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Research Article

The Characteristic of the Minimal Ideals and the Minimal Generalized Ideals in Rings

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In this paper, we prove that the characteristic of a minimal ideal and a minimal generalized ideal, which is meant to be one of minimal left ideal, minimal right ideal, bi-ideal, quasi-ideal, and (m, n)-ideal in a ring, is either zero or a prime number p. When the characteristic is zero, then the minimal ideal (minimal generalized ideal) as additive group is torsion-free, and when the characteristic is p, then every element of its additive group has order p. Furthermore, we give some properties for minimal ideals and for generalized ideals which depend on their characteristics.

1. Introduction

Throughout this paper by ring, we shall mean an associative ring which does not necessarily has an identity element. The motivation for this paper steams from a proposition in [1] which states that the characteristic of a simple ring is either zero or a prime number. This result for the characteristic of a simple ring, which is the analogue of a minimal ideal of ring itself, is extended for the minimal ideals and the minimal generalized ideals. By generalized ideals, we shall mean the left ideals, the right ideals, the quasi-ideals [2], the bi-ideals [3], and (m, n)-ideals for non-negative integers m, n [4].

Firstly, we will prove this theorem: the characteristic of a minimal ideal (generalized ideal) I in a ring $(A, +, \cdot)$ is either zero or a prime number p. If the char$I = 0$, then the minimal ideal (generalized ideal) regarded as an additive group $(I, +)$ is torsion-free. If the char$I = p$, when p is prime number, then every element of additive group $(I, +)$ is of order p. Furthermore, by this theorem, we provide some results about minimal ideals and minimal generalized ideals. Thus, among others, we get that a zero ring (a ring with zero multiplication) whose characteristic is not a prime number cannot be a minimal ideal (minimal generalized ideal) of any ring. By this corollary, we obtain infinitely many counterexamples of an open problem raised in [2] and solved for the first time with only a counterexample in [5].

For the minimal left ideal, the minimal right ideal and the minimal quasi-ideal have their characteristic zero, and it is proved that for every prime number p, their additive groups are p-divisible. Moreover, for the minimal quasi-ideal $(a)_q$ of a ring A which has the intersection property (this means that it is the intersection of a left ideal and a right ideal) and has the characteristic zero (characteristic p), we prove that if $b \in A$ is $\mathscr{L}$-equivalent ($\mathscr{R}$-equivalent, $\mathscr{D}$-equivalent) with the element a, then the minimal quasi-ideal $(b)_q$ has the characteristic zero (characteristic p).

Finally, at the end of this paper, a simplification of the Proposition 5 [6] formulation is given and consequently a shorter proof for it. The original proof in [6] was a bit more complicated, and giving an easier proof of it was another motivation for us.

In this paper, two open problems arise in the last section.

This work is the extension of effort to give some properties for minimal ideals and for generalized ideals in rings which depend on their characteristics.

2. Preliminaries

We present some notions and some auxiliary results that will be used throughout the paper.

Definition 1 (see [7]). An abelian group $(G, +)$ is called p-divisible for a prime p, and if for every $a \in G$, there exists $b \in G$ such that $a = pb$.

Definition 2 (see [8]). An abelian group G is said to be torsion-free group if no element other than the identity e is of finite order.

As generalized ideals in a ring, we shall consider the left ideals, the right ideals, the quasi-ideals, the bi-ideals, and the (m, n)-ideals where m, n are every non-negative integers, whose definitions are as follows (A^m is suppressed if $m = 0$):

Definition 3 (see [2]). A quasi-ideals of a ring $(A, +, \cdot)$ is called an additive subgroups Q of $(A, +)$ if $QA \cap AQ \subseteq Q$.

Definition 4 (see [3]). A subring B of a ring $(A, +, \cdot)$ is called a bi-ideal if $BAB \subseteq B$.

Definition 5 (see [4]). For every non-negative integers m, n, a subring T of the ring $(A, +, \cdot)$ is called a (m, n)-ideal if $T^m A T^n \subseteq T$.

Let a be an element of a ring $(A, +, \cdot)$. The intersection of all ideals (generalized ideals) of A which contains the element a is called the principal ideal (the principal generalized ideal) of A generated by a. The principal ideal (left ideal, right ideal, quasi-ideal) is denoted by (a) ($(a)_l$, $(a)_r$, $(a)_q$, respectively). It is easy to see that for every element a of a ring A, we have

$$\begin{aligned} (a) &= \mathbb{Z}a + aA + Aa + AaA, \\ (a)_l &= \mathbb{Z}a + Aa, \\ (a)_r &= \mathbb{Z}a + aA, \\ (a)_q &= \mathbb{Z}a + aA \cap Aa. \end{aligned} \tag{1}$$

Proposition 1 (see [1]). *Let s be a nonzero element of a prime ring A and suppose that there is a least positive integer m such that ms = 0. Then, A has characteristic m, and m is a prime number.*

In [6], Green's relations $\mathscr{R}, \mathscr{L}, \mathscr{H}$ were introduced and studied, which are the analogues of Green's relations $\mathscr{R}_{\text{plain}}$, $\mathscr{L}_{plain}$, $\mathscr{H}_{plain}$ in plain semigroups [9]. So Green's relations in a ring A are defined by

$$\begin{aligned} &\forall (a, b) \in A^2, a\mathscr{R}b \Leftrightarrow (a)_r = (b)_r \\ &\forall (a, b) \in A^2, a\mathscr{L}b \Leftrightarrow (a)_l = (b)_l \\ &\forall (a, b) \in A^2, a\mathscr{H}b \Leftrightarrow (a)_r = (b)_r \text{ and } (a)_l = (b)_l. \end{aligned} \tag{2}$$

It turns out that $\mathscr{R}, \mathscr{L}, \mathscr{H}$ are equivalence relations. The respective equivalence classes of $a \in A$ are denoted by R_a, L_a, and H_a, respectively.

In [6], it is proved that $\mathscr{R}, \mathscr{L}$ commute, that is, $\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$, and also that $\mathscr{R} \circ \mathscr{L}$ is an equivalence relation. So, we have the forth Green's relation $\mathscr{D}$ on A, which is $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$. The equivalence class mod $\mathscr{D}$ containing $a \in A$ is denoted by D_a.

A quasi-ideal in a ring is said to have the intersection property if it is the intersection of a left ideal and a right ideal.

Theorem 1 (see [6]). *Let a, b be two elements of the ring A such that $a\mathscr{D}b$. The principal quasi-ideal $(a)_q$ is minimal and has the intersection property if and only if the same hold for $(b)_q$.*

In [6], it is also introduced and studied the relation "to generate the same principal quasi-ideal," that is,

$$\forall (a, b) \in A^2, a\mathscr{Q}b \Leftrightarrow (a)_q = (b)_q. \tag{3}$$

It is easy to see that the relation $\mathscr{Q}$ is an equivalence relation and $\mathscr{Q} \subseteq \mathscr{H}$. In [6], differently from semigroups, it is shown that the inclusion $\mathscr{Q} \subseteq \mathscr{H}$ is strict, and the following proposition is proved.

Proposition 2 (Proposition 5 [6]). *If a minimal quasi-idealQ in a ring A is a subring, whose characteristic is not a prime number, then Q/0 is an $\mathscr{H}$-class.*

In [10], the authors have proved that which of Green's relations $\mathscr{L}, \mathscr{R}, \mathscr{H}$ and $\mathscr{D}$ in rings preserve the minimality of quasi-ideal. By this, it was shown the structure of the classes generated by the above relations which have a minimal quasi-ideal.

3. Main Results

Since every simple ring is prime, then by the Proposition 1, it follows that the characteristic of a simple ring is either zero or a prime number. The same is true for every minimal ideal (generalized ideal) as the following theorem shows, which gives also a further property regarding its characteristic.

Theorem 2. *The characteristic of a minimal ideal (generalized ideal) I in a ring $(A, +, \cdot)$ is either zero or a prime number p. If the* char$I = 0$*, then the minimal ideal (generalized ideal), regarded as an additive group, $(I, +)$ is torsion-free. If the* charI $= p$*, then every element of the additive group $(I, +)$ is of order p.*

Proof. We will give the proof only for the minimal ideal, and the proof for others is similarly.

We will consider two possible cases:

First case: char$I = 0$. In this case, for every positive integer n, there exists an element $b \in I$ such that $nb \neq 0$. For every element $0 \neq a \in I$, we have

$$\begin{aligned} I = (a) &= \mathbb{Z}a + aA + Aa + AaA \\ &= (b) = \mathbb{Z}b + bA + Ab + AbA. \end{aligned} \tag{4}$$

Then, there exists an integer k and the elements c_1, c_2, c_3, c_4 of A such that

$$b = ka + ac_1 + c_2 a + c_3 a c_4. \tag{5}$$

If there is a positive integer n, such that $na = 0$, then we shall get

$$nb = k(na) + (na)c_1 + c_2(na) + c_3(na)c_4 = 0, \tag{6}$$

which is a contradiction. Therefore, the order of any nonzero element of additive group $(I, +)$ is infinite, and consequently, the group $(I, +)$ is torsion-free.

Second case: charI $= n$, where n is a positive integer not smaller than two. So, for every element $a \neq 0$ of I, we have $na = 0$, and for every positive integer m such that $2 \leq m < n$, there exists an element $b \neq 0$ of I such that $mb \neq 0$. Since $(a) = (b) = I$, then again the equality (5) holds true. By this equality, it follows that for every element $a \neq 0$ of I, we have $ma \neq 0$. Thus, the order of every element $a \neq 0$ of the additive group $(I, +)$ is $n > 1$. By the Cauchy Theorem, for every prime divisor p of n, there exists an element $c \neq 0$ of additive group $(I, +)$ which has its order p. Since $(a) = I = (c)$, then we find the integers k' and the elements b_1', b_2', b_3', b_4' of the ring A such that

$$a = k'c + b_1'c + cb_2' + b_3'cb_4'. \tag{7}$$

By this equality, we have

$$pa = k'(pc) + b_1'(pc) + (pc)b_2' + b_3'(pc)b_4' = 0. \tag{8}$$

If there is a positive integer p_1 such that $1 < p_1 < p$, then we would have $p_1 a = 0$. Since $(a) = I = (c)$, then there exist an integer k_1 and the elements d_1, d_2, d_3, d_4 of the ring A such that

$$c = k_1 a + ad_1 + d_2 a + d_3 a d_4. \tag{9}$$

By this equality, we get

$$p_1 c = k_1(p_1 a) + (p_1 a)d_1 + d_2(p_1 a) + d_3(p_1 a)d_4 = 0. \tag{10}$$

Thus, we have the equality $p_1 c = 0$ which is a contradiction

Therefore, $n = p$, that is charI $= p$, where p is a prime number and the order of any element of I is p

Since every simple ring A is a minimal ideal of A, then bythe Theorem 2, we have the following two corollaries (the first part of the first corollary derives from Proposition 2): □

Corollary 1

(1) If $(A, +, \cdot)$ is a simple ring, then its characteristic is either zero or a prime number

(2) If char$A = 0$, *then the additive group $(A, +)$ is a torsion-free group*

(3) If char$A = p$, *p prime, then every element of additive group $(A, +)$ has order p*

Corollary 2. *If $(A, +, \cdot)$ is a simple ring, then we have*

(1) char$A = 0$ *if and only if* char$I = 0$ *for every minimal generalized ideal I in A*

(2) char$A = p$, *p prime number, if and only if* char$I = p$ *for every minimal generalized ideal I in A*

By Theorem 2, we get immediately the following corollary:

Corollary 3. *If the characteristic of a ring A is not a prime number, then there is no ring B such that A is a minimal ideal or minimal generalized ideal of B.*

By the above corollary, it follows that the ring of integers modules n, where n is not a prime number, $\mathbb{Z}_n$, the ring of $m \times m$ matrices with elements in $\mathbb{Z}_n$, and also the ring of polynomials with coefficients in $\mathbb{Z}_n$ can not be minimal ideal or minimal generalized ideal of some ring.

As a particular case of Corollary 3, we have that any ring with zero multiplication (zero ring) which have its characteristic not a prime number cannot be a minimal ideal or a minimal generalized ideal of some ring.

For the quasi-ideal, regarded as a generalized ideal, with zero multiplication [2], L. Márki has raised this problem: given a zero ring A does, there exist a ring B such that A is a minimal quasi-ideal of B.

This open problem is resolved negatively with one counterexample in [5]. Here, we find infinity many zero rings which cannot be quasi-ideals of some ring. For example, let we consider the additive groups $\mathbb{Z}_n$, for every not-prime number n, the additive groups of rings of $n \times n$ matrices with elements in $\mathbb{Z}_n$, and the additive groups of polynomials with coefficients in $\mathbb{Z}_n$. They can be made into requested zero rings, by defining their multiplication by $a \cdot b = 0$, where 0 is the identity of their addition.

For the minimal left ideal, the minimal right ideal, and the quasi-ideal, the $(m, 0)$-ideal, $m \geq 2$ and $(0, n)$-ideals, $n \geq 2$, a part of Theorem 2 is sharpen by this.

Theorem 3. *If the characteristic of a minimal left ideal (right ideal, quasi-ideal, $(m, 0)$-ideal, $m \geq 2$, $(0, n)$-ideals, $n \geq 2$) of a ring A is zero, then this minimal left ideal (right ideal, quasi-ideal, $(m, 0)$-ideal, $m \geq 2$, $(0, n)$-ideals, $n \geq 2$) considered as an additive group is p divisible for every prime p.*

Proof. We will give the proof only for minimal left ideal and minimal quasi-ideal: the rest of proof is similarly.

Let L be a minimal left ideal in a ring A such that the $charL = 0$. By Theorem 2, we have that $(L, +)$ is a torsion-free group. Hence, for every prime p and for every element $a \neq 0$, we have $pa \neq 0$. Since L is minimal and

$$0 \neq pa \in L \cap (pa)_l, \tag{11}$$

then we have that $L \subseteq (pa)_l$. Then, there exist an integer k and an element $b \in A$ such that $a = k(pa) + b(pa)$. By this equality, we have $(1 - kp)a = b(pa)$. The integer $1 - kp$ is not zero; therefore, by Theorem 2, we have $(1 - kp)a \neq 0$. Since L is minimal and

$$0 \neq (1 - kp)a \in L \cap (pba)_l, \tag{12}$$

then we get $L \subseteq (pba)_l$. So, there exist an integer k_1 and the element $c \in A$ such that

$$a = k_1 (pba) + c (pba) = (k_1 b + cb)(pa). \tag{13}$$

By this equality, if we denote $d = k_1 b + cb$, then we have $a = p(da)$, where $da \in L$, and consequently, the minimal left ideal L as an additive group is p-divisible for every prime p.

Now, we give the proof for a minimal quasi-idealQ in the ring A. By Theorem 2, for every prime p and every element $a \neq 0$ of Q, we have $pa \neq 0$. Since Q is minimal quasi-ideal and

$$0 \neq pa \in Q \cap (pa)_q, \tag{14}$$

then we have $Q \subseteq (pa)_q$. Then, there exist an integer k_2 and two elements $b_2.b_3$ of A such that

$$a = k_2 (pa) + b_2 (pa) = k_2 (pa) + (pa)b_3. \tag{15}$$

By these equalities, we find

$$(1 - k_2 p)a = b_2 (pa) = (pa)b_3. \tag{16}$$

The integer $1 - k_2 p$ is not zero, and by Theorem 2, we have $(1 - k_2 p)a \neq 0$. Since Q is minimal quasi-ideal and

$$0 \neq (1 - k_2 p)a \in Q \cap \left((pb_2 a)_l \cap (pab_{3r}\right), \tag{17}$$

then we have $Q \subseteq (pb_2 a)_l \cap (pab_3)_r$. Then, there exist the integers k_3, k_4 and the elements c_3, c_4 of A such that

$$a = k_3 pb_2 a + c_3 pb_2 a = k_4 pab_3 + pab_3 c_4. \tag{18}$$

By these equalities, we find $a = p(d_1 a) = p(ad_2)$, where $d_1 = k_3 b_2 + c_3 b_2$ and $d_2 = k_4 b_3 + b_3 c_4$. But

$$d_1 a = d_1 (pad_2) = (pd_1 a)d_2 = ad_2 \in AQ \cap QA \subseteq Q. \tag{19}$$

Thus, $a = p(d_1 a)$, where $d_1 a = ad_2 \in Q$, and consequently, the minimal quasi-idealQ as an additive group is p-divisible for every prime p.

For the minimal generalized ideals in which there are not considered in Theorem 3, the following open problem arise: □

Problem 1. Does the assertion of Theorem 3 hold true for minimal ideal, in particular for simple ring, for minimal bi-ideal, and for minimal (m, n)-ideal where $m \geq 2$ or $n \geq 2$?

It is shown in [1] that there exist simple rings which do not have an identity element. It follows that there are minimal ideals in rings which do not have an identity element. On the other hand, also in [1], it is proved that every simple right Noetherian ring of characteristic zero has an identity element. Indicated by this proposition, the following open problem arises:

Problem 2. Do minimal ideals (minimal left ideals, minimal right ideals) of a right Noetherian ring of characteristic zero have identity elements?

Now, we will show that the characteristic of a minimal quasi-ideal which has the intersection property will be "transferred" from each of Green's relations $\mathscr{L}, \mathscr{R}$ and $\mathscr{D}$ in rings. Precisely, we have the following theorem:

Theorem 4. *Let a, b be two elements of a ring A such that $\mathscr{L}b[a\mathscr{R}b, a\mathscr{D}b]$. Then, the principal quasi-ideal$(a)_q$ is minimal of characteristic zero (characteristic p, p-prime) and has the intersection property if and only if the same holds for $(b)_q$.*

Proof. In view of the duality between $\mathscr{L}$ and $\mathscr{R}$ and the definition of $\mathscr{D}$, it suffices to prove the theorem only for the relation $\mathscr{L}$.

Since $\mathscr{L} \subseteq \mathscr{D}$, then by Theorem 1, the principal quasi-ideal$(a)_q$ is minimal, and it has the intersection property if and only if the same holds for $(b)_q$. So, it remains to show that if $(a)_q$ has the characteristic zero (characteristic p, p-prime), then $(b)_q$ has the characteristic zero (characteristic p, p-prime).

Since $a\mathscr{L}b$, then there exist an integer k and an element $a \in A$ such that $b = ka + ua$. Therefore, by the Proposition 1 [6], the mapping

$$\begin{aligned} f\colon (a)_q &\longrightarrow (b)_q \\ x &\longrightarrow kx + ux, \end{aligned} \tag{20}$$

is a bijection. Furthermore, for every two elements x, y of $(a)_q$, we have

$$f(x + y) = k(x + y) + u(x + y) = (kx + ux) + (ky + uy) = f(x) + f(y). \tag{21}$$

This equalities show that f is an additive groups isomorphism. So, the quasi-ideals$(a)_q$, $(b)_q$ as additive groups of a ring A are isomorphic groups. It now follows that the minimal quasi-ideals$(a)_q$, $(b)_q$ have the same characteristic.

By Theorem 2, the characteristic of a quasi-ideal in a ring is either zero or a prime number, and therefore, the proposition which we will prove below is equivalent with Proposition 2, but the proof that we give here is a little bit simpler and shorter than the proof of Proposition 2 (Proposition 5 [6]). □

Proposition 3. *If a minimal quasi-idealQ of a ring A has its characteristic zero, then Q/0 is a $\mathscr{H}$-class.*

Proof. Since Q is a minimal quasi-ideal with $charQ = 0$, then for every prime p, there exists an element $a \in Q$ such that $pa \neq 0$. As Q is a minimal quasi-ideal, we have

$$0 \neq pa \in (a)_q \cap [(pa)_l \cap (pa)_r] \Rightarrow (a)_q \subseteq (pa)_l \cap (pa)_r. \tag{22}$$

Then, there exist integers k_1, k_2 and elements $a_1, a_2 \in A$ such that

$$a = k_1(pa) + a_1(pa) = k_2(pa) + (pa)a_2. \tag{23}$$

Hence,

$$(1 - k_1 p)a = pa_1 a \text{ and } (1 - k_2 p)a = paa_2. \tag{24}$$

By these equalities, we get

$$(1 - k_1 p)(1 - k_2 p)a = (1 - k_2 p)pa_1 a = (1 - k_1 p)paa_2. \tag{25}$$

The integer $k = (1 - k_1 p)(1 - k_2 p)$ is not zero, and so by Theorem 3, we have $ka \neq 0$. Since

$$0 \neq ka \in Q \cap [(pa_1 a)_l \cap (pa_2 a)_r], \tag{26}$$

and Q is a minimal quasi-ideal, then we get $Q \subseteq (pa_1 a)_l \cap (pa_2 a)_r$. Hence, there exist elements b_1, b_2 of A such that $a = b_1 a = ab_2$. Now, let x be an arbitrary element of H_a. Then, there exist elements $c_1, c_2 \in A$ such that $x = c_1 a = ac_2$. Since $(a)_q = Q$ is a minimal quasi-ideal, then we have $(x)_q = (a)_q$, and consequently, $x \in Q_a$. Thus, $Q_a = H_a$, and finally, we have $Q/0 = H_a$. □

4. Conclusion

In this paper, we have shown that the characteristic of minimal ideals and minimal generalized ideals in rings, which do not necessarily have an identity element, is zero or a prime number. Furthermore, we gave some properties for minimal ideals and for generalized ideals that depend on their characteristics. By using these results and Green's relations, we have shown an additional result, that is, the characteristic of a minimal quasi-ideal that has the intersection property will be transferred from each of Green's relations in rings. More specifically, if a and b are two elements of a ring A such that $\mathscr{L}b[a\mathscr{R}b, a\mathscr{D}b]$, then the principal quasi-ideal $(a)_q$ is minimal of characteristic zero (characteristic p, p-prime) and has the intersection property if and only if the same holds for $(b)_q$. The obtained results led raise the following issues:

(1) What happens between the principal bi–ideal $(a)_b$ and the principal bi–ideal $(c)_b$ if $a\mathscr{D}c$ in a ring A?

(2) Does the assertion of Theorem 3 hold true for minimal ideal, in particular for simple ring, for minimal bi-ideal and for minimal (m, n)-ideal where $m \geq 2$ or $n \geq 2$?

(3) Do minimal ideals (minimal left ideals, minimal right ideals) of a right Noetherian ring of characteristic zero have identity elements?

(4) Can we extend these results to other structures such as hyper-rings, ternary rings, Γ-rings etc.?

These issues will be some the potential aims of our future work.

Data Availability

No data were used to support the findings of the study because this paper is in pure mathematics (algebra) and all results are in these fields.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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