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Research Article

On the Construction of Traveling Water Waves with Constant Vorticity and Infinite Boundary

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The issue of whether there is a closed orbit in the water waves in an infinite boundary condition is an outstanding open problem. In this work, we first discuss the various developments on the structure of water waves in the context of finite bottom conditions. We then focus on the behavior of water for the kinematic boundary for the infinite depth. We present some findings to address this issue by creating a water wave profile for the zero and constant vorticity conditions through the application of the Crandall–Rabinowitz theorem.

1. Introduction

In the nineteenth century, Airy and Stokes started the mathematical study of wave theory within the framework of linear theory. Stokes water waves are nonlinear and periodic surface waves on an inviscid fluid layer of constant mean depth, which were introduced in the mid-nineteenth century. These waves are the progressive periodic waves of permanent form. These water waves, which propagate on the water surface of a sea or river, are periodic two-dimensional waves.

Stokes noticed the actual water wave characteristics and then studied the nonlinear governing equations extensively for water waves [1]. The governing equations for the two-dimensional irrotational gravity water waves have the Hamiltonian structure [2–4]. The irrotational flow (zero vorticity) was considered in most of the theoretical works of surface water waves. In most of the studies, incompressible Euler equations with free boundary conditions are considered governing equations for water waves. The insight into the dynamics of waves of large amplitude is given in [5–8] with the help of a qualitative understanding of periodic traveling waves in an irrotational flow over the flatbed and also the aspects of propagation of irrotational waves of the variable bottom are discussed [9]. The authors in [10–12] dealt with the full nonlinearity of the governing equations for the water waves providing valuable quantitative information.

Constantin and Villari proved that in linear periodic gravity water waves, there are no closed orbits for the water particles in the fluid and the paths are approximately elliptical, and they considered the flow up to the finite depth for the kinematic boundary condition [13].

Ehrnstrom and Villari showed that for the positive vorticity, the situation resembles that of Stokes waves. For large enough vorticity, the particle trajectories are affected. Also, for the negative vorticity, there is the appearance of internal waves, and vortices and the trajectories are not closed ellipses [14]. The author in [15] presented a mathematical formulation of the water wave problem. He gave the existence result for a small amplitude solution based on the bifurcation theory. Also, he described the particle motion in the physical frame and obtained the fluid in a moving frame, and confirmed the prediction from the linear theory [14].

The authors in [16] showed that the steady periodic deep-water waves are symmetric and propagate against a current with vorticity if their profile is monotone between crests and troughs. They discussed the traveling waves for water waves with zero and nonzero vorticity and showed that symmetry is valid, particularly for the irrotational waves. They also showed that vorticity distribution vanishes at the infinite depth. Constantin and Strauss [17] constructed the two-dimensional inviscid periodic traveling waves with vorticity with the consideration of a free surface under the effect of gravity over a flat bottom. These waves are symmetric and monotone between each crest and trough. They used the bifurcation theory and degree theory for the construction.

The authors in [18] constructed the periodic traveling waves with vorticity, which are symmetric waves whose profiles are monotone between each crest and trough. For this, they used bifurcation theory and considered the water wave problem described by the Euler equation with a free surface over a flat bottom. Also, the authors in [19] proved that the profile of a periodic traveling wave propagating on the surface of the water above a flatbed in a flow with real analytic vorticity must be real analytic with the assumption that wave speed exceeds the horizontal fluid velocity throughout the flow.

Recently with a degenerate diffusivity, Eyring–Powell viscosity term, and a Darcy– Forchheimer law in the porous medium, the authors in [20] obtained the traveling wave profile and showed the existence of the asymptotic solution using the geometric perturbation theory. They also showed the existence of an exponential profile of the solution under an asymptotic approximation. Similarly, the authors in [21] explored the solutions in the traveling wave domain with the use of geometric perturbation theory. Furthermore, the authors in [22] transformed the problem into the study of traveling wave-kind solutions. With the use of geometric perturbation theory, they confirmed the existence of an exponentially decaying rate in the traveling wave profile. Both the authors [21, 22] performed numerical simulations to validate their results.

The authors in [23] explored the regularity, existence, and uniqueness of the solutions with the use of variational weak formulations. Then they transformed the Eyring –Powell equation into a traveling wave domain where they obtained the analytical solution with the use of geometric perturbation theory. Their main work was to show the existence of an exponential traveling wave tail together with a certain minimizing error critical speed.

All of these existing results apply only to the finite boundary, but this problem remains open for the infinite boundary. With the ideas in [18, 19], and [17], and using the Crandall–Rabinowitz theorem, we created the water waves profile by considering the kinematic boundary condition up to infinity. We studied this as an extension of the flat bed at a finite depth to an infinite depth. Our findings will be useful for the study of the extension of the solutions from finite boundary conditions to infinite boundary conditions. Similar concepts can be used in other situations while dealing with problems in fluid dynamics with infinite boundaries.

In Section 2, we present the equation of motion with boundary conditions. In Section 3, the use of the Crandall–Rabinowitz theorem is proposed to guarantee the existence of traveling waves for water waves with zero and nonzero vorticity for the kinematic boundary condition up to the infinite depth. Section 4 concludes the paper.

2. Equation of Motion

The equation of motion of the wave is given by the following:

$$\partial_t \vec{u} + (\vec{u}.\nabla)\vec{u} + \nabla P = -g\vec{e}_y, \operatorname{div}\vec{u} = 0, \tag{1}$$

where u is the velocity, P is the pressure, and g is the acceleration due to gravity.

The related boundary conditions are as follows:

(1) Kinematic: The free surface with moving fluid is given by $\eta_t = \sqrt{1 + (\nabla\eta)^2}\vec{u} \cdot \vec{n}$ where $\vec{n}$ is the unit normal to the surface

(2) Dynamic: The dynamic boundary condition is the balance of forces at the free surface and is given by $P = P_{\mathrm{atm}}$, the atmospheric pressure

(3) Bottom: At the bottom, the second component of the fluid is zero. It just moves horizontally i.e.,

$$\lim_{y\to-\infty} \vec{e_y}.\vec{u} = 0. \tag{2}$$

With these, the equation of motion is given by $\Delta\phi = 0$ in D_η, $\phi = \psi$ at $y = \eta(t, x)$, and

$$\lim_{y\to-\infty} \partial_y \phi = 0. \tag{3}$$

Here, ϕ is the velocity potential function, D_η is the domain of the fluid motion, $\eta(t, x)$ is the free surface which describes a wave on the bottom of the fluid dependent on time t and $\psi(t, x)$ is the value of the potential at the free surface.

3. Traveling Waves for Water Waves

In this section, the water waves for the two cases with zero vorticity and constant vorticity are discussed.

3.1. Traveling Waves for Water Waves with Zero Vorticity. The water waves formulation [24, 25] is given by the following:

$$\begin{aligned}\eta_t &= G(\eta)[\psi],\\ \psi_t &= -g\eta - \frac{\psi_x^2}{2} + \frac{1}{2}\frac{\left(G(\eta)\psi + \eta_x\psi_x\right)^2}{1+\eta_x^2},\end{aligned} \tag{4}$$

where $G(\eta)[\psi]$ is a Dirichlet–Neumann operator [26, 27] and is given by

$$G(\eta)\psi = \sqrt{1+\eta_x^2}\partial_{\vec{n}}\phi|_{y=\eta(x)}. \tag{5}$$

Here, we are looking for the small size traveling wave periodic in x. That means we expect the solution of the form $\eta(t,x)=\breve{\eta}(x+ct)$ and $\psi(t,x)=\breve{\psi}(x+ct)$, where c is the phase velocity. These solutions must satisfy the equations as follows:

$$c\eta_x = G(\eta)[\psi],$$
$$c\psi_x = -g\eta - \frac{\psi_x^2}{2} + \frac{1}{2}\frac{\left(G(\eta)\psi + \eta_x\psi_x\right)^2}{1+\eta_x{}^2}. \tag{6}$$

For this, we look for the zeros of the equation, which indicate the situation with zero vorticity

$$F(c,\eta,\psi) = \begin{pmatrix} -c\eta_x + G(\eta)[\psi] \\ c\psi_x + g\eta + \dfrac{\psi_x^2}{2} - \dfrac{1}{2}\dfrac{\left(G(\eta)\psi + \eta_x\psi_x\right)^2}{1+\eta_x^2} \end{pmatrix}. \tag{7}$$

When $\eta=0, \psi=0$, (7) takes the form $F(c,0,0)=0, \forall c$, which is a bifurcation problem. To solve this type of bifurcation problem, the Crandall-Rabinowiz theorem [28] is applied. For this, we must have the following:

(1) $d_{(\eta,\psi)}F(c^*,0,0)$ has the 1-dimensional kernel

(2) Im $d_{(\eta,\psi)}F(c^*,0,0)$ is closed and has codimension 1

(3) It must satisfy the traversability condition i.e., $\partial_{c,(\eta,\psi)}F(c^*,0,0)[(\eta^*,\psi^*)] \notin R$ where $(\eta^*,\psi^*) \in \mathrm{Ker}d_{(\eta,\psi)}F(c^*,0,0)$ where Ker d is the kernel of d and R is the image set.

By linearizing (7) at $\eta=0, \psi=0$, we have

$$d_{(\eta,\psi)}F(c,0,0)[(\widehat{\eta},\widehat{\psi})] = \begin{pmatrix} -c\widehat{\eta}_x + G(0)\widehat{\psi} \\ c\widehat{\psi}_x + g\widehat{\eta} \end{pmatrix}. \tag{8}$$

We assume the series representation for $\widehat{\eta}$ and $\widehat{\psi}$ as

$$\widehat{\eta}(x) = \sum_{n\ge 0}\eta_n \cos nx,$$
$$\widehat{\psi}(x) = \sum_{n\ge 0}\psi_n \sin nx. \tag{9}$$

Which are even and odd, respectively, together with the norms

$$\|\eta\|_s^2 = \sum n^{2s}\left|\eta_n\right|^2,$$
$$\|\psi\|_s^2 = \sum n^{2s}\left|\psi_n\right|^2, \tag{10}$$

which are the norms defined in the homogeneous Sobolev space H^s.

With these series representations, we have

$$d_{(\eta,\psi)}F(c,0,0)[(\widehat{\eta},\widehat{\psi})] = \sum_{n\ge 1}\begin{pmatrix} (cn\eta_n + n\psi_n)\sin nx \\ (cn\psi_n + g\eta_n)\cos nx \end{pmatrix} + \begin{pmatrix} 0 \\ g\eta_0 \end{pmatrix}. \tag{11}$$

We define

$$f = \sum_{n\ge 0} f_n \cos nx,$$
$$g = \sum_{n\ge 0} g_n \sin nx. \tag{12}$$

For these g and f; we solve, $d_{(\eta,\psi)}F(c,0,0)[(\widehat{\eta},\widehat{\psi})] = \begin{pmatrix} g \\ f \end{pmatrix}$

For $n \ge 1$, we have

$$\begin{pmatrix} cn & n \\ g & cn \end{pmatrix}\begin{pmatrix} \eta_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} g_n \\ f_n \end{pmatrix}. \tag{13}$$

We note that det $M_n(c)=0$ iff $c = \pm(\sqrt{g/n})$, where $M_n = \begin{pmatrix} cn & n \\ g & cn \end{pmatrix}$. For fixed n_0 and choosing $c_{n_0} = (\sqrt{g/n_0})$, the n_0-matrix has kernel. Now for c_{n_0}, we have

$$\begin{pmatrix} c_{n_0}n_0 & n_0 \\ g & c_{n_0}n_0 \end{pmatrix}\begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} c_{n_0} \\ -\dfrac{g}{n_0} \end{pmatrix}. \tag{14}$$

Here, we note that if $n_0=0$, then the inverse of that matrix $\begin{pmatrix} c_{n_0}n_0 & n_0 \\ g & c_{n_0}n_0 \end{pmatrix}$ does not exist. So, the matrix becomes singular, and hence the solution is not possible.

Therefore, $\mathrm{Ker}d_{(\eta,\psi)}F(c_{n_0},0,0) = \begin{pmatrix} c_{n_0}\cos n_0x \\ -(g/n_0)\sin n_0x \end{pmatrix}$ which is of dimension 1.

Again

$$\begin{pmatrix} 0 \\ -g\eta_0 \end{pmatrix} = \begin{pmatrix} 0 \\ f_0 \end{pmatrix},$$
$$\begin{pmatrix} cn & n \\ g & cn \end{pmatrix}\begin{pmatrix} \eta_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} g_n \\ f_n \end{pmatrix}; n\ge 1. \tag{15}$$

This implies two possibilities, $n=n_0$ or $n \ne n_0$. In both cases, it can be shown that $\eta, \psi \in H^s$. Again for $n=n_0$, choosing $g_{n_0}=1$, we have

$$R = \mathrm{Im}d_{(\eta,\psi)}F\left(c_{n_0},0,0\right) = \sum_{n\ne n_0}\begin{pmatrix} g_n\sin nx \\ f_n\cos nx \end{pmatrix} + \begin{pmatrix} \sin n_0x \\ c_{n_0}\cos n_0x \end{pmatrix}. \tag{16}$$

But, this is perpendicular to $R^\perp = \begin{pmatrix} c_{n_0}\sin n_0x \\ -\cos n_0x \end{pmatrix}$. This shows that the range is closed and has codimension 1. Also, the traversability condition is satisfied as

$$\partial_{c,(\eta,\psi)}F\left(c_{n_0},0,0\right)[(\eta^*,\psi^*)] = \begin{pmatrix} -c_{n_0}n_0\sin n_0x \\ -g\cos n_0x \end{pmatrix} \notin R. \tag{17}$$

As all the conditions for the Crandall–Rabinowitz theorem [28] are satisfied, we have the following theorem:

Theorem 1. *Let $s>(5/2)$ and $n_0 \in N$: $\exists\epsilon_0>0$ and C^1 functions C_ϵ: $(-\epsilon_0,\epsilon_0)\longrightarrow R$; $\begin{pmatrix} \breve{\eta}_\epsilon \\ \breve{\psi}_\epsilon \end{pmatrix}(-\epsilon_0,\epsilon_0)\longrightarrow H^s\times H^s$ such that $\begin{pmatrix} \breve{\eta}(x-ct) \\ \breve{\psi}(x-ct) \end{pmatrix}$ solves the water wave equation. Moreover, $C_\epsilon = C_{n_0} + 0(\epsilon)$, $\begin{pmatrix} \breve{\eta}_\epsilon(x) \\ \breve{\psi}_\epsilon(x) \end{pmatrix} = \epsilon\begin{pmatrix} c_{n_0}\cos n_0x \\ -g/n_0\sin n_0x \end{pmatrix} + O(\epsilon^2)$.*

Thus, we have constructed traveling waves of the form

$$\eta(t,x) = \epsilon c_{n_0} \cos n_0 (x - c_\epsilon t) + O(\epsilon^2). \tag{18}$$

3.2. Traveling Waves for Water Waves with Constant Vorticity. The water waves formulation [24, 25] is given by

$$\begin{aligned} \eta_t &= G(\eta)[\psi] + \gamma\eta\eta_x, \\ \psi_t &= -g\eta - \frac{\psi_x^2}{2} + \frac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x)^2} + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi, \end{aligned} \tag{19}$$

where $G(\eta)[\psi]$ is a Dirichlet–Neumann operator [26, 27].

Here, we are looking for the small size traveling waves periodic in x. That means we expect the solution of the form $\eta(t,x) = \breve{\eta}(x+ct)$ and $\psi(t,x) = \breve{\psi}(x+ct)$, where c is the phase velocity.

These solutions must satisfy the equations:

$$\begin{aligned} c\eta_x &= G(\eta)[\psi] + \gamma\eta\eta_x, \\ c\psi_x &= -g\eta - \frac{\psi_x^2}{2} + \frac{1}{2}\frac{G(\eta)\psi + \eta_x\psi_x^2}{1+\eta_x^2} + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi. \end{aligned} \tag{20}$$

For this, we look for the zeros of the equation, which indicate the situation with constant vorticity

$$F(c,\eta,\psi) = \begin{pmatrix} -c\eta_x + G(\eta)[\psi] + \gamma\eta\eta_x \\ c\psi_x + g\eta + \frac{\psi_x^2}{2} - \frac{1}{2}\frac{G(\eta)\psi + \eta_x\psi_x^2}{1+\eta_x^2} - \gamma\eta\psi_x - \gamma\partial_x^{-1}G(\eta)\psi \end{pmatrix}. \tag{21}$$

When $\eta = 0$, $\psi = 0$, (21) takes the form $F(c,0,0) = 0$; $\forall c$, which is a bifurcation problem. Here also, the CrandallRabinowitz theorem [28] is applied to solve this type of bifurcation problem. For this, we must have the following:

(1) $d_{(\eta,\psi)}F(c^*,0,0)$ has 1 dimensional kernel
(2) Im $d_{(\eta,\psi)}F(c^*,0,0)$ is closed and has codimension 1
(3) It must satisfy the traversability condition i.e. $\partial_{c,(\eta,\psi)}F(c^*,0,0)[(\eta *, \psi *)] \notin R$ where $(\eta^*,\psi^*) \in \mathrm{Ker}d_{(\eta,\psi)}F(c^*,0,0)$ where Ker d is the kernel of d and R is image set.

By linearizing equation (21) at $\eta = 0$, $\psi = 0$, we have

$$d_{(\eta,\psi)}F(c,0,0)[(\hat{\eta},\hat{\psi})] = \begin{pmatrix} -c\hat{\eta}_x + G(0)\hat{\psi} \\ c\hat{\psi}_x + g\hat{\eta} - \gamma\partial_x^{-1}|D|\psi \end{pmatrix}. \tag{22}$$

We assume the series representation for $\hat{\eta}$ and $\hat{\psi}$ as

$$\begin{aligned} \hat{\eta}(x) &= \sum_{n\geq 0} \eta_n \cos nx, \\ \hat{\psi}(x) &= \sum_{n\geq 0} \psi_n \sin nx, \end{aligned} \tag{23}$$

which are even and odd, respectively, together with the norms

$$\begin{aligned} \|\eta\|_s^2 &= \sum n^{2s}|\eta_n|^2, \\ \|\psi\|_s^2 &= \sum n^{2s}|\psi_n|^2. \end{aligned} \tag{24}$$

These are the norms defined on homogeneous Sobolev space H^s.

With these series representation, we have

$$\begin{aligned} d_{(\eta,\psi)}F(c,0,0)[(\hat{\eta},\hat{\psi})] = \sum_{n\geq 1}&\begin{pmatrix} (cn\eta_n + n\psi_n)\sin nx \\ ((cn+\gamma)\psi_n + g\eta_n)\cos nx \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ (cn_0+\gamma)\psi_0 + g\eta_0 \end{pmatrix}. \end{aligned} \tag{25}$$

Assuming

$$\begin{aligned} g &= \sum_{n\geq 0} g_n \sin nx, \\ f &= \sum_{n\geq 0} f_n \cos nx. \end{aligned} \tag{26}$$

For these g and f; we solve $d_{(\eta,\psi)}F(c,0,0)[(\hat{\eta},\hat{\psi})] = \begin{pmatrix} g \\ f \end{pmatrix}$

.

Then, we have $\begin{pmatrix} cn & n \\ g & cn+\gamma \end{pmatrix}\begin{pmatrix} \eta_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} g_n \\ f_n \end{pmatrix}$.

We note that $\det M_n(c) = 0$ if $c = -(\gamma/2n) \pm \sqrt{(\gamma^2/4n^2) + (g/n)}$, where $M_n = \begin{pmatrix} cn & n \\ g & cn+\gamma \end{pmatrix}$. For fixed n_0 and choosing $c_{n_0} = -(\gamma/2n_0) \pm \sqrt{(\gamma^2/4n_0^2) + (g/n_0)}$, the n_0 -matrix has kernel.

Now,

$$\begin{pmatrix} c_{n_0}n_0 & n_0 \\ g & c_{n_0}n_0 + \gamma \end{pmatrix}\begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \eta_{n_0} \\ \psi_{n_0} \end{pmatrix} = \begin{pmatrix} c_{n_0} + \frac{\gamma}{n_0} \\ -\frac{g}{n_0} \end{pmatrix}. \tag{27}$$

Therefore, $\mathrm{Ker}d_{(\eta,\psi)}F(c_{n_0},0,0) = \begin{pmatrix} (c_{n_0} + (\gamma/n_0))\cos n_0 x \\ -(g/n_0)\sin n_0 x \end{pmatrix}$, which is of dimension 1.

Again

$$\begin{aligned} \begin{pmatrix} 0 \\ -g\eta_0 \end{pmatrix} &= \begin{pmatrix} 0 \\ f_0 \end{pmatrix}, \\ \begin{pmatrix} cn & n \\ g & cn+\gamma \end{pmatrix}\begin{pmatrix} \eta_n \\ \psi_n \end{pmatrix} &= \begin{pmatrix} g_n \\ f_n \end{pmatrix}; n\geq 1. \end{aligned} \tag{28}$$

This implies two possibilities $n=n_0$ or $n\neq n_0$. In both the cases, it can be shown that η, $\psi \in H^s$. Again for $n=n_0$ and choosing $g_{n_0} = 1$, we have,

$$R = \operatorname{Im} d_{(\eta,\psi)}F\left(c_{n_0},0,0\right) = \sum_{n\neq n_0}\begin{pmatrix} g_n \sin nx \\ f_n \cos nx \end{pmatrix} + \begin{pmatrix} \sin n_0 x \\ \left(c_{n_0} + \dfrac{\gamma}{n_0}\right)\cos n_0 x \end{pmatrix}, \tag{29}$$

which is perpendicular to $R^{\perp} = \begin{pmatrix} -\left(c_{n_0} + (\gamma/n_0)\right)\sin n_0 x \\ \cos n_0 x \end{pmatrix}$.

This shows that the range is closed and has codimension 1. Also the traversability condition is satisfied as

$$\partial_{c,(\eta,\psi)}F\left(c_{n_0},0,0\right)\left[(\eta^*,\psi^*)\right] = \begin{bmatrix} -\left(c_{n_0} + (\gamma/n_0)\right)n_0 \sin n_0 x \\ -g \cos n_0 x \end{bmatrix} \notin R. \tag{30}$$

With the help of the Crandall–Rabinowitz theorem [28], we have obtained the following theorem:

Theorem 2. *Let* $s > (5/2)$ *and* $n_0 \in N$: $\exists \epsilon_0 > 0$ *and* C^1 *functions* C_ϵ: $(-\epsilon_0, \epsilon_0) \longrightarrow R$; $\begin{pmatrix} \breve{\eta}_\epsilon \\ \breve{\psi}_\epsilon \end{pmatrix}(-\epsilon_0,\epsilon_0) \longrightarrow H^s \times H^s$ *such that* $\begin{pmatrix} \breve{\eta}(x-ct) \\ \breve{\psi}(x-ct) \end{pmatrix}$ *solves the water wave equation.*

Moreover, $C_\epsilon = C_{n_0} + 0(\epsilon)$, $\begin{pmatrix} \breve{\eta}_\epsilon(x) \\ \breve{\psi}_\epsilon(x) \end{pmatrix} = \epsilon \begin{pmatrix} \left(c_{n_0} + (\gamma/n_0)\right)\cos n_0 x \\ -g/n_0 \sin n_0 x \end{pmatrix} + O(\epsilon^2)$.

Thus, we have constructed traveling waves of the form

$$\eta(t,x) = \epsilon\left(c_{n_0} + \frac{\gamma}{n_0}\right)\cos n_0\left(x - c_\epsilon t\right) + O\left(\epsilon^2\right). \tag{31}$$

Here, we have created the water profile for the zero and constant vorticity cases. We use Newtonian fluid. A similar result may be useful for the non-Newtonian fluid. Also, a similar result may be extended to the positive or negative, or variable viscosity cases.

4. Conclusion

In this work, we studied the various development of the water waves and related facts such as the wave profile and path of the wave in the finite bottom conditions. We have then extended the finite depth condition to the infinite depth condition. In infinite depth conditions, we have constructed the water wave profile for the zero and constant vorticity conditions with the use of the Crandall–Rabinowitz theorem by using an analytical approach. While we focused on the zero and constant vorticity cases, similar work may be established on the positive and negative vorticity cases for the construction of wave profiles. Here we did our work for the Newtonian fluid. This work may be extended for the non-Newtonian fluid.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] G. Stokes, “On the theory of oscillatory waves,” *Trans. Camb. Philos. Soc.*vol. 8, pp. 441–455, 1847.

[2] T. B. Benjamin and P. J. Olver, “Hamiltonian structure, symmetries and conservation laws for water waves,” *Journal of Fluid Mechanics*, vol. 125, no. -1, pp. 137–185, 1982.

[3] W. Craig, “Water waves, Hamiltonian systems and Cauchy integrals,” in *Microlocal Analysis and Nonlinear Waves*, vol. 30, pp. 37–45, Springer, New York, NY, USA, 1991.

[4] W. Craig and M. D. Groves, “Hamiltonian long-wave approximations to the water-wave problem,” *Wave Motion*, vol. 19, no. 4, pp. 367–389, 1994.

[5] C. J. Amick, L. E. Fraenkel, and J. F. Toland, “On the Stokes conjecture for the wave of extreme form,” *Acta Mathematica*, vol. 148, no. 0, pp. 193–214, 1982.

[6] B. Buffoni and J. F. Toland, *Analytic Theory of Global Bifurcation*, Princeton University Press, Princeton, NJ, USA, 2003.

[7] A. Constantin, “The trajectories of particles in Stokes waves,” *Inventiones Mathematicae*, vol. 166, no. 3, pp. 523–535, 2006.

[8] A. Constantin, “Exact traveling periodic water waves in two-dimensional irrotational flows,” *Lecture Notes in Mathematics*, vol. 2158, pp. 1–82, Springer, Berlin, Germany, 2016.

[9] A. Compelli, R. Ivanov, and M. Todorov, “Hamiltonian models for the propagation of irrotational surface gravity waves over a variable bottom,” *Phil. Trans. R. Soc. A*, vol. 376, no. 2111, Article ID 20170091, 2017.

[10] B. Basu, “Wave height estimates from pressure and velocity data at an intermediate depth in the presence of uniform currents,” *Phil. Trans. R. Soc. A*, vol. 376, no. 2111, Article ID 20170087, 2017.

[11] D. Clamond, “New exact relations for steady irrotational two-dimensional gravity and capillary surface waves,” *Phil. Trans. R. Soc. A*, vol. 376, no. 2111, Article ID 20170220, 2017.

[12] F. Kogelbauer, “Existence and amplitude bounds for irrotational water waves in finite depth,” *Phil. Trans. R. Soc. A*, vol. 376, no. 2111, Article ID 20170094, 2017.

[13] A. Constantin and G. Villari, “Particle trajectories in linear water waves,” *Communications to SIMAI Congress*, vol. 1, 2006.

[14] M. Ehrnstrom and G. Villari, “Linear water waves with vorticity rotational features and particle paths,” *Journal of Differential Equations*, vol. 244, no. 8, pp. 1888–1909, 2008.

[15] E. Wahlén, “Steady water waves with a critical layer,” *Journal of Differential Equations*, vol. 246, no. 6, pp. 2468–2483, 2009.

[16] A. Constantin and J. Escher, "Symmetry of steady deep-water waves with vorticity," *European Journal of Applied Mathematics*, vol. 15, no. 6, pp. 755–768, 2004.

[17] A. Constantin and W. Strauss, "Exact steady periodic water waves with vorticity," *Communications on Pure and Applied Mathematics*, vol. 57, no. 4, pp. 481–527, 2004.

[18] A. Constantin and W. Strauss, "Exact periodic traveling water waves with vorticity," *Comptes Rendus Mathematique*, vol. 335, no. 10, pp. 797–800, 2002.

[19] A. Constantin and J. Escher, "Analyticity of periodic traveling free surface water waves with vorticity," *Annals of Mathematics*, vol. 173, no. 1, pp. 559–568, 2011.

[20] J. D. Palencia, S. Rahman, A. N. Redondo, and J. R. Gonzalez, "Regularity and traveling wave profiles for a porous eyring-powell fluid with Darcy-forchhimer law," *Symmetry*, vol. 1451, 2022.

[21] J. D. Palencia and S. Rahman, "Geometric perturbation theory and traveling waves profiles analysis in a Darcy- forchheimer fluid model," *Journal of Nonlinear Mathematical Physics*, vol. 29, pp. 556–572, 2022.

[22] J. D. Palencia, S. Rahman, and A. N. Rodendo, "Analysis of traveling wave solutions for Eyring-Powell fluid formulated with a degenerate diffusivity and a Darcy-Forchheimer law," *AIMs,Mathematics*, vol. 7, no. 8, Article ID 15212, 2022.

[23] J. L. Diaz, S. U. Rahman, J. C. Sánchez Rodríguez, M. A. Simón Rodríguez, G. Filippone Capllonch, and A. Herrero Hernández, "Analysis of solutions, asymptotic and exact profiles to an eyring–powell fluid modell," *Mathematics*, vol. 10, no. 4, p. 660, 2022.

[24] E. Wahlén, "A Hamiltonian formulation of water waves with constant vorticity," *Letters in Mathematical Physics*, vol. 79, no. 3, pp. 303–315, 2007.

[25] A. Constantin, R. I. Ivanov, and E. M. Prodanov, "Nearly-Hamiltonian structure for water waves with constant vorticity," *Journal of Mathematical Fluid Mechanics*, vol. 10, no. 2, pp. 224–237, 2008.

[26] D. P. Nicholls and F. Reitich, "Analytic continuation of dirichlet Neumann operators," *Numerische Mathematik*, vol. 94, pp. 107–146, 2003.

[27] J. Wilkening and V. Vasan, "Comparison of five method of computing the dirichlet neumann operator for the water wave problem," 2014, https://arxiv.org/abs/1406.5226.

[28] M. G. Crandall and P. H. Rabinowitz, "Bifurcation from simple eigenvalues," *Journal of Functional Analysis Water waves, Hamiltonian systems and Cauchy integrals*, vol. 8, no. 2, pp. 321–340, 1971.