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Research Article

Stability Conditions for a Nonlinear Time Series Model

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This research aims to determine whether the proposed time series model is stable or not. To achieve this, Ozaki's approximation method was utilized, where the nonlinear part is a function that approaches zero, similar to Ozaki's imposed function. The study produced examples of both stable and unstable models, and it was discovered that the stability and instability of the model are affected by the enforced optional constants. The researchers used the approximation method to obtain an asymptotic linear model that satisfied the singular point of the proposed model. The study first identified the singular points of the suggested model and then focused on determining the stability conditions, which was the main objective of the study. Last, the stability conditions of the limit cycle were established.

1. Introduction

It is of much importance to view the studies that have been done on Ozaki's method of approximation within the last few years. The method itself has been used differently by different scholars as it will be mentioned later. As for Ozaki himself, "'he used the local linearization method on the nonlinear exponential autoregressive model" [1]. Both "Mohammad and Salim used the same method on the logistic autoregressive model" [2]. Within the same, later "Salem and Ahmad used it on a nonlinear exponential autoregressive model" [3]. Sometimes later, "Mohammad and Mudhir used the same method but on the exponential (GARCH) models" [4]. After that "Khalaf and Mohammad used it on the Burr X autoregressive model" [5]. "Youns and Salim used this method on the nonlinear model with hyperbolic secant function" [6]. "Youns used this method on the nonlinear time series models with fractional functions" [7]. Also, it has been seen that "Noori and Mohammad used the same method on GJR-GARCH (Q, P) Mode l" [8] and "Hamdi et al. used it too but on stability conditions of Pareto autoregressive model " [9]. "Salim et al." applied it in [10]. "Chen et al. studied the stability, estimation, and applications for the generalized exponential autoregressive models of nonlinear time series" [11]. "Gan et al. studied the local linear RBF dependent AR model of nonlinear model" [12]. Linear model is stable if the roots of the deference equation are in the unit circle [13].

In essence, the Ozaki method relies on transforming the proposed nonlinear model into a linear autoregressive model that depends on the singular point which satisfies the nonlinear model. Consequently, the stability of the nonlinear model in the study is heavily influenced by the stability of the singular point.

This study addressed three examples that showcased the method of local linearization and illustrated the stability or instability of the singular point of a proposed nonlinear model of order one through graphing the model's orbits. Example 1 demonstrated the stable singular point, Example 2 demonstrated the unstable singular point, and Example 3 showcased both the unstable singular point and the unstable limit cycle of an order one model. Orbits were graphed for models with various initial values.

2. Ozaki Method

"The nonzero singular point and the limit cycle terms" are defined in references [2, 3].

2.1. Definition. "The exponential nonlinear autoregressive equation is defined as follows:

$$\begin{aligned} y_t &= \sum_{i=1}^{p} \left(u_i + v_i e^{-y_{t-1}^2} \right) y_{t-i} + \varepsilon_t \\ &= \left(u_1 + v_1 e^{-y_{t-1}^2} \right) y_{t-1} + \cdots + \left(u_p + v_p e^{-y_{t-1}^2} \right) y_{t-p} + \varepsilon_t. \end{aligned} \tag{1}$$

ε_t is a white noise, $u_1, \cdots, u_p; v_1, \cdots, v_p$ be the parameters (constants) for this model; then, it is said that the model is an autoregressive exponential model of the p order that symbolized with the symbol EXP AR (p)" [1, 13].

Theorem 1 (see [13]). *"Suppose that $p = 1$ at the above variation equation. So, we have $y_t = (u_1 + v_1 e^{-y_{t-1}^2}) y_{t-1} + \varepsilon_t$. Let q is a positive integer, the limit period of the period q. Then, $y_t, \cdots, y_{t+q}$ of an equation $y_t = (u_1 + v_1 e^{-y_{t-1}^2}) y_{t-1} + \varepsilon_t$ is orbitally stable, if it satisfies $|Z_{t+q}/Z_t| < 1$" [13].*

Theorem 2 (see [13]). *"Assumed that*

$$\begin{aligned} y_t &= \sum_{i=1}^{p} \left(u_i + v_i e^{-y_{t-1}^2} \right) y_{t-i} + \varepsilon_t \\ &= \left(u_1 + v_1 e^{-y_{t-1}^2} \right) y_{t-1} + \cdots + \left(u_p + v_p e^{-y_{t-1}^2} \right) y_{t-p} + \varepsilon_t. \end{aligned} \tag{2}$$

Therefore, $y_t, \cdots, y_{t+q}$ is a limit period of the period q. Wherever q is a positive integer, the previous equation is orbital stable, if the eigenvalues $|\lambda_i|$ for a matrix A, where $= A_q . A_{q-1} \cdots A_2 . A_1$, have the utter values smaller than one, $|\lambda_i| < 1$, so that

$$A_i = \begin{pmatrix} u_1 + \left\{ v_1 - 2\sum_{j=1}^{p} (v_j y_{t+i-j}) y_{t+i-1} \right\} e^{-y_{t+i-1}^2} & u_2 + v_2 e^{-y_{t+i-1}^2} & \cdots & \bullet & u_p + v_p e^{-y_{t+i-1}^2} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}'' \; [13].$$

2.2. Study Model. The suggested research model is defined in the p order such that

$$y_t = \sum_{i=1}^{p} \left[w_i + v_i \left(\frac{1}{e^{y_{t-1}} + 1} \right) \right] y_{t-i} + \varepsilon_t, \tag{3}$$

where $w_1, \cdots, w_p; v_1, \cdots, v_p$ indicate the parameters for the proposed research model and $\{\varepsilon_t\}$ is the white noise.

2.2.1. Nonlinear Research Model Observations. Regarding the research model, the following is noted clearly:

(1) When y_{t-1} approaches to positive infinity, the equation (3) model is transformed into a linear regression model:

$$y_t = \sum_{i=1}^{p} [w_i] y_{t-i} + \varepsilon_t. \tag{4}$$

(2) When y_{t-1} approaches to minus infinity, the equation (3) model is transformed into a linear regression model:

$$y_t = \sum_{i=1}^{p} [w_i + v_i] y_{t-i} + \varepsilon_t. \tag{5}$$

(3) When y_{t-1} equal to zero, the equation (3) model is transformed into a linear regression model:

$$y_t = \sum_{i=1}^{p} \left[w_i + \frac{v_i}{2} \right] y_{t-i} + \varepsilon_t. \tag{6}$$

3. Stability Conditions of the Proposed Study Model

The approximation technique was used to identify the stability for research model study when the order $i = 1, 2, \ldots, p$ in that portion of a publication.

The study model is

$$y_t = \sum_{i=1}^{p} \left[w_i + v_i \left(\frac{1}{e^{y_{t-1}} + 1} \right) \right] y_{t-i} + \varepsilon_t. \tag{7}$$

3.1. Finding the Singular Point Z. Assume you have a model of order one:

$$y_t = \left[w_i + v_i \left(\frac{1}{e^{y_{t-1}} + 1} \right) \right] y_{t-i} + \varepsilon_t. \tag{8}$$

Let $\varepsilon_t = 0$, $Z = f(Z)$ in (8).

Then, the unique point Z can be located:

$$Z = \left[w_1 + v_1 \left(\frac{1}{e^{Z} + 1} \right) \right] Z, \tag{9}$$

where $(Z \neq 0)$, $(v_1 \neq 0)$, and $(w_1 \neq 1)$.

Therefore,

$$Z = \log \left(\frac{v_1}{1 - w_1} - 1 \right). \tag{10}$$

The condition for the existence of a single point Z in equation (8) model is

$$\left(\frac{v_1}{1 - w_1} - 1 \right) > 0; \; 1 - w_1 \neq 0; \; v_1 \neq 0; \; w_1 \neq 1. \tag{11}$$

If the model's order is two,

$$y_t = \left[w_1 + v_1 \left(\frac{1}{e^{y_{t-1}} + 1} \right) \right] y_{t-1} + \left[w_2 + v_2 \left(\frac{1}{e^{y_{t-1}} + 1} \right) \right] y_{t-2} + \varepsilon_t. \tag{12}$$

Consider the following: $\varepsilon_t = 0$, $Z = f(Z)$.

Then,

$$Z = \left[w_1 + v_1\left(\frac{1}{e^Z + 1}\right)\right]Z + \left[w_2 + v_2\left(\frac{1}{e^Z + 1}\right)\right]Z. \quad (13)$$

The singular point Z for (12) is

$$Z = \log\left(\frac{\sum_{j=1}^{2} v_j}{1 - \sum_{i=1}^{2} w_i} - 1\right). \quad (14)$$

The singular point of a non-linear model of an equation (12) exists when

$$\left(\frac{\sum_{j=1}^{2} v_j}{1 - \sum_{i=1}^{2} w_i} - 1\right) > 0; \left(1 - \sum_{i=1}^{2} w_i\right) \neq 0; \left(\sum_{j=1}^{2} v_j\right) \neq 0. \quad (15)$$

The singular point of equation (3) was found by using similar method to (10), such as

$$Z = \log\left(\frac{\left(\sum_{i=1}^{p} v_i\right)}{\left(1 - \sum_{i=1}^{p} w_i\right)} - 1\right). \quad (16)$$

3.2. Stability Singular Point. Let $\varepsilon_t = 0$, and $y_s = Z + Z_s$, $\forall s = t; t-1$, in (6), $Z_s; \forall s = t, t-1$ is very small.

Therefore, $Z_s^n \longrightarrow 0; \forall n \geq 2, \forall s = t; t-1$; $Z_t.Z_{t-1} = 0$. The Taylor expansion series is then used.

$$Z + Z_t = \left[w_1 + v_1\left(\frac{1}{e^{(Z+Z_{t-1})} + 1}\right)\right](Z + Z_{t-1}). \quad (17)$$

To obtain

$$Z_t = \left[\left(\frac{w_1\left(e^Z + 1\right) + w_1 Ze^Z + v_1 - Ze^Z}{\left(e^Z + 1\right)}\right)\right]Z_{t-1}. \quad (18)$$

Therefore,

$$Z_t = a_1 Z_{t-1}; a_1 = \left[\left(\frac{w_1\left(e^Z + 1\right) + w_1 Ze^Z + v_1 - Ze^Z}{\left(e^Z + 1\right)}\right)\right]. \quad (19)$$

As a result, when the root of (16) is inside the unit circle, (9) is a stable model of order one.

$$|r_1| = |a_1| < 1. \quad (20)$$

The following represents the stationary conditions of singular point of (10):

Let $\varepsilon_t = 0$, $\forall s = t, t-1, t-2$; $y_s = Z + Z_s$; $Z_s; \forall s = t, t-1, t-2$ is quite tiny, therefore $Z_s^n \longrightarrow 0; \forall n \geq 2 \forall s = t, t-1, t-2$, $Z_{t-s}.Z_{t-s} = 0, \forall s = 0, 1, 2$, we have that via making use of the Taylor expansion series.

$$Z + Z_t = \left[w_1 + v_1\left(\frac{1}{e^{(Z+Z_{t-1})} + 1}\right)\right](Z + Z_{t-1}) + \left[w_2 + v_2\left(\frac{1}{e^{(Z+Z_{t-1})} + 1}\right)\right](Z + Z_{t-2}). \quad (21)$$

Then,

$$Z_t = \left[\frac{w_1\left(e^Z + 1\right) + w_1 Ze^Z + v_1 - e^Z Z + w_2 Ze^Z}{\left(e^Z + 1\right)}\right]Z_{t-1} + \left[\frac{w_2\left(e^Z + 1\right) + v_2}{\left(e^Z + 1\right)}\right]Z_{t-2}. \quad (22)$$

Therefore,

$$Z_t = a_1 Z_{t-1} + a_2 Z_{t-2},$$
$$a_1 = \left[\frac{w_1\left(e^Z + 1\right) + w_1 Ze^Z + v_1 - e^Z Z + w_2 Ze^Z}{\left(e^Z + 1\right)}\right], \quad (23)$$
$$a_2 = \left[\frac{w_2\left(e^Z + 1\right) + v_2}{\left(e^Z + 1\right)}\right].$$

The distinguishing equation $(v - r_1).(v - r_2) = v^2 - a_1(v) - a_2 = 0$.

Thus, $a_1 = (r_1 + r_2), a_2 = -r_1 r_2$.

Then, the roots of $v^2 - a_1 v - a_2 = 0$ are r_1, r_2.

Hence, the stationary condition is $|r_i| < 1$; $\forall i = 1, 2$.

The singular point in a nonlinear model of an equation (3) has the following condition:

$$\left(\frac{\sum_{i=1}^{p} v_i}{\left(1 - \sum_{i=1}^{p} w_i\right)} - 1\right) > 0; \left(1 - \sum_{i=1}^{p} w_i\right) \neq 0; \left(\sum_{i=1}^{p} v_i\right) \neq 0. \quad (24)$$

The stability of the singular point is defined as follows:

$$Z_t = \sum_{i=1}^{p} a_i Z_{t-i},$$
$$a_1 = \frac{w_1\left(e^Z + 1\right) + w_1 Ze^Z + v_1 - Ze^Z + \sum_{i=2}^{p} w_i Ze^Z}{\left(e^Z + 1\right)},$$
$$a_i = \frac{w_i\left(e^Z + 1\right) + v_i}{\left(e^Z + 1\right)}, \quad \text{for all } i = 2, \cdots, p. \quad (25)$$

Then, when the roots' absolute values $|r_i|$ for $v^p - a_1 v^{p-1} - a_2 v^{p-2} - a_3 v^{p-3} - \cdots - a_p = 0$, must become situated within a unitary circle, a conditional of stability singular point to the suggested model is obtained.

3.3. The Limit Cycle. The limit cycle of the period q of (6) has the following form: $y_t, y_{t+1}, \cdots, y_{t+q} = y_t$; then, all y_s replaced by $y_s = y_s + Z_s$ and then substitute y_t by $y_t + Z_t$, and y_{t-1} by $y_{t-1} + Z_{t-1}$, $\varepsilon_t = 0$.

$$y_t + Z_t = \left[w_1 + v_1 \left(\frac{1}{e^{(y_{t-1}+Z_{t-1})} + 1} \right) \right] (y_{t-1} + Z_{t-1}). \quad (26)$$

To reach at

$$Z_t = \left[\left(\frac{w_1 (e^{y_{t-1}} + 1) + w_1 y_{t-1} e^{y_{t-1}} + v_1 - e^{y_{t-1}} y_t}{(e^{y_{t-1}} + 1)} \right) \right] Z_{t-1}. \quad (27)$$

Analogous way to Theorem 1 for determining $|Z_{t+q}/Z_t| < 1$ [9].

When $t = t + q$ in $Z_t = [(w_1 (e^{y_{t-1}} + 1) + w_1 y_{t-1} e^{y_{t-1}} + v_1 - e^{y_{t-1}} y_t / (e^{y_{t-1}} + 1))] Z_{t-1}$,

$$Z_{t+q} = \left[\left(\frac{w_1 (e^{y_{t+q-1}} + 1) + w_1 y_{t+q-1} e^{y_{t+q-1}} + v_1 - e^{y_{t+q-1}} y_{t+q}}{(e^{y_{t+q-1}} + 1)} \right) \right] Z_{t+q-1}. \quad (28)$$

Then,

$$Z_{t+q} = \prod_{i=1}^{q} \left[\left(\frac{w_1 (e^{y_{t+q-i}} + 1) + w_1 y_{t+q-i} e^{y_{t+q-i}} + v_1 - e^{y_{t+q-i}} y_{t+q-(i-1)}}{(e^{y_{t+q-i}} + 1)} \right) \right] Z_t. \quad (29)$$

As a result, apply Theorem 1 to (29).

Therefore,

$$\left| \frac{Z_{t+q}}{Z_t} \right| = \left| \prod_{i=1}^{q} \left[\left(\frac{w_1 (e^{y_{t+q-i}} + 1) + w_1 y_{t+q-i} e^{y_{t+q-i}} + v_1 - e^{y_{t+q-i}} y_{t+q-(i-1)}}{(e^{y_{t+q-i}} + 1)} \right) \right] \right| < 1. \quad (30)$$

Hence, (30) is

$$\left| \frac{Z_{t+q}}{Z_t} \right| = \left| \prod_{i=1}^{q} \left[\left(\frac{w_1 (e^{y_{t+i-1}} + 1) + w_1 y_{t+i-1} e^{y_{t+i-1}} + v_1 - e^{y_{t+i-1}} y_{t+i}}{(e^{y_{t+i-1}} + 1)} \right) \right] \right| < 1. \quad (31)$$

The formula for the research model (12) is $y_t, \cdots, y_{t+q} = y_t$.

Then, the points y_s approaching the limit cycle are $y_s = y_s + Z_s; \forall s$,

$$y_t + Z_t = \left[w_1 + v_1 \left(\frac{1}{e^{y_{t-1}+Z_{t-1}} + 1} \right) \right] . (y_{t-1} + Z_{t-1}) + \left[w_2 + v_2 \left(\frac{1}{e^{y_{t-1}+Z_{t-1}} + 1} \right) \right] (y_{t-2} + Z_{t-2}) + \varepsilon_t. \quad (32)$$

Then,

$$Z_t = \left[\frac{w_1 (e^{y_{t-1}} + 1) + w_1 y_{t-1} e^{y_{t-1}} + v_1 - e^{y_{t-1}} y_t + w_2 e^{y_{t-1}} y_{t-2}}{(e^{y_{t-1}} + 1)} \right] Z_{t-1} + \left[\frac{w_2 (e^{y_{t-1}} + 1) + v_2}{(e^{y_{t-1}} + 1)} \right] Z_{t-2}. \quad (33)$$

As a result, similar to Theorem 1, calculate a value $|Z_{t+q}/Z_t| < 1$ [13].

When, $t = t + q$ in (33) to obtain

$$Z_{t+q} = \left[\frac{w_1\left(e^{y_{t+q-1}} + 1\right) + w_1 y_{t+q-1} e^{y_{t+q-1}} + v_1 - e^{y_{t+q-1}}. y_{t+q} + w_2 e^{y_{+qt-1}} y_{t+q-2}}{\left(e^{y_{t+q-1}} + 1\right)}\right] Z_{t+q-1} + \left[\frac{w_2\left(1 + e^{y_{t+q-1}}\right) + v_2}{\left(e^{y_{t+q-1}} + 1\right)}\right] Z_{t+q-2}, \tag{34}$$

$$Z_{t+q} = \left\{\prod_{i=1}^{q}\left[\frac{w_1\left(e^{y_{t+i-1}} + 1\right) + w_1 y_{t+i-1} e^{y_{t+i-1}} + v_1 - e^{y_{t+i-1}} y_{t+i} + w_2 e^{y_{t+i-1}} y_{t+i-2}}{\left(e^{y_{t+i-1}} + 1\right)}\right] + \prod_{i=1}^{q-1}\left[\frac{w_2\left(e^{y_{t+i}} + 1\right) + v_2}{\left(e^{y_{t+i}} + 1\right)}\right]\right\} Z_t. \tag{35}$$

When the stability condition is met, then (35) is orbitally stable.

$$\left|\frac{Z_{t+q}}{Z_t}\right| = \left|\left\{\prod_{i=1}^{q}\left[\frac{w_1\left(e^{y_{t+i-1}} + 1\right) + w_1 y_{t+i-1} e^{y_{t+i-1}} + v_1 - e^{y_{t+i-1}} y_{t+i} + w_2 e^{y_{t+i-1}} y_{t+i-2}}{\left(e^{y_{t+i-1}} + 1\right)}\right] + \prod_{i=2}^{q}\left[\frac{w_2\left(e^{y_{t+i-1}} + 1\right) + v_2}{\left(e^{y_{t+i-1}} + 1\right)}\right]\right\}\right| < 1. \tag{36}$$

Theorem 3. *When the suggested model of (3) has a limit cycle of period q* $y_t, \cdots, y_{t+q} = y_t$, *then it is orbitally stable if every eigenvalues of the matrix A,* $A = A_q.A_{q-1} \cdots A_1$ *with an absolute value is smaller than 1,* $|r_i| < 1, i = 1, 2, \cdots, p$ *and*

$$A_i = \begin{pmatrix} \dfrac{w_1 m_1 + w_1 m_2 y_{t+i-1} + v_1 - m_2 y_{t+i} + w_2 m_2 y_{t+i-2} + \sum_{j=3}^{p} w_j m_2 y_{t+i-j}}{m_1} & \dfrac{w_2 m_1 + v_2}{m_1}. & \cdots \cdots & \dfrac{w_p m_1 + v_p}{m_1} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ . & . & \cdots . & . \\ . & . & \cdots . & . \\ . & . & \cdots . & . \\ . & . & \cdots . & . \\ . & . & \cdots . & . \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{37}$$

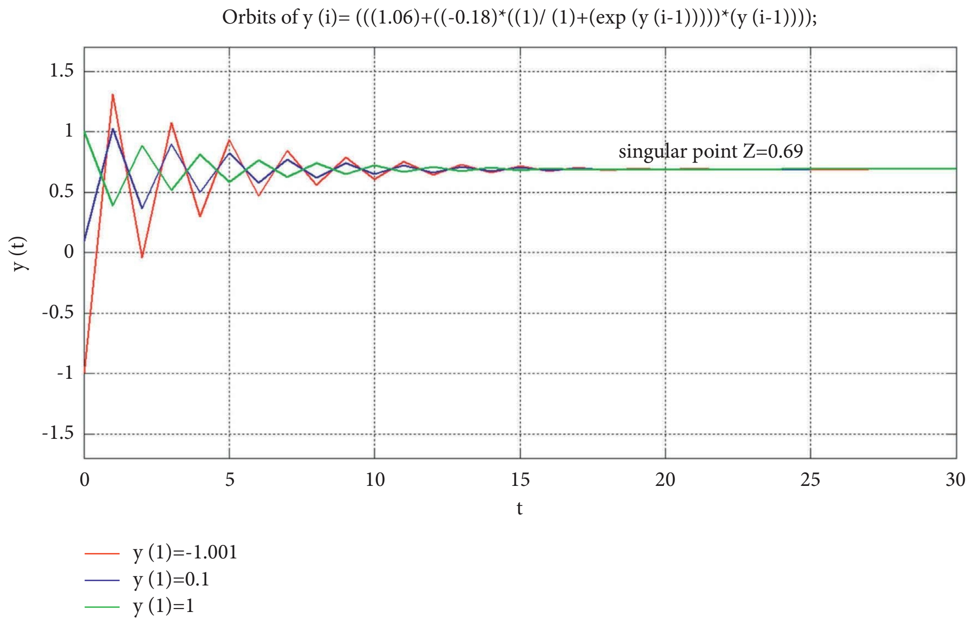

Figure 1: Stable singular point $Z = 0.69$ with different initial values.

where $m_1 = e^{y_{t+i-1}} + 1; m_2 = e^{y_{t+i-1}}$.

4. Examples

Regarding the examples presented in this study, Examples 1 and 2 demonstrate the method of identifying and locating the true singular point of the model under study, as well as satisfying the stability condition for the singular point and graphing the model's orbits. Meanwhile, Example 3 clarifies the concept of an unstable limit cycle.

Example 1. Where $w_1 = 1.06$, $v_1 = -0.18$ in (6) to reach $y_t = [1.06 + (-0.18)(1/1 + e^{y_{t-1}})]y_{t-1} + \varepsilon_t$.

Then, the model meets the condition $(((v_1)/(1 - w_1)) - 1) = (-0.18/1 - 1.06 - 1) = ((-0.18/-0.06) - 1) = (3 - 1) = 2 > 0$.

As a result of using equation (10), we get

$$\begin{aligned} Z &= \log\left(\frac{v_1}{1 - w_1} - 1\right) \\ &= \log(2) \\ &= 0.69. \end{aligned} \tag{38}$$

While = 0.69 and applied (19) to get $Z_t = 0.86Z_{t-1}$.

Since the root of the preceding equation is in the unit circle, a singular (solitary) point for this case is found stable. The following shape Figure 1 illustrate the model's stability with varying beginning values.

Example 2. Let $w_1 = 1.09$, $v_1 = -0.31$ in (6) to reach $y_t = [1.09 + (-0.31)(1/1 + e^{y_{t-1}})]y_{t-1} + \varepsilon_t$ that satisfies the following condition:

$$\begin{aligned} \left(\frac{v_1}{1 - w_1} - 1\right) &= \left(\frac{-0.31}{1 - 1.09} - 1\right) \\ &= \left(\frac{-0.31}{-0.09} - 1\right) \\ &= \left(\frac{0.31}{0.09} - 1\right) \\ &= (3.4444 - 1) \\ &= 2.4444 > 0. \end{aligned} \tag{39}$$

Consequently, Z can be determined, such that by utilizing equation (10).

$$\begin{aligned} Z &= \log\left(\frac{v_1}{1 - w_1} - 1\right) \\ &= \log(2.4444) \\ &= 0.8938. \end{aligned} \tag{40}$$

Since $Z = 0.8938$ and applied (19) to reach $Z_t = 1.0571Z_{t-1}$. Then, $Z = 0.8938$ of Example 2 is not stable because the root $r = 1.0571$ is located outside the unit circle.

Since $Z = 0.8938$ and $Z_t = 1.0571Z_{t-1}$ is deduced, the model of Example 2 is unstable since the root is outside the unity circle.

The figures show the model orbits which are not stable for the different starting values. Figure 2 displays the trajectory of the model with an initial value $y(0) = 1$. Figure 3 displays the trajectory of the model with an initial value $y(0) = 0.001$, and Figure 4 displays the trajectory of the model with an initial value $y(0) = -1$.

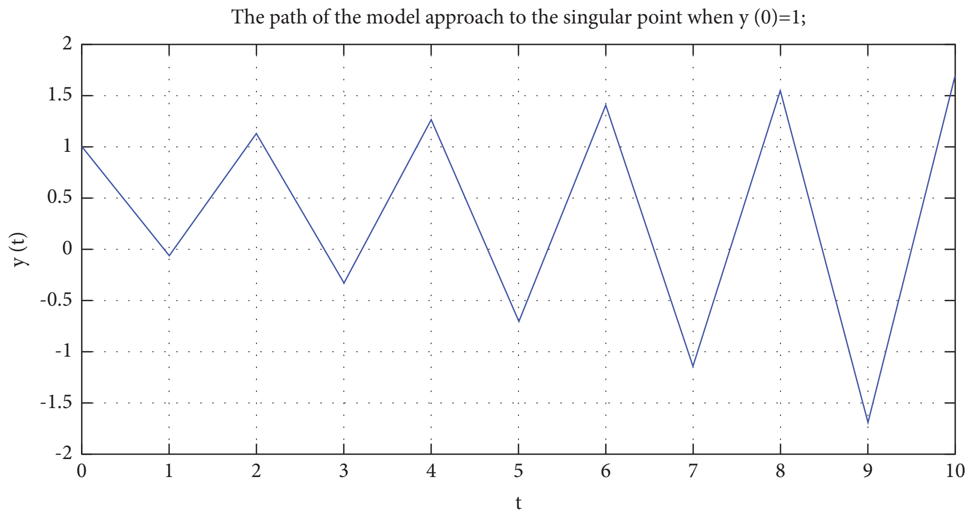

Figure 2: Unstable singular point $Z = 0.8938$ when $y(0) = 1$.

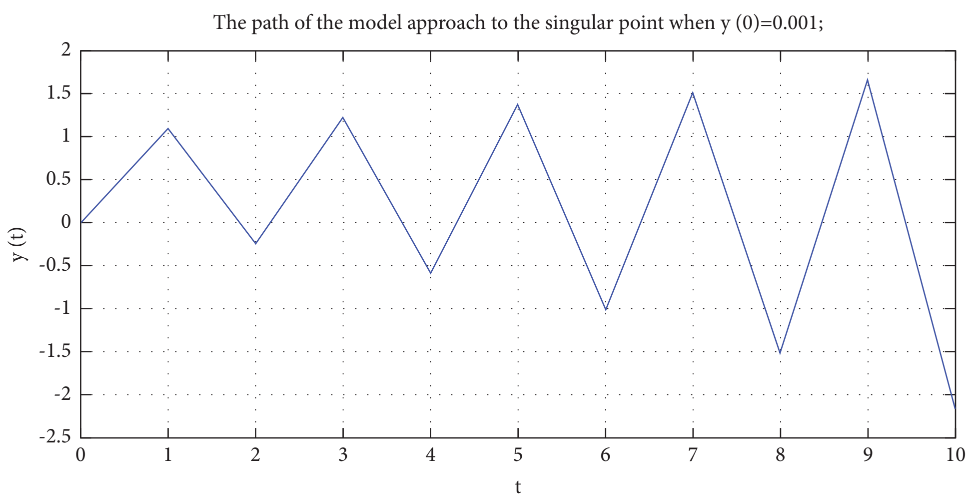

Figure 3: Unstable singular point $Z = 0.8938$ when $y(0) = 0.001$.

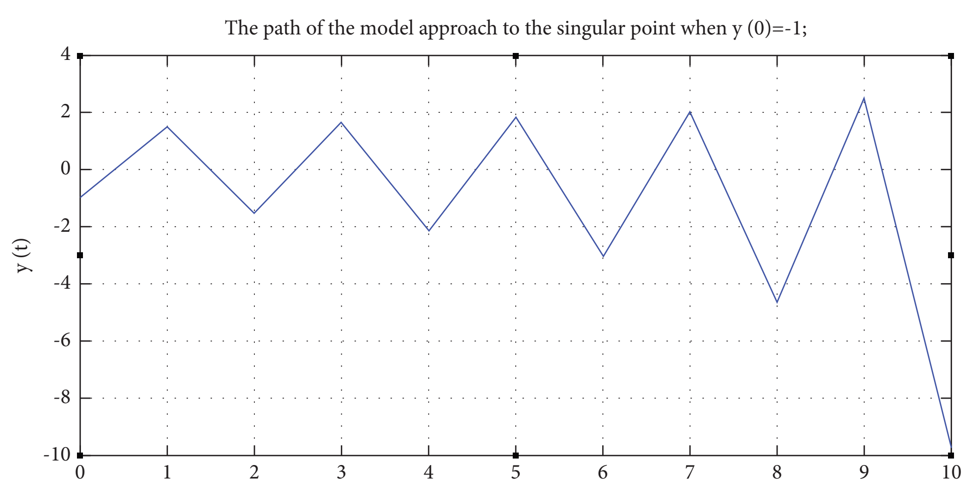

Figure 4: Unstable singular point $Z = 0.8938$ when $y(0) = -1$.

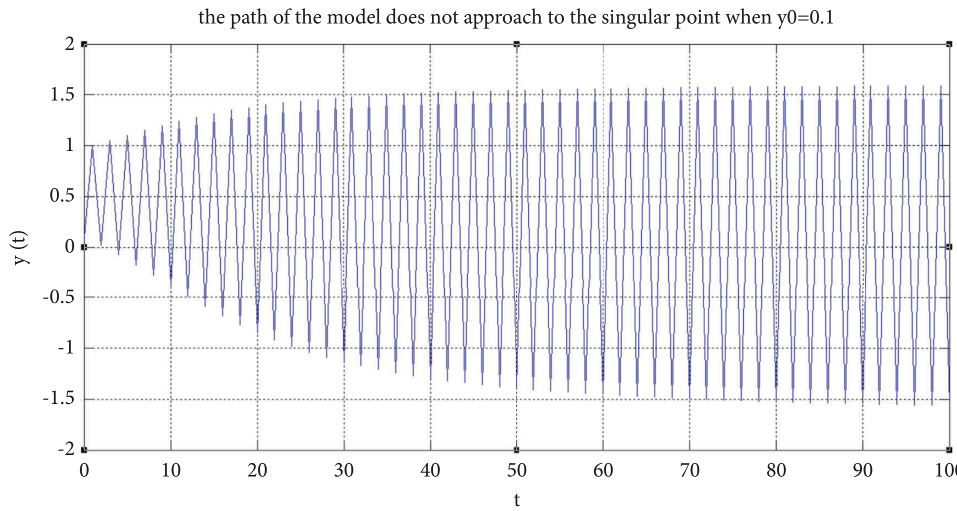

FIGURE 5: Unstable singular point $Z = 1.2$ when $y(0) = 0.1$.

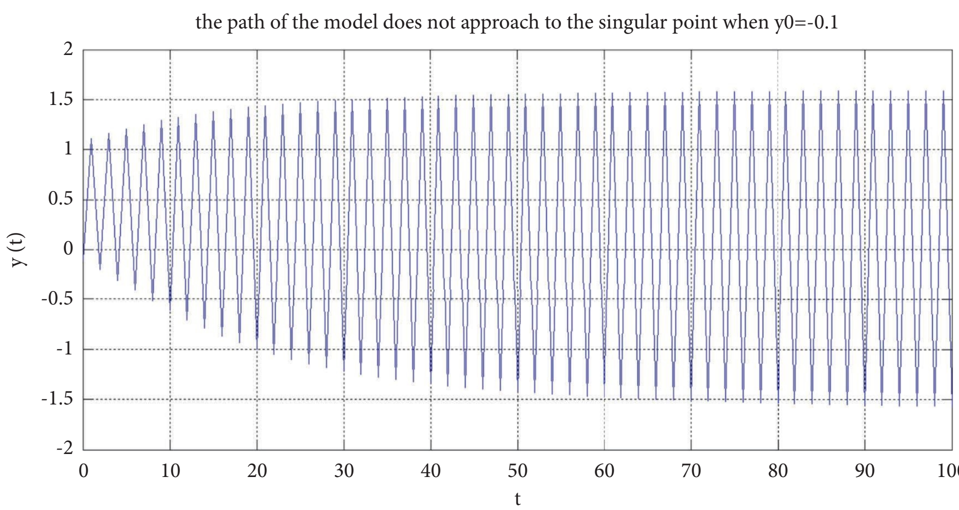

FIGURE 6: Unstable singular point $Z = 1.2$ when $y(0) = -0.1$.

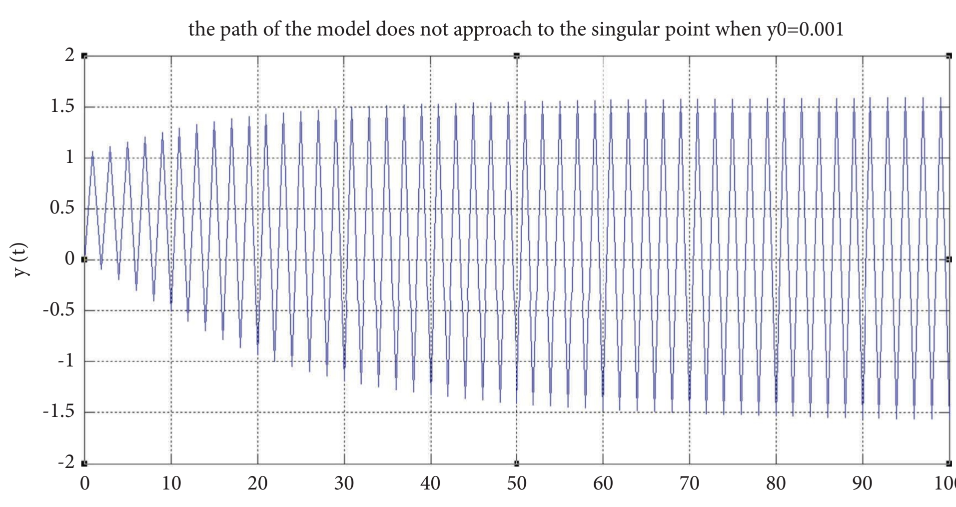

FIGURE 7: Unstable singular point $Z = 1.2$ when $y(0) = 0.001$.

Example 3. Since $w_1 = 1.06$, $v_1 = -0.28$ in (6), then $y_t = [1.06 + (-0.28)(1/1 + e^{y_{t-1}})]y_{t-1} + \varepsilon_t$; it fulfills the requirement such that

$$\begin{aligned}\left(\frac{(v_1)}{(1-w_1)} - 1\right) &= \left(\frac{-0.28}{1-1.06} - 1\right)\\ &= \left(\frac{-0.28}{-0.06} - 1\right)\\ &= (4.6 - 1)\\ &= 3.6 > 0.\end{aligned} \quad (41)$$

In order to obtain the nonzero real singular points such that equation (10), $Z = \log((v_1/1 - w_1) - 1) = \log(3.6) = 1.2$.

Since $Z = 1.2$ and by applying (19), then $Z_t = 1.1(Z_{t-1})$ was obtained. As a result, Example 3 is an unstable model simply because the root $r = 1.1$ lies outside the unity circle. The limit cycle is unstable in the period $q = 2$, which include $\{1.5; -1.5; 1.5\}$, where equation (31) was employed.

$$\left|\prod_{i=1}^{q=2}\left[\frac{1.06(e^{y_{t+i-1}} + 1) + 1.06y_{t+i-1}e^{y_{t+i-1}} + (-0.28) - e^{y_{t+i-1}})y_{t+i}}{(e^{y_{t+i-1}} + 1)}\right]\right| == |(3.5)(-0.78)| = 2.74 > 1. \quad (42)$$

The following figures show the paths inception from various starting values:

Figure 5 displays the trajectory of the model with an initial value $y(0) = 0.1$, Figure 6 displays the trajectory of the model with an initial value $y(0) = -0.1$, and Figure 7 displays the trajectory of the model with an initial value $y(0) = 0.001$.

5. Conclusions

The stability criteria for the suggested nonlinear time series model were established, along with the limit cycle stabilization condition for the model, and numerical examples that met these conditions were identified. It was found that the stability factors of the model depended on the values of its arbitrary constants and the selected nonlinear function.

Furthermore, if the singular point is stable, then the proposed research model is also stable, and the opposite is also true. The proposed model is a unique case among all other general cases as its nonlinear component represents a decreasing function.

As the decreasing nonlinear function proposed by the study model such that $(1/e^{y_{t-1}} + 1)$

It is a special case of all cases $e^{-y_{t-1}}$, $e^{-y_{t-1}^2}$, $(1/y_{t-1}^2 - 1)$, $\sec h(y_{t-1})$, ...

It is also possible to use other methods in the future to analyze and find stability conditions for different nonlinear models.

Data Availability

No data supporting this study are made availabe.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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