{\left|c_m\right| - \left|c_s\right|}{\left|c_m\right| + \left|c_s\right|} + \frac{2s}{1+\left|z_0\right|}\right]|r(z)|, \tag{32}$$

for $z \in T_1$.

Proof. Let $r(z) = (p(z)/w(z))$ where $p(z) = (z-z_0)^s h(z)$ and $h(z) = \sum_{j=0}^{m-s} c_{j+s} z^j$ is a polynomial of degree $m-s$ having all its zeros in $T_1 \cup D_1^-$.

By differentiating with respect to z, we get

$$p'(z) = (z-z_0)^s h'(z) + h(z)s(z-z_0)^{s-1}. \tag{33}$$

It is easy to see that

$$\frac{r'(z)}{r(z)} = \frac{w(z)p'(z) - p(z)w'(z)}{(w(z))^2} \cdot \frac{w(z)}{p(z)} = \frac{h'(z)}{h(z)} + \frac{s}{z-z_0} - \frac{w'(z)}{w(z)}. \tag{34}$$

This implies that

$$\frac{zr'(z)}{r(z)} = \frac{zh'(z)}{h(z)} - \frac{zw'(z)}{w(z)} + \frac{sz}{z-z_0}. \tag{35}$$

Hence,

$$\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) = \mathrm{Re}\left(\frac{zh'(z)}{h(z)}\right) - \mathrm{Re}\left(\frac{zw'(z)}{w(z)}\right) + \mathrm{Re}\left(\frac{sz}{z-z_0}\right). \tag{36}$$

Lemma 6 and Lemma 7 yield

$$\begin{aligned}\mathrm{Re}\left(\frac{zr'(z)}{r(z)}\right) &\ge \left(\frac{m-s-1}{2} + \frac{\left|c_m\right|}{\left|c_m\right| + \left|c_s\right|}\right) - \left(\frac{n - \left|B'(z)\right|}{2}\right) + \mathrm{Re}\left(\frac{sz}{z-z_0}\right)\\ &= \frac{m-s-1-n}{2} + \frac{\left|c_m\right|}{\left|c_m\right| + \left|c_s\right|} + \frac{\left|B'(z)\right|}{2} + \frac{s}{1+\left|z_0\right|}\\ &= \frac{1}{2}\left[\left|B'(z)\right| - (n-m+s) + \frac{\left|c_m\right| - \left|c_s\right|}{\left|c_m\right| + \left|c_s\right|} + \frac{2s}{1+\left|z_0\right|}\right],\end{aligned} \tag{37}$$

for $z \in T_1$. Therefore,

$$\left|\frac{zr'(z)}{r(z)}\right| \geq \frac{1}{2}\left[\left|B'(z)\right| - (n-m+s) + \frac{|c_m| - |c_s|}{|c_m| + |c_s|} + \frac{2s}{1+|z_0|}\right], \tag{38}$$

for $z \in T_1$. That is,

$$\left|r'(z)\right| \geq \frac{1}{2}\left[\left|B'(z)\right| - (n-m+s) + \frac{|c_m| - |c_s|}{|c_m| + |c_s|} + \frac{2s}{1+|z_0|}\right]|r(z)|, \tag{39}$$

for $z \in T_1$. This completes the proof. □

By taking $s = 0$ in Theorem 16, we obtain that Theorem 16 reduces to Theorem 5.

By taking $s = 0$ and $n = m$ in Theorem 16, we obtain that Theorem 16 reduced to Corollary 2 of Wali and Shah [9].

Remark 17. From the conditions of Theorem 16, we have $|z_0| > 1$.

Thus, $(1 - |z_0|/1 + |z_0|)^s < 1$ for $s > 0$. That is, our lower bound in Theorem 16 is better than the lower bound in Theorem 3 of Mir et al. [15].

From Theorem 16, we obtain the following results in term of polar derivative.

Corollary 18. *Suppose $p(z) = (z - z_0)^s h(z)$ has the zeros z_0 of order s with $|z_0| > 1$ and $h(z) = \sum_{j=0}^{m-s} c_{j+s} z^j$ has all its zeros lying in $T_1 \cup D_1^-$. Then, for any complex number α with $|\alpha| \geq 1$,*

$$\left|D_\alpha p(z)\right| \geq \frac{(|\alpha| - 1)}{2}\left[(m-s) + \frac{|c_m| - |c_s|}{|c_m| + |c_s|} + \frac{2s}{1+|z_0|}\right]|p(z)|, \tag{40}$$

for $z \in T_1$.

If $s = 0$ in Corollary 18, we get the result below.

Corollary 19. *Suppose $p(z) \in P_m$ has all its zeros lying in $T_1 \cup D_1^-$. Then, for any complex number α with $|\alpha| \geq 1$,*

$$\left|D_\alpha p(z)\right| \geq \frac{|\alpha| - 1}{2}\left[m + \frac{|c_m| - |c_0|}{|c_m| + |c_0|}\right]|p(z)|, \tag{41}$$

for $z \in T_1$.

Dividing both sides of the inequality (40) by $|\alpha|$ and letting $|\alpha| \longrightarrow \infty$, we get the following result.

Corollary 20. *Suppose $p(z) = (z - z_0)^s h(z)$ has the zeros z_0 of order s with $|z_0| > 1$ and $h(z) = \sum_{j=0}^{m-s} c_{j+s} z^j$ has all its zeros lying in $T_1 \cup D_1^-$. Then,*

$$\left|p'(z)\right| \geq \frac{1}{2}\left[(m-s) + \frac{|c_m| - |c_s|}{|c_m| + |c_s|} + \frac{2s}{1+|z_0|}\right]|p(z)|, \tag{42}$$

for $z \in T_1$.

Corollary 21. *Suppose $r(z) = (p(z)/w(z)) \in R_{m,n}$ where $p(z) = (z - z_1)^{s_1}(z - z_0)^{s_0} h(z)$ has the zeros z_0 of order s_0 and the zeros z_1 of order s_1 with $|z_0|, |z_1| > 1$ and $h(z) = \sum_{j=0}^{m-(s_1+s_0)} c_{j+s_0+s_1} z^j$ has all its zeros lying in $T_1 \cup D_1^-$. Then,*

$$\left|r'(z)\right| \geq \left[\frac{1}{2}\left(\frac{|z_1| - 1}{|z_1| + 1}\right)^{s_1}\left(\left|B'(z)\right| - (n - m + s_0 + s_1) + \frac{|c_m| - |c_{s_0+s_1}|}{|c_m| + |c_{s_0+s_1}|} + \frac{2(s_0 + s_1)}{1+|z_0|}\right) - \frac{s_1}{1+|z_1|}\right]|r(z)|, \tag{43}$$

for $z \in T_1$.

Proof. Let $r(z) = (p(z)/w(z))$ where $p(z) = (z - z_1)^{s_1}(z - z_0)^{s_0} h(z)$ and $h(z) = \sum_{j=0}^{m-(s_0+s_1)} c_{j+s_0+s_1} z^j$ is a polynomial of degree $m - (s_0 + s_1)$ having all its zeros in $T_1 \cup D_1^-$.

Let $r(z) = (z - z_1)^{s_1} r_0(z)$ where $r_0(z) = ((z - z_0)^{s_0} h(z))/(w(z))$.

By differentiating with respect to z, we obtain

$$r'(z) = (z - z_1)^{s_1} r_0'(z) + s_1 r_0(z)(z - z_1)^{s_1 - 1}. \tag{44}$$

The reverse triangle inequality implies that

$$\left|r'(z)\right| \geq \left|(z - z_1)^{s_1}\right|\left|r_0'(z)\right| - s_1\left|r_0(z)(z - z_1)^{s_1-1}\right|. \tag{45}$$

Applying Theorem 16 to $r_0(z)$, we have

$$\left|r'(z)\right| \geq |z - z_1|^{s_1}\left[\frac{1}{2}\left(\left|B'(z)\right| - (n - m + s_0 + s_1) + \frac{|c_m| - |c_{s_0+s1}|}{|c_m| + |c_{s_0+s_1}|} + \frac{2(s_0 + s_1)}{1+|z_0|}\right)\left|r_0(z)\right|\right] - s_1\left|r_0(z)(z - z_1)^{s_1 - 1}\right|$$

$$\geq \frac{\left(|z_1|-1\right)^{s_1}}{2}\left[\left|B'(z)\right| - (n-m+s_0+s_1) + \frac{|c_m|-|c_{s_0+s_1}|}{|c_m|+|c_{s_0+s_1}|} + \frac{2(s_0+s_1)}{1+|z_0|}\right]|r_0(z)| - s_1\left|(z-z_1)\right|^{s_1-1}|r_0(z)|$$
$$\geq \left[\frac{\left(|z_1|-1\right)^{s_1}}{2}\left(\left|B'(z)\right| - (n-m+s_0+s_1) + \frac{|c_m|-|c_{s_0+s_1}|}{|c_m|+|c_{s_0+s_1}|} + \frac{2(s_0+s_1)}{1+|z_0|}\right) - s_1\left(|z_1|+1\right)^{s_1-1}\right]|r_0(z)|, \quad (46)$$

for $z \in T_1$.

Since $r(z) = (z-z_1)^{s_1} r_0(z)$, we get

$$|r_0(z)| \geq \frac{|r(z)|}{\left(1+|z_1|\right)^{s_1}}. \quad (47)$$

Therefore,

$$\left|r'(z)\right| \geq \left[\frac{\left(|z_1|-1\right)^{s_1}}{2}\left(\left|B'(z)\right| - (n-m+s_0+s_1) + \frac{|c_m|-|c_{s_0+s_1}|}{|c_m|+|c_{s_0+s_1}|} + \frac{2(s_0+s_1)}{1+|z_0|}\right) - s_1\left(|z_1|+1\right)^{s_1-1}\right]\frac{|r(z)|}{\left(1+|z_1|\right)^{s_1}}$$
$$\geq \left[\frac{1}{2}\left(\frac{|z_1|-1}{|z_1|+1}\right)^{s_1}\left(\left|B'(z)\right| - (n-m+s_0+s_1) + \frac{|c_m|-|c_{s_0+s_1}|}{|c_m|+|c_{s_0+s_1}|} + \frac{2(s_0+s_1)}{1+|z_0|}\right) - \frac{s_1}{1+|z_1|}\right]|r(z)|, \quad (48)$$

for $z \in T_1$. Thus, the proof is complete. □

Corollary 22. *Suppose $r(z) = (p(z)/w(z)) \in R_{m,n}$ where*

$$p(z) = (z-z_\nu)^{s_\nu}(z-z_{\nu-1})^{s_{\nu-1}}\cdots(z-z_0)^{s_0}h(z). \quad (49)$$

has the zeros $z_0, z_1, \ldots, z_\nu$ with $|z_i| > 1$ for $0 \leq i \leq \nu$ and $h(z) = \sum_{j=0}^{m-(s_0+s_1+\cdots+s_i)} c_{s_0+s_1+\cdots+s_i} z^j$ has all its zeros lying in $T_1 \cup D_1^-$. Then,

$$\left|r'(z)\right| \geq \left[\frac{1}{2}\left(\frac{|z_1|-1}{|z_1|+1}\right)^{s_1}\left(\frac{|z_2|-1}{|z_2|+1}\right)^{s_2}\cdots\left(\frac{|z_i|-1}{|z_i|+1}\right)^{s_i}\right.$$
$$\cdot\left(\left|B'(z)\right| - (n-m+s_0+s_1+\cdots+s_i) + \frac{|c_m|-|c_{s_0+s_1+\cdots+s_i}|}{|c_m|+|c_{s_0+s_1+\cdots+s_i}|} + \frac{2(s_0+s_1+\cdots+s_i)}{1+|z_0|}\right)$$
$$\left.-\frac{s_1}{1+|z_1|} - \cdots - \frac{s_i}{1+|z_i|}\right]|r(z)|, \quad (50)$$

for $0 \leq i \leq \nu$ and $z \in T_1$.

Proof. Let

$$r(z) = \frac{(z-z_\nu)^{s_\nu}(z-z_{\nu-1})^{s_{\nu-1}}\cdots(z-z_0)^{s_0}h(z)}{w(z)}, \quad (51)$$

where $r(z)$ has the zeros $z_0, z_1, \ldots, z_\nu$ with $|z_i| > 1$ for $0 \leq i \leq \nu$ and the remaining $n-(s_0+s_1+\cdots+s_\nu)$ zeros lie in $T_1 \cup D_1^-$.

Let $r_0(z) = (z-z_0)^{s_0}(h(z)/w(z))$ and $r(z) = (z-z_i)^{s_i} r_{i-1}$ for $1 \leq i \leq \nu$.

A lower bound of $|r_0'(z)|$ is obtained by Theorem 16. Using the fact that

$$|r_0(z)| \geq \frac{|r(z)|}{\left(1+|z_1|\right)^{s_1}}. \quad (52)$$

We get a lower bound of $|r(z)|$ as in Corollary 21.

Next, we can find a lower bound of $|r_i'(z)|$ for $1 \le i \le v$ by a similar process by using a lower bound of $|r_{i-1}'(z)|$ from the previous process and the fact

$$\left|r_{i-1}(z)\right| \ge \frac{|r(z)|}{\left(1+\left|z_i\right|\right)^{s_i}}, \tag{53}$$

for $1 \le i \le v$. Finally, we get

$$\begin{aligned}\left|r'(z)\right| \ge \Bigg[\frac{1}{2}\left(\frac{\left|z_1\right|-1}{\left|z_1\right|+1}\right)^{s_1}\left(\frac{\left|z_2\right|-1}{\left|z_2\right|+1}\right)^{s_2}\cdots\left(\frac{\left|z_i\right|-1}{\left|z_i\right|+1}\right)^{s_i}\\ \cdot\left(\left|B'(z)\right|-\left(n-m+s_0+s_1+\cdots+s_i\right)+\frac{\left|c_m\right|-\left|c_{s_0+s_1+\cdots+s_i}\right|}{\left|c_m\right|+\left|c_{s_0+s_1+\cdots+s_i}\right|}+\frac{2\left(s_0+s_1+\cdots+s_i\right)}{1+\left|z_0\right|}\right)\\ -\frac{s_1}{1+\left|z_1\right|}-\cdots-\frac{s_i}{1+\left|z_i\right|}\Bigg]|r(z)|,\end{aligned} \tag{54}$$

for $1 \le i \le v$. □

3. Conclusions

This paper gives an upper bound of a modulus of derivative of rational functions.

$$r(z)=\frac{z^s h(z)}{w(z)} \in R_{m,n}, \tag{55}$$

where $r(z)$ has exactly n poles $a_1, a_2, \ldots, a_n$ and all the zeros of $r(z)$ lie in $T_k \cup D_k^+, k \ge 1$ except the zeros of order s lying in the origin. Moreover, we give a lower bound of a modulus of derivative of rational functions.

$$r(z)=\frac{\left(z-z_v\right)^{s_v}\left(z-z_{v-1}\right)^{s_{v-1}}\cdots\left(z-z_0\right)^{s_0} h(z)}{w(z)}, \tag{56}$$

where $r(z)$ has the zeros $z_0, z_1, \ldots, z_v$ with $|z_i|>1$ for $0 \le i \le v$ and the remaining $n-(s_0+s_1+\cdots+s_v)$ zeros lie in $T_1 \cup D_1^-$.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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