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Research Article

The Intersection Multiplicity of Intersection Points over Algebraic Curves

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In analytic geometry, Bézout's theorem stated the number of intersection points of two algebraic curves and Fulton introduced the intersection multiplicity of two curves at some point in local case. It is meaningful to give the exact expression of the intersection multiplicity of two curves at some point. In this paper, we mainly express the intersection multiplicity of two curves at some point in $\mathbb{R}^2$ and $\mathbb{A}_K^2$ under fold point, where $\text{char}(K) = 0$. First, we give a sufficient and necessary condition for the coincidence of the intersection multiplicity of two curves at some point and the smallest degree of the terms of these two curves in $\mathbb{R}^2$. Furthermore, we show that two different definitions of intersection multiplicity of two curves at a point in $\mathbb{A}_K^2$ are equivalent and then give the exact expression of the intersection multiplicity of two curves at some point in $\mathbb{A}_K^2$ under fold point.

1. Introduction

Analytic geometry or Cartesian geometry is an important branch of algebra, a great invention of Fermat and Descartes, which deals with the modelling of some geometrical objects, such as points, lines, and curves. It is a mathematical subject that uses algebraic symbolism and methods to solve some geometric problems. It establishes the correspondence between the algebraic equations and the geometric curves. In analytic geometry, one of the most important fundamental problems is to find the number of intersection points of two algebraic curves. Bézout's theorem stated that two algebraic curves of degrees m and n intersect in mn points counting multiplicities and cannot meet in more than mn points unless they have a component in common ([1]). In a local case, Fulton ([2], Section 3.3) introduced the intersection multiplicity of two affine algebraic curves at some point. Consequently, many mathematicians focussed on the geometric modelling with curves (cf. [3–12]). In this paper, we consider the intersection multiplicities of two curves at some point in $\mathbb{R}^2, \mathbb{P}_{\mathbb{R}}^2$ and $\mathbb{A}_K^2$ with $\text{char}(K) = 0$, respectively, and we mainly give the exact expression of the intersection multiplicities of two curves at some point in $\mathbb{R}^2$ and $\mathbb{A}_K^2$.

Let $f(x, y) = 0$ and $g(x, y) = 0$ be two algebraic curves in $\mathbb{R}^2$ (resp., $\mathbb{P}_{\mathbb{R}}^2$), and the intersection multiplicity $I_P(f, g)$ of f and g at a point P is the number of times that the curves $f(x, y) = 0$ and $g(x, y) = 0$ intersect at the point P ([4], Chapter 1). Also, there are some other definitions of the intersection multiplicity of algebraic curves at a point (cf. [2, 7, 11, 13–15]). According to the definitions of $\mathbb{P}_{\mathbb{R}}^2$ and projective transformation, we can transfer some difficult cases to some easy cases by projective transformation when we consider the intersection multiplicity in $\mathbb{P}_{\mathbb{R}}^2$; thus, we can connect the intersection multiplicity in $\mathbb{R}^2$ with the intersection multiplicity in $\mathbb{P}_{\mathbb{R}}^2$ by homogenizing polynomials (([4], Theorem 3.7), ([16], Lemma 2.5)). Following the factorization theorem of polynomials, we note that the intersection multiplicity of two curves at a point has a close relation with the fold point, so it is important to give the relation between $I_P(f, g)$ and the fold point. However, for the case of the affine space $\mathbb{A}_K^2$, where K is an algebraically closed field with $\text{char}(K) = 0$, we have two different definitions of intersection multiplicity of two curves at a point: one is given by using independent polynomials (([4], Definition 13.2)), and the other is given by means of the dimension of a local ring (([13], Definition 2.3), ([2], Section

3.3)). So it is meaningful to show that the two different definitions are equivalent. Furthermore, similar as the projective transformation, we use affine transformation to extend the intersection multiplicity of curves at some point from $\mathbb{R}^2$ to $\mathbb{A}_K^2$.

This paper is organized as follows: In Section 2 we first introduce some properties of the intersection multiplicity of algebraic curves at some point in $\mathbb{R}^2$ and $\mathbb{P}_{\mathbb{R}}^2$, respectively, and then give a sufficient and necessary condition for the coincidence of the intersection multiplicities of two curves under fold point and the smallest degree of the terms of these two curves in $\mathbb{R}^2$. In generalization, we consider the intersection multiplicity of affine algebraic curves at some point in $\mathbb{A}_K^2$ in Section 3 and we show an equivalence for two different definitions of intersection multiplicity of algebraic curves in $\mathbb{A}_K^2$. Furthermore, we give some properties for the intersection multiplicity between curves and lines in $\mathbb{A}_K^2$ by using localization and then give some related results about the intersection multiplicity of curves in terms of fold point by using the affine transformation.

2. Intersection Multiplicity of Algebraic Curves in $\mathbb{R}^2$ and $\mathbb{P}_{\mathbb{R}}^2$

In this section, we introduce some properties of the intersection multiplicity of algebraic curves at some point in $\mathbb{R}^2$ and the real projective plane $\mathbb{P}_{\mathbb{R}}^2$, respectively, where $\mathbb{P}_{\mathbb{R}}^2$ is $(\mathbb{R}^3 - \{(0,0,0)\})/\sim$ and $\sim$ is the equivalence relation defined by $(x, y, z) \sim (x', y', z')$ if there exists a nonzero $\lambda \in \mathbb{R}$, such that $(x, y, z) = (\lambda x', \lambda y', \lambda z')$. Consequently, according to the properties of the intersection multiplicities of curves under fold point in $\mathbb{P}_{\mathbb{R}}^2$, we give a sufficient and necessary condition for the coincidence of the intersection multiplicities of two curves under fold point and the smallest degree of the terms of these two curves in $\mathbb{R}^2$.

2.1. Properties of Intersection Multiplicity of Curves in $\mathbb{R}^2$ and $\mathbb{P}_{\mathbb{R}}^2$. An algebraic curve in $\mathbb{R}^2$ is the graph of a polynomial in x, y over $\mathbb{R}$. Let $f(x, y) = 0$ and $g(x, y) = 0$ be algebraic curves (abbreviate to curves) which intersect at a point p in $\mathbb{R}^2$. The *intersection multiplicity* of f and g at p is the number of times that the curves $f = 0$ and $g = 0$ intersect at the point p, denoted by $I_p(f, g)$.

Property 1 ((see [4], Section 1)). Let $f(x, y) = 0$, $g(x, y) = 0$, and $h(x, y) = 0$ be curves and p a point in $\mathbb{R}^2$. Then

(1) $I_p(f, g)$ is a nonnegative integer or ∞, and $I_p(f, g) = I_p(g, f)$.

(2) $I_p(f, g) \geq 1$ if and only if $f(p) = 0$ and $g(p) = 0$.

(3) $I_p(f, g) = I_p(f, g + fh)$.

(4) $I_p(f, gh) = I_p(f, g) + I_p(f, h)$, and $I_p(f, gh) = I_p(f, g)$ if $h(p) \neq 0$.

An algebraic curve (simply curve) in $\mathbb{P}_{\mathbb{R}}^2$ is a homogeneous polynomial in x, y, z. We can extend algebraic curves from $\mathbb{R}^2$ to $\mathbb{P}_{\mathbb{R}}^2$ by homogenizing polynomials. Note that, for any homogeneous polynomial $F(x, y, z)$, we set $f(x, y) = F(x, y, 1)$; then, any point $(x, y) \in \mathbb{R}^2$ lies on the curve $f(x, y) = 0$ if and only if the point $(x, y, 1) \in \mathbb{P}_{\mathbb{R}}^2$ lies on the curve $F(x, y, z) = 0$. Let $F(x, y, z) = 0, G(x, y, z) = 0$ be curves in $\mathbb{P}_{\mathbb{R}}^2$. Similar as the the definition of $I_p(f, g)$, we denote $I_P(F, G)$ the *intersection multiplicity* of F and G at the point $P \in \mathbb{P}_{\mathbb{R}}^2$. It is clear that $I_P(F, G)$ have the similar properties as in Property 1.

Lemma 1 ((see [4], Theorem 3.7)). *Let $F(x, y, z) = 0$ and $G(x, y, z) = 0$ be curves in $\mathbb{P}_{\mathbb{R}}^2$, and we set $f(x, y) = F(x, y, 1), g(x, y) = G(x, y, 1)$. Then, for any point $(a, b, 1) \in \mathbb{P}_{\mathbb{R}}^2$, we have*

$$I_{(a,b,1)}(F(x, y, z), G(x, y, z)) = I_{(a,b)}(f(x, y), g(x, y)). \quad (1)$$

A projective transformation is a linear map $T: \mathbb{P}_{\mathbb{R}}^2 \longrightarrow \mathbb{P}_{\mathbb{R}}^2$ defined by

$$(x, y, z)^T \mapsto A(x, y, z)^T, \quad (2)$$

where $A \in \mathrm{GL}(3, \mathbb{R})$ is an invertible 3×3 matrix. It follows from ([4], Theorem 3.4) that a projective transformation can transform any four points, no three of which are collinear, into any other four such points. Furthermore, projective transformations preserve intersection multiplicities (Lemma 2), and it follows that we can find the intersection multiplicity $I_P(F, G)$ for any point $P \in \mathbb{P}_{\mathbb{R}}^2$ by transforming P to the origin $O = (0, 0, 1)$.

Lemma 2 ((see [4], Property 3.5)). *Let $F(x, y, z) = 0$ and $G(x, y, z) = 0$ be curves and P a point in $\mathbb{P}_{\mathbb{R}}^2$, and T is a projective transformation that maps $P(x, y, z)$ (resp.,$F = 0, G = 0$) to $P'(x', y', z')$ (resp.,$F' = 0, G' = 0$). Then,*

$$I_P(F(x, y, z), G(x, y, z)) = I_{P'}(F'(x', y', z'), G'(x', y', z')). \quad (3)$$

2.2. Intersection Multiplicity of Curves under Fold Point in $\mathbb{R}^2$ and $\mathbb{P}_{\mathbb{R}}^2$. Let $f(x, y)$ be a nonzero polynomial and $f_d(x, y)$ be the sum of the terms of degree d in f. Then, we can write

$$f_d(x, y) = (a_1 x + b_1 y)^{s_1} (a_2 x + b_2 y)^{s_2} \cdots (a_j x + b_j y)^{s_j} r(x, y) \quad (4)$$

for distinct lines $a_i x + b_i y = 0$ uniquely, where s_i is a nonnegative integer and $r(x, y)$ is a polynomial that has no line factors. According to the factorization of f_d, to study the intersection multiplicity of curves in $\mathbb{R}^2$, it is important and convenient to study the intersection multiplicity between curves and lines first.

Lemma 3 ((see [16], Corollary 2.3)). *Let $f(x, y) = 0$ be a curve that contains the origin $o = (0, 0) \in \mathbb{R}^2$, and d is the smallest degree of the terms in f. We assume that $\ell = 0$ is a line through the origin o. Then, $I_o(\ell, f) > d$ if ℓ is a factor of f_d, and $I_o(\ell, f) = d$ if ℓ is not a factor of f_d.*

Definition 1 ((see [4], §4)). Let $f(x, y) = 0$ be a curve and p a point in $\mathbb{R}^2$. We say that the point p is a *d-fold point* of $f = 0$ if there is a nonnegative integer d, such that there are at most d distinct lines that intersect f at p more than d times and that all other lines intersect f at p exactly d times.

It follows from Lemma 3 and Definition that the origin o is a d-fold point of $f = 0$ if and only if d is the smallest degree of the terms in f. Similarly, we can define the fold point of a curve $F = 0$ in $\mathbb{P}^2_{\mathbb{R}}$ and have the following Lemma.

Lemma 4 ((see [16], Theorem 3.4)). *Let $F(x, y, z) = 0$ and $G(x, y, z) = 0$ be curves and P a point in $\mathbb{P}^2_{\mathbb{R}}$. Assume that P is a d-fold point of $F = 0$ and an e-fold point of $G = 0$, then*

$$I_P(F, G) \geq \mathrm{de}. \tag{5}$$

By Lemma 1 and the definition of the projective plane $\mathbb{P}^2_{\mathbb{R}}$, we can consider the intersection multiplicity of curves under fold point in $\mathbb{R}^2$ by setting $z = 1$.

Corollary 1 ((see [16], Corollary 3.5)). *Let $f(x, y) = 0$ and $g(x, y) = 0$ be curves and p a point in $\mathbb{R}^2$. We assume that p is a d-fold point of $f = 0$ and an e-fold point of $g = 0$. Then,*

$$I_p(f, g) \geq \mathrm{de}. \tag{6}$$

Theorem 1. *Let $f(x, y) = 0$ and $g(x, y) = 0$ be curves that both contain the origin o in $\mathbb{R}^2$. We assume that the origin o is a 1-fold point of both $f = 0$ and $g = 0$. Let f_1 (resp.,g_1) be the sum of the terms of degree 1 in f (resp.,g). Then, $I_o(f, g) > 1$ if and only if f_1 and g_1 have a common factor. Equivalently, $I_o(f, g) = 1$ if and only if f_1 and g_1 have no common factors.*

Proof. Since the origin o is a 1-fold point of $f = 0$, by Lemma 3, we can write

$$f(x, y) = xp(x, y) + yq(y), \tag{7}$$

for some polynomials $p(x, y)$ and $q(y)$ in $\mathbb{R}[x, y]$. Note that $p(0, 0)$ and $q(0)$ are not all zeros, and we may assume that $p(0, 0) \neq 0$. Similarly, we can write

$$g(x, y) = xu(x, y) + yv(y), \tag{8}$$

for some polynomials $u(x, y)$ and $v(y)$ in $\mathbb{R}[x, y]$ with $(u(0, 0), v(0)) \neq (0, 0)$. By Property 1, we have

$$I_o(f, g) = I_o(xp + yq, xu + yv) = 1 + I_o(xp + yq, pv - qu). \tag{9}$$

It follows that $I_o(f, g) > 1$ if and only if $pv - qu$ contains the origin o, i.e., the determinant of the matrix $\begin{pmatrix} p(0,0) & q(0) \\ u(0,0) & v(0) \end{pmatrix}$ is zero. This is equivalent to the condition that f_1 and g_1 have a common factor. Hence, we obtain the assertion. □

Theorem 2. *Let $f(x, y) = 0$ and $g(x, y) = 0$ be curves in $\mathbb{R}^2$. We assume that the origin o is a d-fold point of $f = 0$ and an e-fold point of $g = 0$. Let f_d (resp., g_e) be the sum of the terms of degree d (resp., e) in f (resp., g). Then, $I_o(f, g) > \mathrm{de}$ if and only if f_d and g_e have a common factor of positive degrees. Equivalently, $I_o(f, g) = \mathrm{de}$ if and only if f_d and g_e have no common factors of positive degrees.*

Proof. Let $P(d, e)$ denote the assertion that $I_o(f, g) > \mathrm{de}$ if the origin o is a d-fold point of $f = 0$ and an e-fold point of $g = 0$ and that f_d and g_e have a common factor of positive degrees. When $d = 1$ and $e = 1$, it is obvious that $P(1, 1)$ holds following Theorem 1.

First, assume that $P(1, e)$ holds for $1 \leq e \leq l$. We prove that $P(1, l + 1)$ also holds. We assume that the origin o is a $(l + 1)$-fold point of $g = 0$ and that f_1 and g_{l+1} have a common factor of positive degrees. By Lemma 3, we can write

$$f(x, y) = x\alpha(x) + y\beta(x, y), \tag{10}$$

for some polynomials $\alpha(x)$ and $\beta(x, y)$. Note that $\alpha(0)$ and $\beta(0, 0)$ are not all zeros. Also, we can write

$$g(x, y) = x^{l+1}\gamma(x) + y\delta(x, y), \tag{11}$$

for some polynomials $\gamma(x)$ and $\delta(x, y)$, and every term of $\delta(x, y)$ has degree at least l. By linear transformation, we may assume that $\alpha(0) = 1$ and $\gamma(0) = 1$. From Property 1, we have

$$I_o(f, g) = 1 + I_o\left(f, \delta\alpha - \beta x^l\gamma\right). \tag{12}$$

We set $w = \delta\alpha - \beta x^l\gamma$. Then the origin o is an r-fold point of $w = 0$ for $r \geq l$ following Lemma 3. Let w_r be the sum of the terms of degree r in w. If $r = l$, then

$$yw_r = g_{l+1} - f_1 x^l, \tag{13}$$

which implies that f_1 and w_l have a common factor of positive degree, and by assumption, $P(1, l + 1)$ holds. If $r > l$, then $I_o(f, w) \geq r > l$ from Corollary 1, and we have that $P(1, l + 1)$ holds clearly. By induction, we obtain that $P(1, e)$ holds for any e.

Second, we assume that $P(m, n)$ holds for some positive integers m and n satisfying $1 \leq m < d$ and $1 \leq n \leq e$. We prove that $P(m + 1, n)$ also holds. We assume that the origin o is a $(m + 1)$-fold point of $f = 0$ and that f_{m+1} and g_n have a common factor of positive degree. Then, we can write

$$f(x, y) = x^{m+1}\alpha'(x) + y\beta'(x, y), g(x, y) = x^n\gamma'(x) + y\delta'(x, y), \tag{14}$$

for some polynomials $\alpha'(x), \beta'(x, y)$ and $\gamma'(x), \delta'(x, y)$, and every term of $\beta'(x, y)$ (resp., $\delta'(x, y)$) has degree at least m (resp., $n - 1$). Note that $\alpha'(0)$ and $\beta'(0, 0)$ (resp., $\gamma'(0)$ and $\delta'(0, 0)$) are not all zeros. By linear transformation, we may assume that $\alpha'(0) = 1$ and $\gamma'(0) = 1$.

We assume that $m + 1 \geq n$ (similarly for the case $m + 1 < n$), and following Property 1, we have

$$I_o(f, g) = n + I_o\left(\beta'\gamma' - \delta'\alpha' x^{m+1-n}, g\right). \tag{15}$$

We set $w' = \beta'\gamma' - \delta'\alpha' x^{m+1-n}$. Then, the origin o is an r'-fold point of $w' = 0$ for $r' \geq m$. Let $w_{r'}'$ be the sum of the terms of degree r' in w'. If $r' = m$, then

$$yw_m' = f_{m+1} - g_n x^{m+1-n}, \tag{16}$$

which implies that w_m' and g_n have a common factor of positive degrees, and by assumption, we obtain that $P(m+1,n)$ holds. If $r' > m$, then

$$I_o(w', g) \geq r'n > mn, \tag{17}$$

following Corollary 1, and clearly we have that $P(m+1,n)$ holds. Thus, we obtain that $P(m+1,n)$ holds. Therefore, by induction, we proved that the sufficient condition holds.

Conversely, following Corollary 1, it is suffice to show that if f_d and g_e have no common factors of positive degree, then $I_o(f,g) = de$.

Let $P'(d,e)$ denote the assertion that $I_o(f,g) = \mathrm{de}$ if the origin o is a d-fold point of $f = 0$ and an e-fold point of $g = 0$ and that f_d and g_e have no common factors of positive degrees. When $d = 1$ and $e = 1$, it is clear that $P'(1,1)$ holds following Theorem 1.

First, we assume that $P'(1,e)$ holds for $1 \leq e \leq l'$. We prove that $P'(1,l'+1)$ also holds. We suppose that the origin o is a $(l'+1)$-fold point of $g = 0$ and that f_1 and $g_{l'+1}$ have no common factors of positive degree. By Lemma 3, we can write

$$f(x,y) = xp(x) + ys(x,y), \tag{18}$$

for some polynomials $p(x)$ and $s(x,y)$. Note that $p(0)$ and $s(0,0)$ are not all zeros. Also, we can write

$$g(x,y) = x^{l'+1}q(x) + yt(x,y), \tag{19}$$

for some polynomials $q(x)$ and $t(x,y)$, and every term of t has a degree at least l'. By linear transformation, we may assume that $p(0) = 1$ and $q(0) = 1$. Following Property 1, we have

$$I_o(f,g) = 1 + I_o\left(f, tp - sx^{l'}q\right). \tag{20}$$

We set $h = tp - sx^{l'}q$. Thenthe origin o is a k-fold point of $h = 0$ for $k \geq l'$ from Lemma 3. Let h_k be the sum of the terms of degree k in h. If $k = l'$, then

$$yh_{l'} = g_{l'+1} - f_1 x^{l'}, \tag{21}$$

which implies that f_1 and $h_{l'}$ have no common factors of positive degree, and by the assumption, we know that $P'(1,l'+1)$ holds. If $k > l'$, then

$$h_{l'} = 0, \tag{22}$$

which implies that f_1 and $g_{l'+1}$ have a common factor of positive degree. This is a contradiction. Therefore, by induction, we prove that $P'(1,e)$ holds for any e.

Second, we assume that $P'(m',n')$ holds for some positive integers m', n' satisfying $1 \leq m' < d$ and $1 \leq n' \leq e$. We prove that $P'(m'+1,n')$ also holds. We suppose that the origin o is a $(m'+1)$-fold point of $f = 0$ and that $f_{m'+1}$ and $g_{n'}$ have no common factors of positive degrees. Following Lemma 3, we can write

$$f(x,y) = x^{m'+1}p'(x) + ys'(x,y), g(x,y) = x^{n'}q'(x) + yt'(x,y), \tag{23}$$

for some polynomials $p'(x)$, $s'(x,y)$, $q'(x)$, and $t'(x,y)$, and every term of s' (resp., t') has a degree at least m' (resp., $n'-1$). Note that $p'(0)$ and $s'(0,0)$ (resp., $q'(0)$ and $t'(0,0)$) are not all zeros. By linear transformation, we may assume that $p'(0) = 1$ and $q'(0) = 1$.

If $m'+1 \geq n'$ (similarly for the case $m'+1 < n'$), following Property 1, we have

$$I_o(f,g) = n' + I_o\left(s'q' - t'p'x^{m'+1-n'}, g\right). \tag{24}$$

We set $h' = s'q' - t'p'x^{m'+1-n'}$. Then, the origin o is a k'-fold point of $h' = 0$ for $k' \geq m'$ by Lemma 3. Let $h_{k'}'$ be the sum of the terms of degree k' in h'. If $k' = m'$, then

$$yh_{m'}' = f_{m'+1} - g_{n'}x^{m'+1-n'}, \tag{25}$$

which implies that $h_{m'}'$ and $g_{n'}$ have no common factor of positive degree, and $P'(m'+1,n')$ holds by the assumption. If $k' > m'$, then

$$h_{m'}' = 0, \tag{26}$$

which implies that $f_{m'+1}$ and $g_{n'}$ have a common factor of positive degree. This is a contradiction.

Therefore, by induction, we proved that the necessary condition holds. □

Example 1. We compute the intersection multiplicity $I_o(f,g)$ of two algebraic curves $f = x + y$ and $g = x^2y^3 + y^3 + x^2 - y^2$ in $\mathbb{R}^2$, where f_1 and g_2 have a common factor $x + y$. Note that the origin o is a 1-fold point of $f = 0$ and a 2-fold point of $g = 0$. According to Theorem 2, we have that $I_o(f,g) > 2$. In fact, by Property 1, we have

$$\begin{aligned} I_o\left(x+y, x^2y^3 + y^3 + x^2 - y^2\right) &= I_o\left(x+y, x^2y^3 + y^3\right) \\ &= I_o\left(x+y, y^3\right) \\ &\quad + I_o\left(x+y, 1+x^2\right) \\ &= I_o\left(x+y, y^3\right) \\ &= 3. \end{aligned} \tag{27}$$

Example 2. We evaluate the intersection multiplicity $I_o(f,g)$ of two algebraic curves $f = x^2 + y^2$ and $g = x^5 + x^3y + x^2 - y^2$ in $\mathbb{R}^2$, where f_2 and g_2 have no common factors. Note that the origin o is a 2-fold point of $f = 0$ and a 2-fold point of $g = 0$. According to Theorem 2, we have that $I_o(f,g) = 4$. In fact, following Property 1, we also have

$$\begin{aligned} I_o\left(x^2+y^2,x^5+x^3y+x^2-y^2\right) &= I_o\left(x^2+y^2,x^5+x^3y+x^2-y^2+\left(x^2+y^2\right)\right) \\ &= I_o\left(x^2+y^2,x^2\left(x^3+xy+2\right)\right) \\ &= I_o\left(x^2+y^2,x^2\right)+I_o\left(x^2+y^2,x^3+xy+2\right) \\ &= I_o\left(x^2+y^2,x^2\right) \\ &= 4. \end{aligned} \tag{28}$$

3. Intersection Multiplicity of Affine Curves in $\mathbb{A}_K^2$

Following the structure of the affine plane $\mathbb{A}_K^2$ with $\mathrm{char}(K)=0$, it is important to consider the intersection multiplicity of affine curves (abbreviate to curves) at some point in $\mathbb{A}_K^2$. Since there are two different definitions of intersection multiplicity of curves at some point in $\mathbb{A}_K^2$, in this section, we first prove that the two different definitions of intersection multiplicity are equivalent. Also, in general, it is not easy to give the intersection multiplicity of curves in $\mathbb{A}_K^2$, and we use the affine transformation to transfer the difficult case to easy case when we consider the intersection multiplicity of curves under fold point in $\mathbb{A}_K^2$. In this section, we mainly extend the intersection multiplicity of curves at some point from $\mathbb{R}^2$ to $\mathbb{A}_K^2$. For the case $\mathrm{char}(K)=p$ with the prime number p, the intersection multiplicity of curves at some point in $\mathbb{A}_K^2$ is still an open question.

3.1. Equivalent Definitions of Intersection Multiplicity of Curves in $\mathbb{A}_K^2$. Let K be an algebraically closed field with $\mathrm{char}(K)=0$, and $\mathbb{A}_K^n$ is the affine n-space over K. In $\mathbb{A}_K^2$, we have two different definitions (([13], Definition 2.3), ([4], Definition 13.2), ([2], Section 3.3)) of intersection multiplicity $I_P(f,g)$ of curves $f=0, g=0$ at a point P; however, the intersection multiplicities of curves have the same properties (see ([4], Section 13), ([2], Section 3.3)), so it is important to show that the two different definitions of intersection multiplicity are equivalent.

Definition 2 ((see [4], Definition 13.1)). Let n be a positive integer, and let $f(x,y)$, $g(x,y)$, and $q_1(x,y),\ldots,q_n(x,y)$ be polynomials in $K[x,y]$ and $P\in\mathbb{A}_K^2$. We call that $q_1,\ldots,q_n$ are *dependent* with respect to f and g at the point P if there are polynomials $u(x,y)$, $s(x,y)$, $t(x,y)\in K[x,y]$, and $b_1,\ldots,b_n\in K$ such that

$$u\left(b_1q_1+\cdots+b_nq_n\right)=sf+tg, \tag{29}$$

where $u(P)\neq 0$ and $b_1,\ldots,b_n$ are not all zeros. In other words, $q_1,\ldots,q_n$ are *independent* with respect to f and g at the point P if they do not satisfy any equation of the form (29), where $u(P)\neq 0$ and $b_1,\ldots,b_n$ are not all zeros.

Using independent polynomials, we can determine the intersection multiplicities of affine algebraic curves in $\mathbb{A}_K^2$ at any point P.

Definition 3 ((see [4], Definition 13.2)). Let $f(x,y), g(x,y)$ be polynomials in $K[x,y]$ and $P\in\mathbb{A}_K^2$. The *intersection multiplicity* of f and g at P, denoted by $I_P(f,g)$, is defined as follows: $I_P(f,g)=e$ if e is the largest integer such that there are e polynomials which are independent with respect to f and g at P; $I_P(f,g)=\infty$ if for every positive integer n, there are at least n polynomials which are independent with respect to f and g at P.

Let $\mathcal{O}_{\mathbb{A}_K^2,P}$ be the local ring of $\mathbb{A}_K^2$ at a point P which is defined as

$$\mathcal{O}_{\mathbb{A}_K^2,P}:=\left\{\frac{s}{u}\colon s,u\in K[x,y]\ \text{with}\ u(P)\neq 0\right\}. \tag{30}$$

Note that $\mathcal{O}_{\mathbb{A}_K^2,P}/\langle f,g\rangle$ is a vector space under K for some polynomials $f,g\in K[x,y]$ and that $\langle f,g\rangle\mathcal{O}_{\mathbb{A}_K^2,P}$ is an ideal of $\mathcal{O}_{\mathbb{A}_K^2,P}$.

Definition 4 ((see [13], Definition 2.3), ([2], Theorem 3)). Let $f(x,y), g(x,y)\in K[x,y]$, and $P\in\mathbb{A}_K^2$. The *intersection multiplicity* of f and g at P is the dimension of the vector space $\mathcal{O}_{\mathbb{A}_K^2,P}/\langle f,g\rangle$, denoted by

$$\widetilde{I}_P(f,g):=\dim_K\left(\frac{\mathcal{O}_{\mathbb{A}_K^2,P}}{\langle f,g\rangle}\right). \tag{31}$$

From Definitions 3 and 4, we have the following theorem which shows that the two different definitions of intersection multiplicity are equivalent.

Theorem 3. *Let $f(x,y)$, $g(x,y)\in K[x,y]$, and $P\in\mathbb{A}_K^2$, and then, $I_P(f,g)=\widetilde{I}_P(f,g)$.*

Proof. We assume that $I_P(f,g)=n$, and following Definition 3, we know that n is the largest number of polynomials which are independent with respect to f and g at P. We suppose that $q_1(x,y),\ldots,q_n(x,y)\in K[x,y]$ are polynomials which are independent with respect to f and g at P. Thenfor any other polynomial $e(x,y)\in K[x,y]$,

$$q_1(x,y),\ldots,q_n(x,y),e(x,y), \tag{32}$$

are dependent with respect to f and g at P.

Let $\varphi\colon K[x,y]\longrightarrow\mathcal{O}_{\mathbb{A}_K^2,P}/\langle f,g\rangle$ be a map defined by

$$p(x,y)\mapsto\frac{p(x,y)}{1}+\langle f,g\rangle\mathcal{O}_{\mathbb{A}_K^2,P}:==\left[\frac{p(x,y)}{1}\right]. \tag{33}$$

For any $p(x,y)$, $q(x,y)\in K[x,y]$, if $p(x,y)=q(x,y)$, then

$$\frac{(p(x,y)-q(x,y))}{1}+\langle f,g\rangle\mathcal{O}_{\mathbb{A}_K^2,P}=0+\langle f,g\rangle\mathcal{O}_{\mathbb{A}_K^2,P}, \tag{34}$$

which follows that

$$\frac{p(x,y)}{1}+\langle f,g\rangle\mathcal{O}_{\mathbb{A}_K^2,P}=\frac{q(x,y)}{1}+\langle f,g\rangle\mathcal{O}_{\mathbb{A}_K^2,P}. \tag{35}$$

This means that

$$\left[\frac{p(x,y)}{1}\right]=\left[\frac{q(x,y)}{1}\right]. \tag{36}$$

Therefore, φ is well defined. Thus, by Definition 4, it is suffice to prove that $\{[q_1/1],\ldots,[q_n/1]\}$ is a basis of the vector space $\mathcal{O}_{\mathbb{A}_K^2,P}/\langle f,g\rangle$ over K.

(1) First, we prove that $\{[q_1/1],\ldots,[q_n/1]\}$ are linearly independent over K.

We suppose that there exist $b_1,\ldots,b_n\in K$ such that

$$b_1\left[\frac{q_1}{1}\right]+\cdots+b_n\left[\frac{q_n}{1}\right]=0, \tag{37}$$

that is,

$$\frac{(b_1q_1+\cdots+b_nq_n)}{1}+\langle f,g\rangle\mathcal{O}_{\mathbb{A}_K^2,P}=\langle f,g\rangle\mathcal{O}_{\mathbb{A}_K^2,P}. \tag{38}$$

It follows that

$$\frac{(b_1q_1+\cdots+b_nq_n)}{1}\in\langle f,g\rangle\mathcal{O}_{\mathbb{A}_K^2,P}. \tag{39}$$

Thus, there exist $s(x,y),t(x,y),r(x,y)\in K[x,y]$ with $r(P)\neq 0$ such that

$$\frac{(b_1q_1+\cdots+b_nq_n)}{1}=\frac{(sf+tg)}{r}, \tag{40}$$

that is,

$$r(b_1q_1+\cdots+b_nq_n)=sf+tg. \tag{41}$$

Since $q_1,\ldots,q_n$ are independent with respect to f and g at P and $r(P)\neq 0$, we have $b_1=\cdots=b_n=0$ by Definition 2, which tells that $\{[q_1/1],\ldots,[q_n/1]\}$ are linearly independent over K.

(2) Second, we prove that $\{[q_1/1],\ldots,[q_n/1]\}$ are the generators of the vector space $\mathcal{O}_{\mathbb{A}_K^2,P}/\langle f,g\rangle$; that is, any $[e/r]\in\mathcal{O}_{\mathbb{A}^2,P}/\langle f,g\rangle$ is a linear combination of $\{[q_1/1],\ldots,[q_n/1]\}$ for any $e(x,y),r(x,y)\in K[x,y]$ with $r(P)\neq 0$.

Since for any polynomial $e(x,y),r(x,y)\in K[x,y]$ with $r(P)\neq 0$,

$$r(x,y)q_1(x,y),\ldots,r(x,y)q_n(x,y),e(x,y), \tag{42}$$

are dependent with respect to f and g at P, so there are polynomials

$$u(x,y),v(x,y),w(x,y)\in K[x,y], \tag{43}$$

and $b_0',\ldots,b_n'\in K$ such that

$$u(b_0'e+b_1'rq_1+\cdots+b_n'rq_n)=vf+wg, \tag{44}$$

where $u(P)\neq 0$ and $(b_0',\ldots,b_n')\neq(0,\ldots,0)$ by Definition 2. Thus, we have

$$b_0'\left[\frac{e}{r}\right]+b_1'\left[\frac{q_1}{1}\right]+\cdots+b_n'\left[\frac{q_n}{1}\right]=0. \tag{45}$$

Since $b_0'\neq 0$, we have

$$\left[\frac{e}{r}\right]=-\frac{b_1'}{b_0'}\left[\frac{q_1}{1}\right]-\cdots-\frac{b_n'}{b_0'}\left[\frac{q_n}{1}\right]. \tag{46}$$

In fact, if $b_0'=0$, then $b_1'=\cdots=b_n'=0$ following the fact that $q_1(x,y),\ldots,q_n(x,y)$ are dependent with respect to f and g at P. This is a contradiction.

Therefore, we obtain that $\{[q_1/1],\ldots,[q_n/1]\}$ is a basis of the vector space $\mathcal{O}_{\mathbb{A}_K^2,P}/\langle f,g\rangle$ over K, that is,

$$\widetilde{I}_P(f,g)=\dim_K\left(\frac{\mathcal{O}_{\mathbb{A}_K^2,P}}{\langle f,g\rangle}\right)=n=I_P(f,g). \tag{47}$$

In general, it is difficult to give the intersection multiplicity $\widetilde{I}_P(f,g)$ of two curves $f=0,g=0$ at some point P in $\mathbb{A}_K^2$. Since the intersection multiplicity is a geometry problem, according to the fact that the properties of figures of geometry are invariant under affine transformations which preserve the intersection multiplicity, we should try to transfer the intersection multiplicity to another easy case by using an affine transformation.

An *affine transformation* of $\mathbb{A}_K^2$ is a map $\varphi\colon\mathbb{A}_K^2\longrightarrow\mathbb{A}_K^2$ defined by

$$\begin{pmatrix}x\\y\end{pmatrix}\mapsto A\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}e\\k\end{pmatrix}, \tag{48}$$

where $A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$ is a 2×2 invertible matrix, i.e., $a_{11}a_{22}-a_{12}a_{21}\neq 0$, and $e,k\in K$. Note that an affine transformation preserves collinearity, i.e., all points lying on a line initially still lie on a line after the affine transformation, and the images $\varphi(f)=0,\varphi(g)=0$ of two curves $f=0,g=0$ intersecting at a point p intersect at the point $\varphi(p)$ under an affine transformation φ. In fact, every affine transformation can be represented as the composition of a linear transformation and a translation.

Following ([2], Section 3.3), we have the properties of the intersection multiplicity $\widetilde{I}_P(f,g)$ of curves f,g at the point $P\in\mathbb{A}_K^2$, which are same as the properties of the intersection multiplicity $I_P(f,g)$. □

Property 2. Let $f=0,g=0$ and $h=0$ be curves in $\mathbb{A}_K^2$ and $P\in\mathbb{A}_K^2$. Then, we have the following:

(1) $\widetilde{I}_P(f,g)$ is a nonnegative integer or ∞.

(2) $\widetilde{I}_P(f,g)\geq 1$ if and only if f and g both contain the point P.

(3) Let ϕ be an affine transformation that maps $P(x,y)\in\mathbb{A}_K^2$ (resp., $f=0,g=0$) *to* $P'(x',y')$ (resp., $f'=0,g'=0$). Then,

$$\widetilde{I}_P(f(x,y), g(x,y)) = \widetilde{I}_{P'}(f'(x',y'), g'(x',y')). \tag{49}$$

(4) $\widetilde{I}_P(f,g) = \widetilde{I}_P(g,f)$.

(5) $\widetilde{I}_P(f,gh) = \widetilde{I}_P(f,g) + \widetilde{I}_P(f,h)$.

(6) $\widetilde{I}_P(f,g) = \widetilde{I}_P(f,g+fh)$.

3.2. Intersection Multiplicity between Curves and Lines in $\mathbb{A}_K^2$. To consider the intersection multiplicities $\widetilde{I}_P(f,g)$ of curves in $\mathbb{A}_K^2$ with $\mathrm{char}(K) = 0$, it is important to consider some properties of intersection multiplicity between curves and lines in $\mathbb{A}_K^2$. Let $f(x,y)$ be an affine algebraic curve in $\mathbb{A}_K^2$, and we can write that

$$f(x,y) = f_d + f_{d+1} + \cdots, \tag{50}$$

where $f_k(x,y)$ is the sum of the terms of degree k in $f(x,y)$. Let $\mathrm{ord}(f)$ be the smallest degree of the terms of f with respect to the origin (0,0), and if $\mathrm{ord}(f) = d$, then $f_d(x,y)$ can be written as

$$f_d(x,y) = (a_1x + b_1y)^{s_1}(a_2x + b_2y)^{s_2}\cdots(a_kx + b_ky)^{s_k}, \tag{51}$$

for distinct lines $a_ix + b_iy = 0$, where s_i is a positive integer.

According to Definition 4, in order to complete the proof of Lemma 5, we should introduce the localization which is a very powerful technique in commutative algebra, and localization always allows us to reduce questions on rings and modules to a union of smaller local problems.

Definition 5 ((see [17], Lecture 9)). Let A be an integral domain, and $\mathfrak{p}$ is a prime ideal in A. The localization of A at $\mathfrak{p}$, denoted by $A_{\mathfrak{p}}$, is defined as

$$A_{\mathfrak{p}} := \left\{\frac{a}{b}\colon a \in A, b \in A - \mathfrak{p}\right\}, \tag{52}$$

where a/b is the equivalent class under the equivalence relation $\sim$ which is defined by $a/b \sim c/d$ if there exists $t \in A - \mathfrak{p}$ such that $t(ad - bc) = 0$.

Note that the equivalence relation $\sim$ owned the addition operation:

$$\frac{a}{b} + \frac{c}{d} = \frac{(ad + bc)}{bd}, \tag{53}$$

and the multiplication operation is

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}, \tag{54}$$

and the localization $A_{\mathfrak{p}}$ is a local ring with a maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$. For the integral domain $K[x,y]$, we have the localization $K[x,y]_{\langle x,y\rangle}$ of $K[x,y]$ at the maximal ideal $\langle x,y\rangle \subset K[x,y]$; i.e., $K[x,y]_{\langle x,y\rangle} = \{f/g\colon f,g \in K[x,y] \text{ and } g(0,0) \neq 0\}$, and $K[x,y]_{\langle x,y\rangle}$ is a local ring with the unique maximal ideal $\mathfrak{m}_{\langle x,y\rangle} := \{f/g \in K[x,y]_{\langle x,y\rangle}\colon f(0,0) = 0\}$.

Lemma 5. *Let $f(x,y) = 0$ be a curve, and $O = (0,0)$ is the origin in $\mathbb{A}_K^2$. We assume that $\mathrm{ord}(f) = d$ and that $\ell(x,y) = 0$ is a line through the origin O. Then, $\widetilde{I}_O(f,\ell) > d$ if ℓ is a factor of f_d; otherwise, $\widetilde{I}_O(f,\ell) = d$.*

Proof. From Definition 4, we have

$$\widetilde{I}_O(f,\ell) = \dim_K\left(\frac{\mathcal{O}_{\mathbb{A}_K^2,O}}{\langle f,\ell\rangle}\right). \tag{55}$$

□

Case 1. We suppose that ℓ is not a factor of f_d. Since $\langle x,y\rangle K[x,y]$ is a prime ideal in $K[x,y]$, from Definitions 4 and 5, we have

$$\dim_K\left(\frac{\mathcal{O}_{\mathbb{A}_K^2,O}}{\langle f,\ell\rangle}\right) = \dim_K\left(\frac{K[x,y]_{\langle x,y\rangle}}{\langle f,\ell\rangle}\right), \tag{56}$$

where

$$\frac{K[x,y]_{\langle x,y\rangle}}{\langle f,\ell\rangle} = \left\{\frac{u_1}{u_2} + \langle f,\ell\rangle K[x,y]_{\langle x,y\rangle}\colon u_1,u_2 \in K[x,y], u_2(0,0) \neq 0\right\}, \tag{57}$$

which have addition operation

$$\left(\frac{u_1}{u_2} + \langle f,\ell\rangle K[x,y]_{\langle x,y\rangle}\right) + \left(\frac{v_1}{v_2} + \langle f,\ell\rangle K[x,y]_{\langle x,y\rangle}\right) = \frac{(u_1v_2 + v_1u_2)}{u_2v_2} + \langle f,\ell\rangle K[x,y]_{\langle x,y\rangle}, \tag{58}$$

and multiplication operation is

$$\left(\frac{u_1}{u_2} + \langle f,\ell\rangle K[x,y]_{\langle x,y\rangle}\right) + \left(\frac{v_1}{v_2} + \langle f,\ell\rangle K[x,y]_{\langle x,y\rangle}\right) = \frac{(u_1v_1)}{u_2v_2} + \langle f,\ell\rangle K[x,y]_{\langle x,y\rangle}. \tag{59}$$

Similarly, we have

$$K[x]_{\langle x\rangle} = \left\{\frac{w_1}{w_2}\colon w_1, w_2 \in K[x], w_2(0) \neq 0\right\},$$
$$\frac{K[x]_{\langle x\rangle}}{\langle w(x)\rangle} = \left\{\frac{w_1}{w_2} + \langle w(x)\rangle K[x]_{\langle x\rangle}\colon w_1, w_2 \in K[x], w_2(0) \neq 0\right\}. \quad (60)$$

Note that $K[x] \subsetneq K[x]_{\langle x\rangle}$ and $\langle w(x)\rangle K[x]_{\langle x\rangle}$ is an ideal of $K[x]_{\langle x\rangle}$. If $w(0) = 0$, we can write that

$$w(x) = x^{\mathrm{ord}(w)} \cdot q(x), q(0) \neq 0. \quad (61)$$

Thus, we have $1/q(x) \in K[x]_{\langle x\rangle}$ and $x^{\mathrm{ord}(w)} = w(x)/q(x)$. It tells that

$$\langle w(x)\rangle K[x]_{\langle x\rangle} = \langle x^{\mathrm{ord}(w)}\rangle K[x]_{\langle x\rangle},$$
$$\dim_K\left(\frac{K[x]_{\langle x\rangle}}{\langle w(x)\rangle}\right) = \dim_K\left(\frac{K[x]_{\langle x\rangle}}{\langle x^{\mathrm{ord}(w)}\rangle}\right). \quad (62)$$

Since $\ell = 0$ is a line through the origin O in $\mathbb{A}_K^2$, we can write $\ell = 0$ as

$$ax + by = 0, \quad (63)$$

where a and b are not all zeros. We may assume that b is nonzero, then the line $\ell = 0$ is $y = \mathrm{d}x, d = -a/b$. Thus, we have

$$\frac{K[x,y]_{\langle x,y\rangle}}{\langle f,\ell\rangle} \cong \frac{K[x]_{\langle x\rangle}}{\langle f(x,dx)\rangle}. \quad (64)$$

In fact, let

$$T\colon \frac{K[x,y]_{\langle x,y\rangle}}{\langle f,\ell\rangle} \longrightarrow \frac{K[x]_{\langle x\rangle}}{\langle f(x,dx)\rangle}, \quad (65)$$

be the map defined by

$$\frac{u_1}{u_2} + \langle f,\ell\rangle K[x,y]_{\langle x,y\rangle} \mapsto \frac{u_1(x,dx)}{u_2(x,dx)} + \langle f(x,dx)\rangle K[x]_{\langle x\rangle}. \quad (66)$$

Let $[u_1/u_2] \coloneqq u_1/u_2 + \langle f,\ell\rangle K[x,y]_{\langle x,y\rangle}$. For any

$$\left[\frac{u_1}{u_2}\right], \left[\frac{v_1}{v_2}\right] \in \frac{K[x,y]_{\langle x,y\rangle}}{\langle f,\ell\rangle}, \quad (67)$$

if $[u_1/u_2] = [v_1/v_2]$, that is,

$$\frac{u_1}{u_2} - \frac{v_1}{v_2} \in \langle f,\ell\rangle K[x,y]_{\langle x,y\rangle}, \quad (68)$$

which implies that there are $s_1/s_2, t_1/t_2 \in K[x,y]_{\langle x,y\rangle}$ such that

$$\frac{u_1}{u_2} - \frac{v_1}{v_2} = \left(\frac{s_1}{s_2}\right) f + \left(\frac{t_1}{t_2}\right)\ell, \quad (69)$$

and then, under the map T, we have

$$\frac{u_1(x,dx)}{u_2(x,dx)} - \frac{v_1(x,dx)}{v_2(x,dx)} = \left(\frac{s_1(x,dx)}{s_2(x,dx)}\right) f(x,dx), \quad (70)$$

which follows that

$$\frac{u_1(x,dx)}{u_2(x,dx)} - \frac{v_1(x,dx)}{v_2(x,dx)} \in \langle f(x,dx)\rangle K[x]_{\langle x\rangle}. \quad (71)$$

Hence, we have

$$T\left(\left[\frac{u_1}{u_2}\right]\right) = \frac{u_1(x,dx)}{u_2(x,dx)} + \langle f(x,dx)\rangle K[x]_{\langle x\rangle}$$
$$= \frac{v_1(x,dx)}{v_2(x,dx)} + \langle f(x,dx)\rangle K[x]_{\langle x\rangle} \quad (72)$$
$$= T\left(\left[\frac{v_1}{v_2}\right]\right),$$

which means that the map T is well defined.

Furthermore, for any $[u_1/u_2], [v_1/v_2] \in K[x,y]_{\langle x,y\rangle}/\langle f,\ell\rangle$, it is clear that

$$T\left(\left[\frac{u_1}{u_2}\right] + \left[\frac{v_1}{v_2}\right]\right) = T\left(\left[\frac{u_1}{u_2}\right]\right) + T\left(\left[\frac{v_1}{v_2}\right]\right),$$
$$T\left(\left[\frac{u_1}{u_2}\right] \cdot \left[\frac{v_1}{v_2}\right]\right) = T\left(\left[\frac{u_1}{u_2}\right]\right) \cdot T\left(\left[\frac{v_1}{v_2}\right]\right), \quad (73)$$

which tells that the map T is a ring homomorphism.

On the other hand, if $[u_1/u_2] \neq [v_1/v_2]$, that is,

$$\frac{u_1}{u_2} + \langle f,\ell\rangle K[x,y]_{\langle x,y\rangle} \neq \frac{v_1}{v_2} + \langle f,\ell\rangle K[x,y]_{\langle x,y\rangle}, \quad (74)$$

which implies that

$$\frac{u_1}{u_2} - \frac{v_1}{v_2} \notin \langle f,\ell\rangle K[x,y]_{\langle x,y\rangle}, \quad (75)$$

and thus, for any $s_1/s_2, t_1/t_2 \in K[x,y]_{\langle x,y\rangle}$, we have

$$\frac{u_1}{u_2} - \frac{v_1}{v_2} \neq \left(\frac{s_1}{s_2}\right) f + \left(\frac{t_1}{t_2}\right)\ell. \quad (76)$$

Consequently, under the map T, we have

$$\frac{u_1(x,dx)}{u_2(x,dx)} - \frac{v_1(x,dx)}{v_2(x,dx)} \neq \left(\frac{s_1(x,dx)}{s_2(x,dx)}\right) f(x,dx), \quad (77)$$

which means that $T([u_1/u_2]) \neq T([v_1/v_2])$. This tells that the map T is injective. It is obvious that the map T is surjective. Therefore, we have

$$\frac{K[x,y]_{\langle x,y\rangle}}{\langle f,\ell\rangle} \cong \frac{K[x]_{\langle x\rangle}}{\langle f(x,dx)\rangle}. \quad (78)$$

Following (62), we have

$$\frac{K[x,y]_{\langle x,y\rangle}}{\langle f,\ell\rangle} \cong \frac{K[x]_{\langle x\rangle}}{\langle x^{\mathrm{ord}(f)}\rangle} = \frac{K[x]_{\langle x\rangle}}{\langle x^d\rangle}. \quad (79)$$

Next, we prove that $\dim_K(K[x]_{\langle x\rangle}/\langle x^d\rangle) = d$. Since $K[x]\subsetneq K[x]_{\langle x\rangle}$ and $K[x]_{\langle x\rangle}/\langle x^d\rangle$ are the quotient ring of $K[x]_{\langle x\rangle}$, we know that the following diagram is commutative.

$$\begin{array}{ccc} K[x] & \longrightarrow & K[x]_{\langle x\rangle} \\ & \overset{\phi}{\searrow} & \downarrow \\ & & K[x]_{\langle x\rangle}/\langle x^d\rangle \end{array}$$

Naturally, we can let

$$\phi\colon K[x] \longrightarrow \frac{K[x]_{\langle x\rangle}}{\langle x^d\rangle}, \tag{80}$$

be the map defined by

$$q(x)\mapsto \frac{q(x)}{1} + \langle x^d\rangle K[x]_{\langle x\rangle} := \left[\frac{q(x)}{1}\right]. \tag{81}$$

It is clear that ϕ is well defined. To prove $\dim_K(K[x]_{\langle x\rangle}/\langle x^d\rangle) = d$, it is suffice to show that $\{[1/1],[x/1],\ldots,[x^{d-1}/1]\}$ is a basis of $K[x]_{\langle x\rangle}/\langle x^d\rangle$ over K.

(i) First, $\{[1/1],[x/1],\ldots,[x^{d-1}/1]\}$ are linearly independent over K. In fact, we assume that there exist $b_1,\ldots,b_d \in K$ such that

$$b_1\left[\frac{1}{1}\right] + b_2\left[\frac{x}{1}\right] + \cdots + b_d\left[\frac{x^{d-1}}{1}\right] = 0, \tag{82}$$

that is,

$$\frac{\left(b_1 + b_2x + \cdots + b_dx^{d-1}\right)}{1} + \langle x^d\rangle K[x]_{\langle x\rangle} = \langle x^d\rangle K[x]_{\langle x\rangle}. \tag{83}$$

It follows that

$$\frac{\left(b_1 + b_2x + \cdots + b_dx^{d-1}\right)}{1} \in \langle x^d\rangle K[x]_{\langle x\rangle}. \tag{84}$$

This means that there exists $s(x)/r(x) \in K[x]_{\langle x\rangle}$ such that

$$\frac{\left(b_1 + b_2x + \cdots + b_dx^{d-1}\right)}{1} = \left(\frac{s(x)}{r(x)}\right)\cdot x^d, \tag{85}$$

which tells that

$$r(x)\left(b_1 + b_2x + \cdots + b_dx^{d-1}\right) = s(x)x^d. \tag{86}$$

Since $1, x, \ldots, x^{d-1}$ are linearly independent over K, following Definition 2 we have that $\{[1/1],[x/1],\ldots,[x^{d-1}/1]\}$ are independent with respect to x^d at 0 over K.

(ii) Second, we have that $\{[1/1],[x/1],\ldots,[x^{d-1}/1]\}$ generate the vector space $K[x]_{\langle x\rangle}/\langle x^d\rangle$. In other words, any $[e/r] \in K[x]_{\langle x\rangle}/\langle x^d\rangle$ is a linear combination of $\{[1/1],[x/1],\ldots,[x^{d-1}/1]\}$, where $e(x), r(x) \in K[x], r(0)\neq 0$.

By Definition 2, we know that for any polynomial $e(x)\in K[x]$,

$$r(x), r(x)x, \ldots, r(x)x^{d-1}, e(x), \tag{87}$$

are dependent with respect to x^d at 0, so there are polynomials $u(x), v(x) \in K[x]$, and $b'_0,\ldots,b'_d \in K$ such that

$$u\left(b'_0e + b'_1r + b'_2rx + \cdots + b'_drx^{d-1}\right) = vx^d, \tag{88}$$

where $u(0)\neq 0$ and $(b'_0,\ldots,b'_d)\neq(0,\ldots,0)$. Thus, we have

$$b'_0\left[\frac{e}{r}\right] + b'_1\left[\frac{1}{1}\right] + b'_2\left[\frac{x}{1}\right] + \cdots + b'_d\left[\frac{x^{d-1}}{1}\right] = 0. \tag{89}$$

Since $b'_0\neq 0$, we have

$$\left[\frac{e}{r}\right] = -\frac{b'_1}{b'_0}\left[\frac{1}{1}\right] - \frac{b'_2}{b'_0}\left[\frac{x}{1}\right] - \cdots - \frac{b'_d}{b'_0}\left[\frac{x^{d-1}}{1}\right]. \tag{90}$$

In fact, if $b'_0 = 0$, according to the fact that $\{[1/1],[x/1],\ldots,[x^{d-1}/1]\}$ are independent with respect to x^d at 0 over K, we have $b'_1 = \cdots = b'_d = 0$, which is a contradiction.

Therefore, $\{[1/1],[x/1],\ldots,[x^{d-1}/1]\}$ is a basis of the vector space $K[x]_{\langle x\rangle}/\langle x^d\rangle$ over K. Hence, we have

$$\dim_K\left(\frac{K[x]_{\langle x\rangle}}{\langle x^d\rangle}\right) = d. \tag{91}$$

Following (56) and (79), we have

$$\widetilde{I}_O(f,\ell) = \dim_K\left(\frac{K[x,y]_{\langle x,y\rangle}}{\langle f,\ell\rangle}\right) = d. \tag{92}$$

Case 2. If ℓ is a factor of f_d, by Property 2, we have

$$\widetilde{I}_O(f,\ell) = \widetilde{I}_O(f - f_d,\ell). \tag{93}$$

Since $\operatorname{ord}(f - f_d) > d$, according to the above proof, we have $\widetilde{I}_O(f,\ell) > d$ easily. Therefore, we obtain the assertion.

3.3. Intersection Multiplicity of Curves under Fold Point in $\mathbb{A}_K^2$. Similar as in Section 2.2, we still have some similar results about the intersection multiplicities of curves under fold point in $\mathbb{A}_K^2$. For the fold point, Hirschfeld et al. (([18], Definition 1.8)) also give the definition of the m_0-fold point of an affine curve. In this paper, we still use the similar definition of fold point as in Definition 1.

Definition 6. Let $f(x,y) = 0$ be a curve and P a point in $\mathbb{A}_K^2$. We say that the point P is a *d-fold point* of $f = 0$ if there is a nonnegative integer d, such that there are at most d distinct lines that intersect f at P more than d times and that all other lines intersect f at P exactly $d$times.

By Property2 we know that the affine transformations preserve the intersection multiplicity of curves in $\mathbb{A}_K^2$. Following Definition 6, we note that the definition of the fold point depends on lines, so it is a natural question to ask whether the intersection multiplicity $\widetilde{I}_P(f,\ell)$ between curves and lines in $\mathbb{A}_K^2$ can be transformed into a simple case. We give the following lemma which is helpful in completing the proof of Theorem 4.

Lemma 6. *Let $f(x,y)=0$ be a curve and P a point in $\mathbb{A}_K^2$. If P is a d-fold point of $f(x,y)=0$, then there exist a line $\ell=0$ through the point P and an affine transformation T, such that*

$$\widetilde{I}_P(f,\ell)=\widetilde{I}_O(f',y)=d, \tag{94}$$

where $T(f)=f'$,$T(\ell)=y$, and $T(P)=O=(0,0)$.

Proof. Since P is a d-fold point of $f=0$ and d is finite, by Lemma 5, we know that there exist infinite lines $\{\ell_i=0, i=1,2,\ldots\}$ through the point P such that $\widetilde{I}_P(f,\ell_i)=d$. It is obvious that there exist an affine transformation T: $\mathbb{A}_K^2\longrightarrow\mathbb{A}_K^2$ and some ℓ_j with $\widetilde{I}_P(f,\ell_j)=d$, $j\in\{1,2,\ldots\}$ such that $T(P)=O=(0,0)$ and $T(\ell_j)=y$. According to Property 2, it states that the affine transformation preserves the intersection multiplicity of curves in $\mathbb{A}_K^2$. Thus, we obtain the assertion. □

Example 3. Let $f(x,y)=x^2+y^2-x^3$ be an algebraic curve, and we take $P=(1,0)$. Thus, we have $f(P)=0$, and we can write

$$f(x,y)=x^2+y^2-x^3=-(x-1)^3+y^2-2(x-1)^2-(x-1). \tag{95}$$

Since $-(x-1)=-x+1=0$ is the only line such that $\widetilde{I}_P(f,-x+1)=2>1$, any other line $\ell'=0$ which is different from the line $-x+1=0$ satisfies $\widetilde{I}_P(f,\ell')=1$. Thus, by Definition 6, we know that P is a 1-fold point of $f=0$.

Taking $\ell=x+2y-1$, we have $\widetilde{I}_P(f,\ell)=1$ from Lemma 5. Since affine transformations preserve intersection multiplicity of curves, our aim is to find an affine transformation T such that

$$\widetilde{I}_P(f,\ell)=\widetilde{I}_O(f',y)=1. \tag{96}$$

By the proof of Lemma 6, we can easy give the affine transformation T_1: $\mathbb{A}_K^2\longrightarrow\mathbb{A}_K^2$ defined by

$$\begin{pmatrix} x\\ y\end{pmatrix}\mapsto\begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}+\begin{pmatrix} 1\\ 0\end{pmatrix}, \tag{97}$$

such that $T_1(P)=O$ and

$$T_1(\ell)=(1-x)+2y-1=2y-x. \tag{98}$$

Also, we have the affine transformation T_2 defined by

$$\begin{pmatrix} x\\ y\end{pmatrix}\mapsto\begin{pmatrix} 2 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}, \tag{99}$$

such that $T_2(O)=O$ and

$$T_2(2y-x)=2(x+y)-(2x+y)=y. \tag{100}$$

Hence, setting $T=T_2{}^\circ T_1$, we have $T(P)=O$ and

$$T(f)=8x^3+y^3+12x^2y+6xy^2-7x^2-y^2-6xy+2x+y. \tag{101}$$

Following Property 2, we have

$$\begin{aligned}\widetilde{I}_P(f,\ell)&=\widetilde{I}_O\big(8x^3+y^3+12x^2y+6xy^2\\&\quad-7x^2-y^2-6xy+2x+y,y\big)\\&=\widetilde{I}_O\big(8x^3-7x^2+2x,y\big)\\&=1.\end{aligned} \tag{102}$$

Theorem 4. *Let $f(x,y)=0$ and $g(x,y)=0$ be curves and P a point in $\mathbb{A}_K^2$. We suppose that P is a d-fold point of $f(x,y)=0$ and an e-fold point of $g(x,y)=0$, respectively. If $1\le d\le e$, then there is a curve $h(x,y)=0$ such that*

$$\widetilde{I}_P(f,g)=d+\widetilde{I}_P(f,h), \tag{103}$$

where P is an m-fold point of $h(x,y)=0$ and $m\ge e-1$.

Proof. When $d=e=1$, that is, P is, respectively, a 1-fold point of $f(x,y)=0$ and a 1-fold point of $g(x,y)=0$. By Definition 6, we know that there exists a line $\ell'=0$ such that

$$\widetilde{I}_P(f,\ell')=1=\widetilde{I}_P(g,\ell'). \tag{104}$$

Following Lemma 6, there exists an affine transformation T_1: $\mathbb{A}_K^2\longrightarrow\mathbb{A}_K^2$ defined by

$$\begin{pmatrix} x\\ y\end{pmatrix}\mapsto\begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}+\begin{pmatrix} m\\ n\end{pmatrix}, b_{11}b_{22}-b_{12}b_{21}\neq 0, \tag{105}$$

such that

$$\widetilde{I}_P(f,\ell')=\widetilde{I}_O(f',y)=1, I_P(g,\ell')=\widetilde{I}_O(g',y)=1, \tag{106}$$

where $T_1(f)=f'$, $T_1(g)=g'$, $T_1(\ell')=y$, and $T_1(P)=O=(0,0)$. Hence, we can write

$$f'(x,y)=x\alpha(x)+y\delta(x,y), \tag{107}$$

for some polynomials $\alpha(x)$ and $\delta(x,y)$ in $K[x,y]$ with $\alpha(0)\neq 0$ and $\mathrm{ord}(\delta)\ge 0$. Also, we can write

$$g'(x,y)=x\alpha'(x)+y\delta'(x,y), \tag{108}$$

for some polynomials $\alpha'(x)$ and $\delta'(x,y)$ in $K[x,y]$ with $\alpha'(0)\neq 0$ and $\mathrm{ord}(\delta')\ge 0$. From Property 2, we have

$$\widetilde{I}_O(f',g')=1+\widetilde{I}_O(f',h'), \tag{109}$$

where $h'=\delta'\alpha-\delta\alpha'$ with $\mathrm{ord}(h')\ge 0$ and O is an s-fold point of h' with $s\ge 0$.

By (48) and the definition of T_1, we know that there exists an affine transformation $\widetilde{T}_1$: $\mathbb{A}_K^2 \longrightarrow \mathbb{A}_K^2$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}^{-1} \begin{pmatrix} m \\ n \end{pmatrix}, \tag{110}$$

such that $\widetilde{T}_1(f') = f$, $\widetilde{T}_1(g') = g$, and $\widetilde{T}_1(h') = h$. Therefore, according to the fact that $\widetilde{I}_O(f', g') = 1 + \widetilde{I}_O(f', h')$, we have

$$\widetilde{I}_P(f, g) = 1 + \widetilde{I}_P(f, h), \tag{111}$$

where P is an s-fold point of $h = 0$ with $s \geq 0$. Thus, we obtain the assertion when $d = e = 1$.

When $e \geq 2$, that is, P is, respectively, a d-fold point of $f(x, y) = 0$ and an e-fold point of $g(x, y) = 0$. Following Definition 6, we know that there exists a line $\ell = 0$ such that

$$\widetilde{I}_P(f, \ell) = d, \widetilde{I}_P(g, \ell) = e. \tag{112}$$

By Lemma 6 there exists an affine transformation T: $\mathbb{A}_K^2 \longrightarrow \mathbb{A}_K^2$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ k \end{pmatrix}, a_{11}a_{22} - a_{12}a_{21} \neq 0, \tag{113}$$

such that

$$\widetilde{I}_P(f, \ell) = \widetilde{I}_O(f', y) = d, \widetilde{I}_P(g, \ell) = \widetilde{I}_O(g', y) = e, \tag{114}$$

where $T(f) = f', T(g) = g'$, $T(\ell) = y$, and $T(P) = O$. From Lemma 5, we have that O is a d-fold point of $f' = 0$ if and only if $\operatorname{ord}(f') = d$, which means that y is not a factor of f'_d. This tells that the coefficient of the term x^d in f'_d is nonzero, and thenwe can write

$$f'(x, y) = x^d q(x) + y\beta(x, y), \tag{115}$$

where $q(x)$ and $\beta(x, y)$ are polynomials in $K[x, y]$ with $q(0) \neq 0$ and $\operatorname{ord}(\beta) \geq d - 1$. Similarly, we can write

$$g'(x, y) = x^e q'(x) + y\beta'(x, y), \tag{116}$$

where $q'(x)$ and $\beta'(x, y)$ are polynomials in in $K[x, y]$ with $q'(0) \neq 0$ and $\operatorname{ord}(\beta') \geq e - 1$. Thus, according to (115) and (116), we have

$$x^{e-d} \cdot q' f' - q g' = y h', \tag{117}$$

where $h'(x, y) = x^{e-d} \cdot \beta q' - q\beta'$ with $h'(0, 0) = 0$ and $\operatorname{ord}(h') = s \geq e - 1$. On the other hand, following (117), we also have that

$$g' = \left(x^{e-d} \cdot \frac{q'}{q} \right) f' - \left(\frac{1}{q} \right) y h', \tag{118}$$

where $x^{e-d} \cdot q'/q, 1/q \in \mathcal{O}_{\mathbb{A}_K^2, O}$. Therefore, we have

$$\langle f', g' \rangle \mathcal{O}_{\mathbb{A}_K^2, O} = \langle f', y h' \rangle \mathcal{O}_{\mathbb{A}_K^2, O}. \tag{119}$$

It follows that

$$\begin{aligned} \widetilde{I}_O(f', g') &= \dim_K \left(\frac{\mathcal{O}_{\mathbb{A}_K^2, O}}{\langle f', g' \rangle} \right) \\ &= \dim_K \left(\frac{\mathcal{O}_{\mathbb{A}_K^2, O}}{\langle f', y h' \rangle} \right) = \widetilde{I}_O(f', y h'). \end{aligned} \tag{120}$$

Hence, by Property 2, we have that

$$\begin{aligned} \widetilde{I}_O(f', g') &= \widetilde{I}_O(f', y) + \widetilde{I}_O(f', h') \\ &= \widetilde{I}_O\left(x^d q(x), y\right) + \widetilde{I}_O(f', h') \\ &= d + \widetilde{I}_O(f', h'). \end{aligned} \tag{121}$$

Since $\operatorname{ord}(h') = s \geq e - 1$, following Lemma 5, we have that O is an s-fold point of $h' = 0$. By (48) and the definition of T, we know that there exists an affine transformation $\widetilde{T}$: $\mathbb{A}_K^2 \longrightarrow \mathbb{A}_K^2$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} e \\ k \end{pmatrix}, \tag{122}$$

such that $\widetilde{T}(f') = f$, $\widetilde{T}(g') = g$, and $\widetilde{T}(h') = h$. Therefore, according to the fact that $\widetilde{I}_O(f', g') = d + \widetilde{I}_O(f', h')$, we have

$$\widetilde{I}_P(f, g) = d + \widetilde{I}_P(f, h), \tag{123}$$

where P is an s-fold point of $h = 0$ and $s \geq e - 1$. Thus, we obtain the assertion.

Hartshorne (([7], Chapter 1)) proved that the intersection multiplicity $\widetilde{I}_P(f, g)$ of distinct curves $f = 0, g = 0$ at a point P in $\mathbb{A}_K^2$ is not less than $\operatorname{ord}_P(f) \cdot \operatorname{ord}_P(g)$. Similar as the proof of Theorem 2, we also have the following corollaries by using the fold point. □

Corollary 2. *Let $f(x, y) = 0$ and $g(x, y) = 0$ be curves and P a point in $\mathbb{A}_K^2$. Suppose that P is a d-fold point of $f(x, y) = 0$ and an e-fold point of $g(x, y) = 0$, respectively, then*

$$\widetilde{I}_P(f, g) \geq de. \tag{124}$$

Corollary 3. *Let $f(x, y) = 0$ and $g(x, y) = 0$ be curves in $\mathbb{A}_K^2$. We assume that the origin O is a d-fold point of $f = 0$ and an e-fold point of $g = 0$, respectively. Let f_d (resp.,g_e) be the sum of the terms of the degree d (resp.,e) in f (resp.,g). Then, $\widetilde{I}_O(f, g) > de$ if and only if f_d and g_e have a common factor of positive degrees. Equivalently, $\widetilde{I}_O(f, g) = de$ if and only if f_d and g_e have no common factors of positive degrees.*

Example 4. We calculate the intersection multiplicity $\widetilde{I}_O(f, g)$ at the origin O of two algebraic curves $f = x^3 y^3 + 2x^3 y - 2x^2$ and $g = x^5 + 3x^2 y^2 - x^2 - 3y^2$ in $\mathbb{A}_K^2$.

It is obvious that $\mathrm{ord}(f) = 2$ and $\mathrm{ord}(g) = 2$. From Definition 6, we know that the origin O is a 2-fold point of both f and g. By Corollary 2, we know that $\widetilde{I}_O(f, g) \geq 4$. Since $f_2 = -2x^2$ and $g_2 = -x^2 - 3y^2$ have no common factors, we have $\widetilde{I}_O(f, g) = 4$ following Corollary 3.

In fact, we have

$$\begin{aligned} f \cdot \left(x^3 - 1\right) + 2g &= yh, \\ g &= \frac{1}{2} yh - \frac{x^3 - 1}{2} f, \end{aligned} \tag{125}$$

where $h(x, y) = (x^3 - 1)(x^3 y^2 + 2x^3) + 6x^2 y - 6y$ and $\mathrm{ord}(h) = 1, h(0, 0) = 0$, which implies that

$$\langle f, yh\rangle \mathcal{O}_{\mathbb{A}_K^2, O} = \langle f, g\rangle \mathcal{O}_{\mathbb{A}_K^2, O}. \tag{126}$$

Then,

$$\widetilde{I}_O\langle f, g\rangle = \dim_K\left(\frac{\mathcal{O}_{\mathbb{A}_K^2, O}}{(f, g)}\right) = \dim_K\left(\frac{\mathcal{O}_{\mathbb{A}_K^2, O}}{(f, yh)}\right) = \widetilde{I}_O\langle f, yh\rangle. \tag{127}$$

Thus, following Property 2 we have

$$\begin{aligned} \widetilde{I}_O(f, g) &= \widetilde{I}_O(f, y) + \widetilde{I}_O(f, h) \\ &= \widetilde{I}_O\left(-2x^2, y\right) + \widetilde{I}_O(f, h) \\ &= 2 + \widetilde{I}_O(f, h). \end{aligned} \tag{128}$$

Note that the origin O is a 1-fold point of h. Thus, from Theorem 4, we can also obtain that $\widetilde{I}_O(f, g) = 2 + \widetilde{I}_O(f, h)$. Similarly, we have

$$\begin{aligned} x\left(x^3 - 1\right) f + h &= yh', \\ h &= yh' - x\left(x^3 - 1\right) f, \end{aligned} \tag{129}$$

where $h'(x, y) = x(x^3 - 1)(x^3 y^2 + 2x^3) + x^6 y - x^3 y + 6x^2 - 6$ and $\mathrm{ord}(h') = 0, h'(0, 0) \neq 0$, which implies that

$$\langle f, yh'\rangle \mathcal{O}_{\mathbb{A}_K^2, O} = \langle f, h\rangle \mathcal{O}_{\mathbb{A}_K^2, O}. \tag{130}$$

This means that

$$\begin{aligned} \widetilde{I}_O(f, h) &= \dim_K\left(\frac{\mathcal{O}_{\mathbb{A}_K^2, O}}{(f, h)}\right) \\ &= \dim_K\left(\frac{\mathcal{O}_{\mathbb{A}_K^2, O}}{(f, yh')}\right) = \widetilde{I}_O(f, yh'). \end{aligned} \tag{131}$$

Following Proposition 3.5, since $h(0, 0) \neq 0$, we have $\widetilde{I}_O(f, h') = 0$ and

$$\begin{aligned} \widetilde{I}_O(f, h) &= \widetilde{I}_O(f, y) + \widetilde{I}_O(f, h') \\ &= \widetilde{I}_O\left(-2x^2, y\right) + \widetilde{I}_O(f, h') \\ &= 2 + \widetilde{I}_O(f, h') = 2. \end{aligned} \tag{132}$$

Therefore, we have $\widetilde{I}_O(f, g) = 4$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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