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Research Article

M_{ve}—Polynomial of Cog-Special Graphs and Types of Fan Graphs

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The study of topological indices in graph theory is one of the more important topics, as the scientific development that occurred in the previous century had an important impact by linking it to many chemical and physical properties such as boiling point and melting point. So, our interest in this paper is to study many of the topological indices “generalized indices’ network” for some graphs that have somewhat strange structure, so it is called the cog-graphs of special graphs “molecular network”, by finding their polynomials based on vertex − edge degree then deriving them with respect to x, y, and $x\,y$, respectively, after substitution $x = y = 1$ of these special graphs are cog-path, cog-cycle, cog-star, cog-wheel, cog-fan, and cog-hand fan graphs; the importance of some types of these graphs is the fact that some vertices have degree four, which corresponds to the stability of some chemical compounds. These topological indices are first and second Zagreb, reduced first and second Zagreb, hyper Zagreb, forgotten, Albertson, and sigma indices.

1. Introduction

A graph $G = (V, E)$ is a pair order of vertex set $V = V(G)$ and edge set $E = E(G)$ where the cardinality of V and E are p and q, respectively. The degree of a vertex v represents the number of edges incident to that vertex and is denoted by dv. The maximum and minimum degrees of graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The neighborhood of vertex v, which is a set of all neighbors of v and denoted by $N_G(v)$, is called open neighborhood, while the closed neighborhood of vertex v, denoted by $N_G[v]$, is the set $N_G(v)$ union the set $\{v\}$. The degrees sum of neighbors of v in G is denoted by δ_v. For more information on many concepts in graph theory, see [1–3]. Chellali et al. were the first to introduce the vertex − edge degree of graph G [4]. The vertex − edge degree (or *ve*− degree) of vertex v is equal to the number of elements in a set $N_G[v]$, and τ_v is denoted by the *ve*− degree of vertex v in G. The maximum and minimum *ve*− degrees of graph G are denoted by $\Delta_{ve}(G)$ and $\delta_{ve}(G)$, respectively. Deutsch and Klavžar [5] first introduced the M− polynomial as follows:

$$M(G) = \sum_{i\leq j} m_{ij} x^i y^j, \tag{1}$$

where m_{ij} is the number of edges $uv \in E(G)$ such that $\{d_u, d_v\} = \{i, j\}$. The M− polynomial is generally polynomial and may generate degree-based topological indices [6–10]. For applications in chemistry and networks, see [11–14]. Developed by Mondal et al., the M− polynomial is called the neighborhood M− polynomial and is defined as follows:

$$NM(G) = \sum_{i\leq j} nm_{ij} x^i y^j, \tag{2}$$

where nm_{ij} is the total number of edges $uv \in E(G)$ such that $\{\delta_u, \delta_v\} = \{i, j\}$. There are many recent works about neighborhood M− polynomials [15, 16]. Because of the importance of these topics, in 2023, Kavi et al. [17] proposed a new polynomial based on *ve*− degree which is called M_{ve}− polynomial, and it is defined as follows:

$$M_{ve}(G) = \sum_{i\leq j} c_{ij} x^i y^j, \tag{3}$$

Table 1: Some vertex-edge-degree-based topological indices for $M_{ve}-$ polynomial.

Topological index	Symbol index	Formula $f(\tau_u, \tau_v)$	Derivation from M_{ve} (G)
First Zagreb	M_{ve}^1	$\sum_{uv\in E(G)}(\tau_u+\tau_v)$	$(D_x+D_y)M_{ve}(G)\|_{x=y=1}$
Second Zagreb	M_{ve}^2	$\sum_{uv\in E(G)}(\tau_u\tau_v)$	$(D_xD_y)M_{ve}(G)\|_{x=y=1}$
Reduced first Zagreb	RM_{ve}^1	$\sum_{uv\in E(G)}(\tau_u+\tau_v)-2$	$(D_x+D_y-2)M_{ve}(G)\|_{x=y=1}$
Reduced second Zagreb	RM_{ve}^2	$\sum_{uv\in E(G)}(\tau_u-1)(\tau_v-1)$	$(D_x-1)(D_y-1)M_{ve}(G)\|_{x=y=1}$
Hyper Zagreb index	Hyp_{ve}	$\sum_{uv\in E(G)}(\tau_u+\tau_v)^2$	$D_x^2JM_{ve}(G)\|_{x=y=1}$
Forgotten index	F_{ve}	$\sum_{uv\in E(G)}((\tau_u)^2+(\tau_v)^2)$	$(D_x^2+D_y^2)M_{ve}(G)\|_{x=y=1}$
Albertson index	Alb_{ve}	$\sum_{uv\in E(G)}\|\tau_u-\tau_v\|$	$D_xIM_{ve}(G)\|_{x=y=1}$
Sigma index	σ_{ve}	$\sum_{uv\in E(G)}(\tau_u-\tau_v)^2$	$D_x^2IM_{ve}(G)\|_{x=y=1}$

where c_{ij} is the total number of edges $uv \in E(G)$ such that $\{\tau_u, \tau_v\} = \{i, j\}$. There are many recent works on $ve-$ degree and $ev-$ degree indices (see [18–20]).

In Table 1, we list many topological indices with respect to the vertex – edge-degree of graph G.

In this article, we calculate a closed form of some ve-degree dependent topological indices mentioned in Table 1 by applying some mathematical operation on Mve-polynomial of the graph, where the operators used are defined as follows:

$D_xM_{ve}(G; x, y) = x\partial M_{ve}(x, y)/\partial x$, $D_yM_{ve}(G; x, y) = y\partial M_{ve}(x, y)/\partial y$, $JM_{ve}(x, y) = M_{ve}(x, x)$, and $IM_{ve}(x, y) = M_{ve}(x, x^{-1})$. To illustrate the abovementioned concepts, we will take the following example, the $M-$ polynomial, $NM-$ polynomial, and $M_{ve}-$ polynomial for a graph are shown in Figure 1, respectively:

(i) $M(G) = 2x^2y^2 + 4x^2y^3 + 2x^2y^4 + 2x^3y^3 + 4x^3y^4 + x^4y^4$

(ii) $NM(G) = x^5y^5 + x^5y^6 + x^5y^7 + 2x^5y^9 + x^6y^{12} + x^7y^7 + x^7y^9 + x^7y^{12} + x^9y^{11} + 2x^9y^{12} + 2x^{11}y^{12} + x^{12}y^{12}$

(iii) $M_{ve}(G) = x^5y^5 + x^5y^6 + x^5y^7 + x^5y^8 + x^5y^9 + x^6y^{10} + x^7y^7 + x^7y^9 + x^7y^{11} + x^8y^9 + x^8y^{10} + x^9y^{10} + 2x^9y^{11} + x^{10}y^{11}$

In Table 2, we give some vertex-edge-degree-based topological indices for the graph in Figure 1.

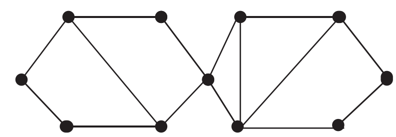

Figure 1: Graph G of order 11 and size 9.

Table 2: Some vertex-edge-degree-based topological indices.

	Symbol index	Index value
1	$M_{ve}^1(G)$	239
2	$M_{ve}^2(G)$	974
3	$RM_{ve}^1(G)$	209
4	$RM_{ve}^2(G)$	750
5	$\text{Hyp}_{ve}(G)$	3977
6	$F_{ve}(G)$	2029
7	$\text{Alb}_{ve}(G)$	29
8	$\sigma_{ve}(G)$	81

2. $M_{ve}-$ Polynomial of Some Cog-Special Graphs

In this section, we find the $M_{ve}-$ polynomials and $M_{ve}-$ indices for some cog-special graphs, such as path, cycle, star, complete, and wheel graphs [21].

2.1. Cog-Path Graph

Definition 1. A cog-path graph P_p^c is the graph constructed from a path P_p, $p \geq 3$, of a set vertex $V(P_p) = \{v_1, v_2, \ldots, v_p\}$, and with $(p-1)$ additional vertices $U = \{u_1, u_2, \ldots, u_{p-1}\}$, and $2(p-1)$ edges $\{u_iv_i, u_iv_{i+1}: i = 1, 2, \ldots, p-1\}$, as shown in Figure 2.

2.1.1. Some Properties of a Cog-Path Graph P_p^c

(i) The order and the size are $p(P_p^c) = 2p-1$ and $q(P_p^c) = 3(p-1)$, respectively

(ii) The degrees of vertices v_i, $2 \leq i \leq p-1$ are 4 which represent the maximum degree "$\Delta(P_p^c) = 4$" and the degrees of vertices v_1, v_p, and u_i; $1 \leq i \leq p-1$ are 2 which represent the minimum degree "$\delta(P_p^c) = 2$"

(iii) The maximum and minimum $ve-$ degrees are $\Delta_{ve}(P_p^c) = 10$ and $\delta_{ve}(P_p^c) = 5$, respectively

In the following theorem, we will find the $M_{ve}-$ polynomial of the cog-path graph, P_p^c, $p \geq 5$.

Theorem 2. *Let P_p^c be the cog-path graph of order $2p-1$, $p \geq 5$. Then,*

$$M_{ve}(P_p^c; x, y) = 2x^5y^5 + 4x^5y^8 + 2x^7y^8 + 2x^8y^{10} + 2(p-4)x^7y^{10} + (p-5)x^{10}y^{10}. \quad (4)$$

Proof. From Definition 1, the vertex set is $V(P_p^c) = V \cup U$, where $V = \{v_1, v_2, \ldots, v_p\}$ and $U = \{v_1, v_2, \ldots, v_{p-1}\}$. Since the vertices v_{i-1}, u_{i-1}, u_i, and v_{i+1} are adjacent to v_i, for $i = 2, 3, \ldots, p-1$, the two vertices v_1 and u_1 are adjacent and the two vertices v_p and u_{p-1} are adjacent, then

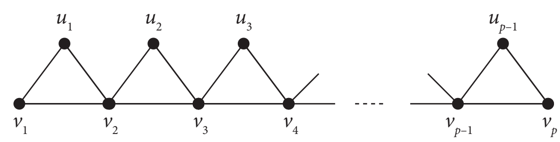

FIGURE 2: A cog-path graph P_p^c.

$$
\begin{aligned}
&\tau_{(v_i)} = 5, \text{for all } i = 1, p,\\
&\tau_{(v_i)} = 8, \text{for all } i = 2, p-1,\\
&\tau_{(v_i)} = 10, \text{for all } 3 \le i \le p-2,\\
&\tau_{(u_i)} = 5, \text{for all } i = 1, p-1,\\
&\tau_{(u_i)} = 7, \text{for all } 2 \le i \le p-2.
\end{aligned} \tag{5}
$$

Hence, for all $z, w \in V(P_p^c)$ such that $\tau_{(z)} \le \tau_{(w)}$, we have

$$
\begin{aligned}
M_{ve}\left(P_p^c; x, y\right) &= \sum_{zw \in E\left(P_p^c\right)} x^{\tau(z)} y^{\tau(w)}\\
&= x^{\tau(u_1)} y^{\tau(v_1)} + x^{\tau(u_1)} y^{\tau(v_2)} + x^{\tau(u_2)} y^{\tau(v_2)} + x^{\tau(v_1)} y^{\tau(v_2)}\\
&\quad + x^{\tau(v_2)} y^{\tau(v_3)} + \sum_{i=3}^{p-2} x^{\tau(u_i)} y^{\tau(v_i)} + \sum_{i=2}^{p-3} x^{\tau(u_i)} y^{\tau(v_{i+1})}\\
&\quad + \sum_{i=3}^{p-3} x^{\tau(v_i)} y^{\tau(v_{i+1})} + x^{\tau(u_{p-1})} y^{\tau(v_p)} + x^{\tau(u_{p-1})} y^{\tau(v_{p-1})}\\
&\quad + x^{\tau(u_{p-2})} y^{\tau(v_{p-1})} + x^{\tau(v_p)} y^{\tau(v_{p-1})} + x^{\tau(v_{p-1})} y^{\tau(v_{p-2})}\\
&= x^5 y^5 + x^5 y^8 + x^7 y^8 + x^5 y^8 + x^8 y^{10} + (p-4) x^7 y^{10}\\
&\quad + (p-4) x^7 y^{10} + (p-5) x^{10} y^{10} + x^5 y^5 + x^5 y^8 + x^7 y^8\\
&\quad + x^5 y^8 + x^8 y^{10}\\
&= 2x^5 y^5 + 4x^5 y^8 + 2x^7 y^8 + 2x^8 y^{10} + 2(p-4) x^7 y^{10}\\
&\quad + (p-5) x^{10} y^{10}.
\end{aligned} \tag{6}
$$

□

Remark 3

(i) $M_{ve}(P_3^c; x, y) = 2x^5 y^5 + 4x^5 y^6$

(ii) $M_{ve}(P_4^c; x, y) = 2x^5 y^5 + 4x^5 y^8 + 2x^7 y^8 + x^8 y^8$.

In the next corollary, we can easily get the topological indices of $M_{ve}^x(P_p^c)$, $M_{ve}^y(P_p^c)$, and $M_{ve}^{xy}(P_p^c)$ from the derivatives with respect to x, y, and xy, respectively, after substitution $x = y = 1$.

Corollary 4. *Let P_p^c be the cog-path graph of order $2p-1$, $p \ge 5$, then we have*

(1) $M_{ve}^1(P_p^c) = 54p - 98$

(2) $M_{ve}^2(P_p^c) = 240p - 578$

(3) $RM_{ve}^1(P_p^c) = 48p - 92$

(4) $RM_{ve}^2(P_p^c) = 189p - 483$

(5) $\mathrm{Hyp}_{ve}(P_p^c) = 978p - 2338$

(6) $F_{ve}(P_p^c) = 498p - 1182$

(7) $\mathrm{Alb}_{ve}(P_p^c) = 6(p-1)$

(8) $\sigma_{ve}(P_p^c) = 18p - 26$.

2.2. Cog-Cycle Graph

Definition 5. Let C_p: $v_1, v_2, \ldots, v_p, v_1$, $p \ge 3$, be a cycle of order p, $p \ge 3$. The cog-cycle C_p^c is obtained from C_p by adding p new vertices $U = \{u_1, u_2, \ldots, u_p\}$ and $2p$ edges $\{u_i v_i, u_i v_{i+1}: i = 1, 2, \ldots, p\}$, $v_{p+1} = v_1$, as shown in Figure 3.

2.2.1. Some Properties of a Cog-Cycle Graph C_p^c

(i) The order and the size are $p(C_p^c) = 2p$ and $q(C_p^c) = 3p$, respectively

(ii) The degrees of vertices v_i, $1 \le i \le p$ are 4 which represent the maximum degree "$\Delta(C_p^c) = 4$" and the

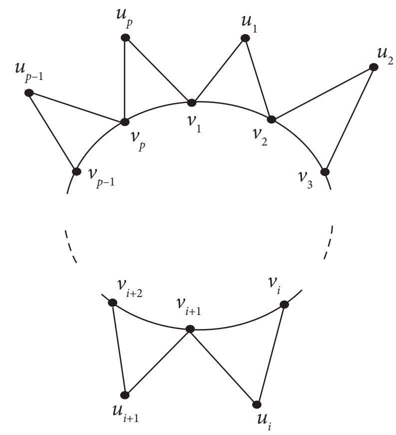

Figure 3: A cog-cycle graph C_p^c.

degrees of vertices u_i, $1 \leq i \leq p$ are 2 which represent the minimum degree "$\delta(C_p^c) = 2$"

(iii) The maximum and minimum *ve*− degrees are $\Delta_{ve}(C_p^c) = 10$ and $\delta_{ve}(C_p^c) = 7$, respectively

Theorem 6. *Let C_p^c be the cog-cycle graph of order $2p$, $p \geq 4$. Then,*

$$M_{ve}\left(C_p^c; x, y\right) = 2px^7y^{10} + px^{10}y^{10}. \tag{7}$$

Proof. From Definition 5, the vertex set is $V(C_p^c) = V \cup U$, where $V = \{v_1, v_2, \ldots, v_p\}$ and $U = \{v_1, v_2, \ldots, v_p\}$. Since the vertices v_{i-1}, u_{i-1}, u_i, and v_{i+1} are adjacent to v_i, for $i = 1,2,3,\ldots,p$, where $u_0 = u_p$, $v_0 = v_p$, and $v_{p+1} = v_1$, then

$$\begin{aligned} &\tau_{(u_i)} = 7, \text{for all } 1 \leq i \leq p, \\ &\tau_{(v_i)} = 10, \text{for all } 1 \leq i \leq p. \end{aligned} \tag{8}$$

Hence, for all $z, w \in V(C_p^c)$ such that $\tau_{(z)} \leq \tau_{(w)}$, we have

$$\begin{aligned} M_{ve}\left(C_p^c; x, y\right) &= \sum_{zw \in E\left(C_p^c\right)} x^{\tau(z)} y^{\tau(w)} \\ &= \sum_{i=1}^{p} x^{\tau(u_i)} y^{\tau(v_i)} + \sum_{i=1}^{p} x^{\tau(u_i)} y^{\tau(v_{i+1})} + \sum_{i=1}^{p} x^{\tau(v_i)} y^{\tau(v_{i+1})} \\ &= px^7y^{10} + px^7y^{10} + px^{10}y^{10} \\ &= 2px^7y^{10} + px^{10}y^{10}. \end{aligned} \tag{9}$$

□

Remark 7

$$M_{ve}\left(C_3^c; x, y\right) = 2px^7y^9 + px^9y^9. \tag{10}$$

From easy to obtain many indices topologically in the following corollary by the derivatives with respect to x, y, and xy, respectively, after substitution $x = y = 1$ for $M_{ve}(C_p^c; x, y)$.

Corollary 8. *Let C_p^c be the cog-cycle graph of order $2p$, $p \geq 4$, then we have*

(1) $M_{ve}^1(C_p^c) = 54p$

(2) $M_{ve}^2(C_p^c) = 240p$

(3) $RM_{ve}^1(C_p^c) = 48p$

(4) $RM_{ve}^2(C_p^c) = 189p$

(5) $\mathrm{Hyp}_{ve}(C_p^c) = 978p$

(6) $F_{ve}(C_p^c) = 498p$

(7) $\mathrm{Alb}_{ve}(C_p^c) = 6p$

(8) $\sigma_{ve}(C_p^c) = 18p$

2.3. Cog-Star Graph

Definition 9. A cog-star graph S_p^c is the graph constructed from a star graph S_p, $p \geq 4$, of a vertex set $V(S_p) = \{v_1, v_2, \ldots, v_p\}$ with $(p-1)$ additional vertices $U = \{u_1, u_2, \ldots, u_{p-1}\}$, and $2(p-1)$ edges $\{u_iv_{i+1}, u_iv_{i+2}: i = 1, 2, \ldots, p-1\}$, $v_{p+1} = v_2$, as shown in Figure 4.

2.3.1. Some Properties of a Cog-Star Graph $\mathbf{S_p^c}$

(i) The order and the size: $p(S_p^c) = 2p-1$ and $q(S_p^c) = 3(p-1)$, respectively

(ii) The degree of vertex v_1 is $(p-1)$ which represent the maximum degree "$\Delta(S_p^c) = p-1$," the degrees of vertices v_i, $2 \le i \le p$ are 3 and the degrees of vertices u_i, $1 \le i \le p-1$ are 2 which represent the minimum degree "$\delta(S_p^c) = 2$"

(iii) The maximum and minimum *ve*− degrees are $\Delta_{ve}(S_p^c) = 3(p-1)$ and $\delta_{ve}(S_p^c) = 6$, respectively

Theorem 10. *Let S_p^c be the cog-star graph of order $2p-1$, $p \ge 4$. Then,*

$$M_{ve}\left(S_p^c; x, y\right) = 2(p-1)x^6 y^{p+3} + (p-1)x^{p+3} y^{3(p-1)}. \quad (11)$$

Proof. Since the vertex v_1 is adjacent to v_i, for $i = 2, \ldots, p$ and every vertices u_i are adjacent to v_{i+1} and v_{i+2}, for $i = 1, 2, \ldots, p-1$, where $(v_{p+1} = v_2)$, see Figure 4. Then,

$$\begin{aligned} \tau_{(v_1)} &= 3(p-1), \\ \tau_{(v_i)} &= p+3, \text{for all } 2 \le i \le p, \\ \tau_{(u_i)} &= 6, \text{for all } 1 \le i \le p-1. \end{aligned} \quad (12)$$

Hence,

$$\begin{aligned} M_{ve}\left(S_p^c; x, y\right) &= \sum_{i=1}^{p-1} x^{\tau(u_i)} y^{\tau(v_{i+1})} + \sum_{i=1}^{p-1} x^{\tau(u_i)} y^{\tau(v_{i+2})} + \sum_{i=2}^{p} x^{\tau(v_i)} y^{\tau(v_1)} \\ &= (p-1)x^6 y^{p+3} + (p-1)x^6 y^{p+3} + (p-1)x^{p+3} y^{3(p-1)} \\ &= 2(p-1)x^6 y^{p+3} + (p-1)x^{p+3} y^{3(p-1)}. \end{aligned} \quad (13)$$

□

Corollary 11. *Let S_p^c be a cog-star graph of order $2p-1$, $p \ge 4$, then we have*

(1) $M_{ve}^1(S_p^c) = 6(p-1)(p+3)$

(2) $M_{ve}^2(S_p^c) = 3(p-1)(p+3)^2$

(3) $RM_{ve}^1(S_p^c) = 6(p-1)(p+2)$

(4) $RM_{ve}^2(S_p^c) = 3(p-1)(p+2)^2$

(5) $\text{Hyp}_{ve}(S_p^c) = 18(p-1)(p^2+2p+9)$

(6) $F_{ve}(S_p^c) = 12(p-1)(p^2+9)$

(7) $\text{Alb}_{ve}(S_p^c) = 4(p-1)(p-3)$

(8) $\sigma_{ve}(S_p^c) = 6(p-1)(p-3)^2$

2.4. Cog-Complete Graph

Definition 12. A cog-complete graph K_p^c is the graph constructed from a complete graph K_p, $p \ge 3$, of a vertex set $V(K_p) = \{v_1, v_2, \ldots, v_p\}$ with p additional vertices $U = \{u_1, u_2, \ldots, u_p\}$, and $2p$ edges $\{u_i v_i, u_i v_{i+1}: i = 1, 2, \ldots, p\}$, $v_{p+1} = v_1$, as shown in Figure 5.

2.4.1. Some Properties of a Cog-Complete Graph K_p^c

(i) The order and the size are $p(K_p^c) = 2p$ and $q(K_p^c) = p(p+3)/2$, respectively

(ii) The degrees of vertices v_i, $1 \le i \le p$ are $p+1$ which represent the maximum degree "$\Delta(K_p^c) = p+1$" and the degrees of vertices u_i, $1 \le i \le p$ are 2 which represent the minimum degree "$\delta(K_p^c) = 2$"

(iii) The maximum and minimum *ve*− degrees are $\Delta_{ve}(K_p^c) = p(p+3)/2$ and $\delta_{ve}(K_p^c) = 2p+1$, respectively

Theorem 13. *Let K_p^c be the cog-complete graph of order $2p$, $p \ge 4$. Then,*

$$M_{ve}\left(K_p^c; x, y\right) = 2p x^{2p+1} y^{p(p+3)/2} + \binom{p}{2} x^{p(p+3)/2} y^{p(p+3)/2}. \quad (14)$$

Proof. For all $i = 2, 3, \ldots, p-1$, the vertex v_i is adjacent to $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p, u_{i-1}, u_i$, the vertex v_1 is adjacent to $v_2, v_3, \ldots, v_p, u_1, u_p$, and the vertex v_p is adjacent to $v_1, v_2, \ldots, v_{p-1}, u_{p-1}, u_p$, then

$$\begin{aligned} \tau_{(u_i)} &= 2p+1, \text{for all } 1 \le i \le p, \\ \tau_{(v_i)} &= \frac{p(p+3)}{2}, \text{for all } 1 \le i \le p. \end{aligned} \quad (15)$$

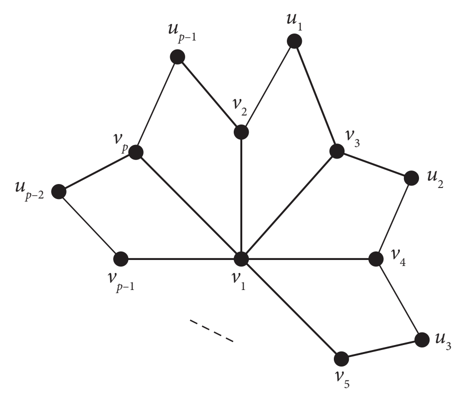

FIGURE 4: A cog-star graph S_p^c.

Hence, for all $z, w \in V(K_p^c)$ such that $\tau_{(z)} \leq \tau_{(w)}$, we have

$$
\begin{aligned}
M_{ve}\left(K_p^c; x, y\right) &= \sum_{zw \in E\left(K_p^c\right)} x^{\tau(z)} y^{\tau(w)} \\
&= \sum_{i=1}^{p} x^{\tau(u_i)} y^{\tau(v_i)} + \sum_{i=1}^{p-1} x^{\tau(u_i)} y^{\tau(v_{i+1})} + x^{\tau(u_p)} y^{\tau(v_1)} \\
&\quad + \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} x^{\tau(v_i)} y^{\tau(v_j)} \\
&= p x^{2p+1} y^{p(p+3)/2} + p x^{2p+1} y^{p(p+3)/2} + \sum_{i=1}^{p-1} (p-i) x^{p(p+3)/2} y^{p(p+3)/2} \\
&= 2p x^{2p+1} y^{p(p+3)/2} + \binom{p}{2} x^{p(p+3)/2} y^{p(p+3)/2}.
\end{aligned}
\tag{16}
$$

□

Remark 14

It is clear that $M_{ve}(K_3^c; x, y) = M_{ve}(C_3^c; x, y)$.

Corollary 15. *Let K_p^c be a cog-complete graph of order $2p$, $p \geq 4$, then we have*

(1) $M_{ve}^1(K_p^c) = 1/2p(p^3 + 4p^2 + 11p + 4)$
(2) $M_{ve}^2(K_p^c) = 1/8p^2(p+3)(p^3 + 2p^2 + 13p + 8)$
(3) $RM_{ve}^1(K_p^c) = 1/2p(p^3 + 4p^2 + 9p - 2)$
(4) $RM_{ve}^2(K_p^c) = 1/8p \quad (p^5 + 5p^4 + 15p^3 + 31p^2 - 16p - 4)$
(5) $\mathrm{Hyp}_{ve}(K_p^c) = 1/2p \quad (p^5 + 6p^4 + 17p^3 + 44p^2 + 28p + 4)$
(6) $F_{ve}(K_p^c) = 1/4p(p^5 + 7p^4 + 15p^3 + 41p^2 + 32p + 8)$
(7) $\mathrm{Alb}_{ve}(K_p^c) = p(p-2)(p+1)$
(8) $\sigma_{ve}(K_p^c) = 1/2p(p^2 - p - 2)^2$

2.5. Cog-Wheel Graph

Definition 16. A cog-wheel graph W_p^c is the graph constructed from a wheel W_p, $p \geq 5$, of a vertex set $V(W_p) = \{v_1, v_2, \ldots, v_p\}$, and with $(p-1)$ additional vertices $U = \{u_1, u_2, \ldots, u_{p-1}\}$, and $2(p-1)$ edges $\{u_i v_{i+1}, u_i v_{i+2} : i = 1, 2, \ldots, p-1\}$, $v_{p+1} \equiv v_2$, as shown in Figure 6.

2.5.1. Some Properties of a Cog-Wheel Graph W_p^c

(i) The order and the size are $p(W_p^c) = 2p - 1$ and $q(W_p^c) = 4(p-1)$, repectively
(ii) The degree of vertex v_1 is $(p-1)$ which represent the maximum degree "$\Delta(W_p^c) = p - 1$," the degrees of vertices v_i, $2 \leq i \leq p$ are 5 and the degrees of vertices u_i, $1 \leq i \leq p-1$ are 2 which represent the minimum degree "$\delta(W_p^c) = 2$"

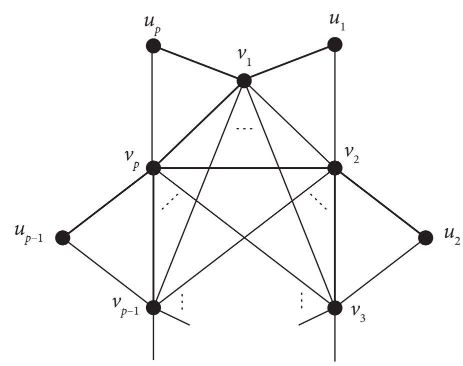

Figure 5: A cog-complete graph K_p^c.

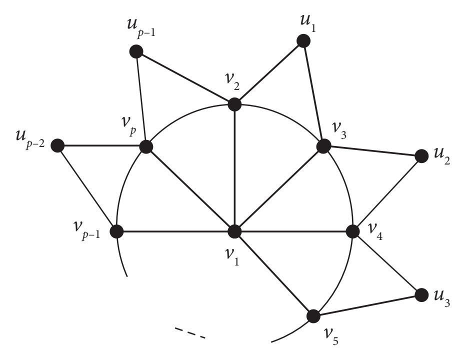

Figure 6: A cog-wheel graph W_p^c.

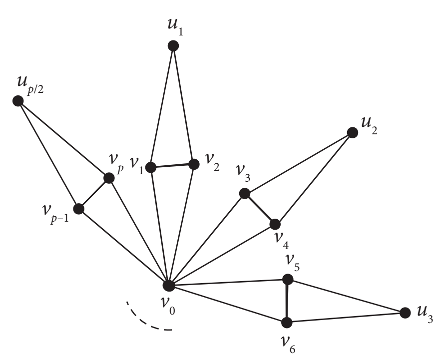

Figure 7: Cog-fan graph F_p^c.

(iii) The maximum and minimum *ve*− degrees are $\Delta_{ve}(W_p^c) = 4(p-1)$ and $\delta_{ve}(W_p^c) = 9$, respectively

Theorem 17. *Let W_p^c be the cog-wheel graph of order $2p-1$, $p \geq 6$. Then,*

$$M_{ve}\left(W_p^c; x, y\right) = 2(p-1)x^9 y^{p+9} + (p-1)x^{p+9} y^{p+9} + (p-1)x^{p+9} y^{4(p-1)}. \tag{17}$$

Proof. Since the vertex v_1 is adjacent to v_i, for $i = 2, \dots, p$, the vertex u_i is adjacent to two adjacent vertices v_{i+1} and v_{i+2}, for $i = 1, 2, \dots, p-1$, where $(v_{p+1} = v_2)$, see Figure 6, then

$$\begin{aligned} \tau_{(v_1)} &= 4(p-1), \\ \tau_{(u_i)} &= 9, \text{for all } 1 \le i \le p-1, \\ \tau_{(v_i)} &= p+9, \text{for all } 2 \le i \le p. \end{aligned} \tag{18}$$

Hence, for all $z, w \in V(W_p^c)$ such that $\tau_{(z)} \le \tau_{(w)}$, we have

$$\begin{aligned} M_{ve}\left(W_p^c; x, y\right) &= \sum_{zw \in E\left(W_p^c\right)} x^{\tau(z)} y^{\tau(w)} \\ &= \sum_{i=1}^{p-1} x^{\tau(u_i)} y^{\tau(v_{i+1})} + \sum_{i=1}^{p-2} x^{\tau(u_i)} y^{\tau(v_{i+2})} + x^{\tau(u_{p-1})} y^{\tau(v_2)} \\ &\quad + \sum_{i=2}^{p-1} x^{\tau(v_i)} y^{\tau(v_{i+1})} + x^{\tau(v_p)} y^{\tau(v_2)} + \sum_{i=2}^{p} x^{\tau(v_i)} y^{\tau(v_1)} \\ &= (p-1)x^9 y^{p+9} + (p-1)x^9 y^{p+9} + (p-1)x^{p+9} y^{p+9} \\ &\quad + (p-1)x^{p+9} y^{4(p-1)} \\ &= 2(p-1)x^9 y^{p+9} + (p-1)x^{p+9} y^{p+9} + (p-1)x^{p+9} y^{4(p-1)}. \end{aligned} \tag{19}$$

□

Remark 18

(i) $M_{ve}(W_4^c; x, y) = 6x^9 y^{12} + 6x^{12} y^{12}$

(ii) $M_{ve}(W_5^c; x, y) = 8x^9 y^{14} + 4x^{14} y^{14} + 4x^{14} y^{16}$.

In the next corollary, we can easily obtain the topological indices of the $M_{ve}(W_p^c; x, y)$ from the derivative with respect to x, y, and $x\,y$ by the substitution $x = y = 1$.

Corollary 19. *Let W_p^c be a cog-wheel graph of order $2p-1$, $p \ge 5$, then we have*

(1) $M_{ve}^1(W_p^c) = (p-1)(9p+59)$

(2) $M_{ve}^2(W_p^c) = (p-1)(p+9)(5p+23)$

(3) $RM_{ve}^1(W_p^c) = (p-1)(9p+51)$

(4) $RM_{ve}^2(W_p^c) = (p-1)(p+8)(5p+19)$

(5) $\mathrm{Hyp}_{ve}(W_p^c) = (p-1)(31p^2+194p+997)$

(6) $F_{ve}(W_p^c) = (p-1)(21p^2+58p+583)$

(7) $\mathrm{Alb}_{ve}(W_p^c) = (p-1)(5p-13)$

(8) $\sigma_{ve}(W_p^c) = (p-1)(11p^2-78p+169)$

3. $M_{ve}-$ Polynomial of Two Types of Cog-Fan Graphs

In this section, we find $M_{ve}-$ polynomials of two types of cog-fan graphs, such as fan and hand fan graphs [17].

3.1. Cog-Fan Graph

Definition 20. A cog-fan graph F_p^c is a graph constructed from a fan graph F_p, $p \ge 4$ (p is an even number) of a vertex set $V = \{v_0, v_1, \dots, v_p\}$ by adding $p/2$ vertices $U = \{u_1, u_2, \dots, u_{p/2}\}$ and p edges $\{u_i v_{2i-1}, u_i v_{2i}\colon i = 1, 2, \dots, p/2\}$ to the fan graph F_p, see Figure 7.

3.1.1. Some Properties of a Cog-Fan Graph F_p^c

(i) The order and the size are $p(F_p^c) = 3/2p+1$ and $q(F_p^c) = 5/2p$, respectively

(ii) The degree of vertex v_0 is p which represent the maximum degree "$\Delta(F_p^c) = p$," the degrees of vertices v_i, $1 \le i \le p$ are 3 and the degrees of vertices u_i, $1 \le i \le p/2$ are 2 which represent the minimum degree "$\delta(F_p^c) = 2$"

(iii) The maximum and minimum $ve-$ degrees are $\Delta_{ve}(F_p^c) = 5/2p$ and $\delta_{ve}(F_p^c) = 5$, respectively

Theorem 21. *Let F_p^c be the cog-fan graph of order $(3/2p+1)$, $p \ge 4$ and p is even. Then,*

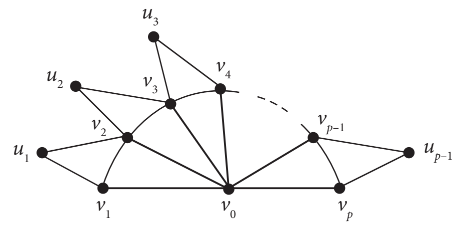

Figure 8: Cog-hand fan graph HF_p^c.

$$M_{ve}\left(F_p^c; x, y\right) = px^5 y^{p+3} + \frac{p}{2} x^{p+3} y^{p+3} + px^{p+3} y^{5p/2}. \quad (20)$$

Proof. Since the vertices u_0 and v_i are adjacent to u_{2i-1} and u_{2i}, for all $1 \le i \le p/2$, then

$$\begin{aligned} \tau_{(v_0)} &= \frac{5p}{2}, \\ \tau_{(v_i)} &= p + 3, \text{for all } 1 \le i \le p, \\ \tau_{(u_i)} &= 5, \text{for all } 1 \le i \le \frac{p}{2}. \end{aligned} \quad (21)$$

Hence,

$$\begin{aligned} M_{ve}\left(F_p^c; x, y\right) &= \sum_{i=1}^{p/2} x^{\tau(u_i)} y^{\tau(v_{2i-1})} + \sum_{i=1}^{p/2} x^{\tau(u_i)} y^{\tau(v_{2i})} + \sum_{i=1}^{p/2} x^{\tau(v_{2i-1})} y^{\tau(v_{2i})} \\ &\quad + \sum_{i=1}^{p} x^{\tau(v_i)} y^{\tau(v_0)} \\ &= \frac{p}{2} x^5 y^{p+3} + \frac{p}{2} x^5 y^{p+3} + \frac{p}{2} x^{p+3} y^{p+3} + px^{p+3} y^{5p/2} \\ &= px^5 y^{p+3} + \frac{p}{2} x^{p+3} y^{p+3} + px^{p+3} y^{5p/2}. \end{aligned} \quad (22)$$

□

Corollary 22. *Let F_p^c be the cog-fan graph of order $(3/2p + 1)$, $p \ge 4$. Then we have*

(1) $M_{ve}^1(F_p^c) = 1/2p(11p + 28)$
(2) $M_{ve}^2(F_p^c) = 1/2p(p + 3)(6p + 13)$
(3) $RM_{ve}^1(F_p^c) = 1/2p(11p + 18)$
(4) $RM_{ve}^2(F_p^c) = 1/2p(6p^2 + 20p + 16)$
(5) $\mathrm{Hyp}_{ve}(F_p^c) = 1/4p(61p^2 + 196p + 364)$
(6) $F_{ve}(F_p^c) = 1/4p(37p^2 + 72p + 208)$
(7) $\mathrm{Alb}_{ve}(F_p^c) = 5/4p(p - 2)$
(8) $\sigma_{ve}(F_p^c) = 13/4p(p^2 - 4p + 4)$

3.2. Cog-Hand Fan Graph

Definition 23. A cog-hand fan graph HF_p^c is the graph constructed from a hand fan graph HF_p, $p \ge 3$ of vertex set $V = \{v_0, v_1, \ldots, v_p\}$ by adding $p - 1$ vertices $U = \{u_1, u_2, \ldots, u_{p-1}\}$ and $2(p - 1)$ edges $\{u_i v_i, u_i v_{i+1}: i = 1, 2, \ldots, p - 1\}$ to the graph HF_p, see Figure 8.

3.2.1. Some Properties of a Cog-Hand Fan Graph HF_p^c

(i) The order and the size are $p(\mathrm{HF}_p^c) = 2p$ and $q(HF_p^c) = 4p - 3$, respectively

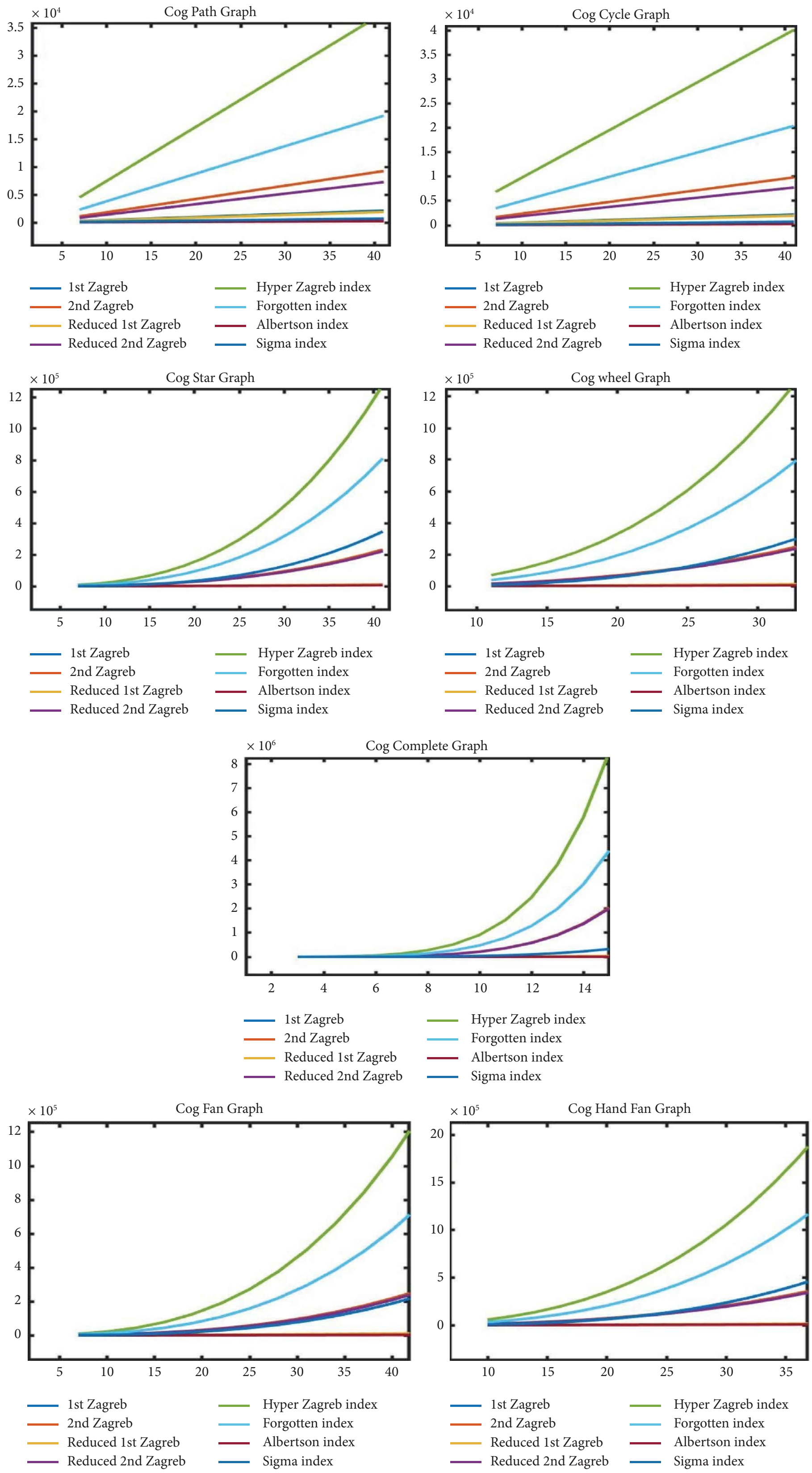

Figure 9: Comparison of some vertex-edge-degree-based topological indices for some cog-special graphs.

(ii) The degree of vertex v_0 is p which represent the maximum degree "$\Delta(\mathrm{HF}_p^c) = p$," the degree of vertices v_1 and v_p are 3, the degrees of vertices v_i, $2 \leq i \leq p-1$ are 5 and the degrees of vertices u_i, $1 \leq i \leq p-1$ are 2 which represent the minimum degree "$\delta(\mathrm{HF}_p^c) = 2$"

(iii) The maximum and minimum $ve-$ degrees are $\Delta_{ve}(F_p^c) = 4p-3$ and $\delta_{ve}(F_p^c) = 7$, respectively

Theorem 24. *Let HF_p^c be the cog-hand fan graph of order $2p$, $p \geq 5$. Then,*

$$\begin{aligned} M_{ve}\left(HF_p^c; x, y\right) = {} & 2x^7 y^{p+5} + 2x^7 y^{p+8} + 2x^9 y^{p+8} + 2x^9 y^{p+10} + 2(p-5)x^9 y^{p+10} \\ & + 2x^{p+5} y^{p+8} + 2x^{p+8} y^{p+10} + (p-5)x^{p+10} y^{p+10} + 2x^{p+5} y^{4p-3} \\ & + 2x^{p+8} y^{4p-3} + (p-4)x^{p+10} y^{4p-3}. \end{aligned} \tag{23}$$

Proof. Since the vertex v_0 is adjacent to v_i, for all $1 \leq i \leq p$. Also, the vertex u_i is adjacent to two adjacent vertices v_i and v_{i+1}, for all $1 \leq i \leq p-1$, then,

$$\begin{aligned} \tau_{(u_0)} &= 4p-3, \\ \tau_{(u_i)} &= p+5, \text{for all } i = 1, p, \\ \tau_{(u_i)} &= p+8, \text{for all } i = 2, p-1, \\ \tau_{(u_i)} &= p+10, \text{for all } 3 \leq i \leq p-2, \\ \tau_{(v_i)} &= 7, \text{for all } i = 1, p-1, \\ \tau_{(v_i)} &= 9, \text{for all } 2 \leq i \leq p-2. \end{aligned} \tag{24}$$

Hence, for all $z, w \in V(HF_p^c)$ such that $\tau_{(z)} \leq \tau_{(w)}$, we have

$$
\begin{aligned}
M_{ve}\left(HF_p^c; x, y\right) &= \sum_{zw\in E\left(HF_p^c\right)} x^{\tau(z)} y^{\tau(w)} \\
&= x^{\tau(u_1)} y^{\tau(v_1)} + x^{\tau(u_1)} y^{\tau(v_2)} + x^{\tau(u_2)} y^{\tau(v_2)} \\
&\quad + \sum_{i=2}^{p-3} x^{\tau(u_i)} y^{\tau(v_{i+1})} + \sum_{i=3}^{p-2} x^{\tau(u_i)} y^{\tau(v_i)} \\
&\quad + x^{\tau(u_{p-1})} y^{\tau(v_p)} + x^{\tau(u_{p-1})} y^{\tau(v_{p-1})} + x^{\tau(u_{p-2})} y^{\tau(v_{p-1})} \\
&\quad + x^{\tau(v_1)} y^{\tau(v_2)} + x^{\tau(v_2)} y^{\tau(v_3)} + \sum_{i=3}^{p-3} x^{\tau(v_i)} y^{\tau(v_{i+1})} \\
&\quad + x^{\tau(v_p)} y^{\tau(v_{p-1})} + x^{\tau(v_{p-1})} y^{\tau(v_{p-2})} + x^{\tau(v_1)} y^{\tau(v_0)} \\
&\quad + x^{\tau(v_2)} y^{\tau(v_0)} + \sum_{i=3}^{p-2} x^{\tau(v_i)} y^{\tau(v_0)} + x^{\tau(v_p)} y^{\tau(v_0)} \\
&\quad + x^{\tau(v_{p-1})} y^{\tau(v_0)} \\
&= x^7 y^{p+5} + x^7 y^{p+8} + x^9 y^{p+8} + (p-4) x^9 y^{p+10} \\
&\quad + (p-4) x^9 y^{p+10} + x^7 y^{p+5} + x^7 y^{p+8} + x^9 y^{p+8} \\
&\quad + x^{p+5} y^{p+8} + x^{p+8} y^{p+10} + (p-5) x^{p+10} y^{p+10} \\
&\quad + x^{p+5} y^{p+8} + x^{p+8} y^{p+10} + x^{p+5} y^{4p-3} + x^{p+8} y^{4p-3} \\
&\quad + (p-4) x^{p+10} y^{4p-3} + x^{p+5} y^{4p-3} + x^{p+8} y^{4p-3} \\
&= 2x^7 y^{p+5} + 2x^7 y^{p+8} + 2x^9 y^{p+8} + 2(p-4) x^9 y^{p+10} \\
&\quad + 2x^{p+5} y^{p+8} + 2x^{p+8} y^{p+10} + (p-5) x^{p+10} y^{p+10} \\
&\quad + 2x^{p+5} y^{4p-3} + 2x^{p+8} y^{4p-3} + (p-4) x^{p+10} y^{4p-3}.
\end{aligned} \tag{25}
$$

□

Remark 25

$$
M_{ve}\left(HF_4^c; x, y\right) = 2x^7 y^9 + 2x^7 y^{12} + 4x^9 y^{12} + 2x^9 y^{13} + x^{12} y^{12} + 2x^{12} y^{13}. \tag{26}
$$

Corollary 26. *Let HF_p^c be a cog-hand fan graph of order 2 p, $p \geq 5$. Then we have*

(1) $M_{ve}^1(\mathrm{HF}_p^c) = 9p^2 + 61p - 116$
(2) $M_{ve}^2(\mathrm{HF}_p^c) = 5p^3 + 74p^2 + 130p - 612$
(3) $RM_{ve}^1(HF_p^c) = 9p^2 + 53p - 110$
(4) $RM_{ve}^2(HF_p^c) = 5p^3 + 65p^2 + 73p - 499$
(5) $\mathrm{Hyp}_{ve}(\mathrm{HF}_p^c) = 31p^3 + 220p^2 + 751p - 2724$
(6) $F_{ve}(\mathrm{HF}_p^c) = 21p^3 + 72p^2 + 491p - 1500$
(7) $\mathrm{Alb}_{ve}(\mathrm{HF}_p^c) = 5p^2 - 13p + 12$
(8) $\sigma_{ve}(\mathrm{HF}_p^c) = 11p^3 - 76p^2 + 231p - 276$

4. Conclusions

The topological indices obtained by presenting a new definition of the $M_{ve}-$ polynomial that depends on the $ve-$ degree of a vertex v, as presented by Chellali et al., are distinguished by the fact that they lie between the topological indices of degrees of a vertex and the topological indices of the neighborhoods of vertex v. We noticed that the topological indicators mentioned in Table 1 for both the cog-path and the cog-cycle have the same slope, only the start points are different. In addition to that, the first Zagreb and reduced first Zagreb indices are almost identical, and the Albertson and sigma indices are also almost identical in all cog-path and cog-cycle graphs. The first Zagreb and reduced first Zagreb indices are almost identical, the second Zagreb and reduced second Zagreb indices are almost identical, and the Albertson index is very close to both the second Zagreb and reduced second Zagreb indices in all cog-star, cog-wheel, cog-complete, cog-fan, and cog-hand fan graphs with very few differences, see Figure 9. Finally, the topological indices mentioned in Table 1 have the same behavior in all the private graphs studied in the paper.

Data Availability

The data used to support the findings of this study are available at the corresponding author upon request.

Conflicts of Interest

The authors declare that they do not have any conflicts of interest.

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