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Research Article

Improved Finite Difference Technique via Adomian Polynomial to Solve the Coupled Drinfeld's–Sokolov–Wilson System

Israa Th. Younis and Ekhlass S. Al-Rawi

Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Mosul, Iraq

Correspondence should be addressed to Ekhlass S. Al-Rawi; drekhlass-alrawi@uomosul.edu.iq

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This study presents a new algorithm for effectively solving the nonlinear coupled Drinfeld's–Sokolov–Wilson (DSW) system using a hybrid explicit finite difference technique with the Adomian polynomial (EFD-AP). The suggested approach addresses the problem of accurately solving the DSW system. Numerical results are obtained by comparing the exact solution with absolute and mean square errors using a test problem to assess the EFD-AP accuracy against the exact solution and the conventional explicit finite difference (EFD) method. The results exhibit excellent agreement between the approximate and exact solutions at different time values, and the results showed that the proposed EFD-AP method achieves superior accuracy and efficiency compared to the EFD method, which makes it a promising method for solving nonlinear partial differential systems of higher order.

1. Introduction

The exploration of nonlinear equations has a long and storied history, dating back to ancient civilizations. Mathematicians of the past were driven to address challenges involving curved shapes and nonlinear phenomena. However, it was not until the 17th century when Sir Isaac Newton and Gottfried Wilhelm Leibniz developed calculus that the fundamental groundwork for solving nonlinear equations began to take shape. Throughout the centuries, renowned figures in the world of mathematics and science, such as Lagrange, Gauss, and Poincaré, made substantial contributions to the theory and methods for solving nonlinear equations [1].

In the 18th and 19th centuries, the finite difference method emerged as a numerical technique for approximating solutions to differential equations [2]. Initially, it was applied to linear equations, but its popularity surged with the advent of digital computers in the mid-20th century. The finite difference method involves discretizing the continuous domain by dividing it into a grid of discrete points and approximating derivatives using finite difference formulas. This approach effectively transforms differential equations into systems of algebraic equations, enabling numerical solutions. The scope of the finite difference method expanded to encompass nonlinear equations, rendering it a potent tool for addressing challenges in fields such as physics, engineering [3], and finance. Various fields, including mathematical physics, theoretical physics, and integrable systems, heavily rely on the coupled Drinfeld's–Sokolov–Wilson (DSW) system. Its applications encompass understanding soliton solutions in nonlinear wave equations, shedding light on quantum fields and string theories, and exploring nonlinear dynamics, chaos theory, and nonlinear optical phenomena, particularly in optical fiber technology. Moreover, it plays a vital role in information theory, quantum computing, and mathematical modeling across numerous physical domains. In the academic realm, the DSW system serves as an invaluable resource for advanced studies and instruction in mathematics and physics [4].

In summary, nonlinear equations boast a rich historical background and the development of the finite difference method has been pivotal in enabling numerical solutions. These advancements have significantly enhanced our grasp of intricate nonlinear phenomena and our capacity to

address real-world challenges. For instance, methods such as the Adomian decomposition method with modified Bernstein polynomials [5], Bernstein polynomials in conjunction with artificial neural networks [6], and the Elzaki decomposition method for tackling higher-order integrodifferential equations have all contributed to this progress [7–14]. The formula for the Drinfeld's–Sokolov–Wilson (DSW) system also plays a critical role in this context [14–18].

$$\frac{\partial u}{\partial t} + \mathrm{pv}\frac{\partial v}{\partial x} = 0, \tag{1}$$

$$\frac{\partial v}{\partial t} + q\frac{\partial^3 v}{\partial x3} + r\,u\frac{\partial v}{\partial x} + \mathrm{sv}\frac{\partial u}{\partial x} = 0. \tag{2}$$

In the system of (1) and (2), p, q, r, and s are nonzero constants, originally introduced by Drinfeld, Sokolov [8], and Wilson [1], and they represent a model for water waves and find extensive applications in the field of fluid dynamics. Over the years, various numerical and analytical methods have been developed to address these equations, including the Adomian decomposition method [16], the exp-function method [10], the improved F-expansion method [19], the bifurcation method [20], and qualitative theory [21]. These methods offer diverse approaches to solve systems (1) and (2), contributing to a better understanding of their behavior and properties. Traditionally, techniques such as the inverse scattering transform, Hirota's bilinear method, and other integrable methods have been employed to analyze the DSW system.

The motivation behind this research is to introduce a novel hybrid approach that combines the finite difference method with the Adomian polynomial. This hybrid method aims to address the nonlinear term in the DSW system and handle cases where points fall outside the solution region. By doing so, this innovative technique enhances the accuracy and efficiency of the solution while dealing with the nonlinearities present in the system. In this research, we cover fundamental concepts, the mathematical formulation of the EFD and EFD-AP methods, their respective algorithms, the convergence analysis of EFD-AP, and their practical application, see [22–24].

2. Main Concepts

2.1. Explicit Finite Difference (EFD) Method. The equations in question contain a third derivative term, which presents a distinctive challenge that calls for the adoption of a specialized numerical solution method. The peculiar impact of the third derivative on these equations and their behavior makes it necessary to employ a different approach or method to ensure accurate and efficient problem solving. To address this, we will utilize the following set of discrete equations [25].

Forward formulas:

$$u_t = \frac{u_i^{j+1} - u_i^j}{k}, \tag{3}$$

$$u_x = \frac{-3u_i^j + 4u_{i+1}^j - u_{i+2}^j}{2H}, \tag{4}$$

$$u_{xx} = \frac{2u_i^j - 5u_{i+1}^j + 4u_{i+2}^j - u_{i+3}^j}{H^2}, \tag{5}$$

$$u_{xxx} = \frac{-5u_i^j + 18u_{i+1}^j - 24u_{i+2}^j + 14u_{i+3}^j - 3u_{i+4}^j}{2H^3}. \tag{6}$$

Central formulas:

$$u_x = \frac{u_{i+1}^j - u_{i-1}^j}{2H}, \tag{7}$$

$$u_{xx} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{H^2}, \tag{8}$$

$$u_{xxx} = \frac{u_{i+2}^j - 2u_{i+1}^j + 2u_{i-1}^j - u_{i-2}^j}{2H^3}. \tag{9}$$

Backward formulas:

$$u_x = \frac{3u_i^j - 4u_{i+1}^j + u_{i+2}^j}{2H}, \tag{10}$$

$$u_{xx} = \frac{2u_i^j - 5u_{i-1}^j + 4u_{i-2}^j - u_{i-3}^j}{H^2}, \tag{11}$$

$$u_{xxx} = \frac{5u_i^j - 18u_{i-1}^j + 24u_{i-2}^j - 14u_{i-3}^j + 3u_{i-4}^j}{2H^3}. \tag{12}$$

2.2. The Adomian Polynomial (AP). The method assumes that the nonlinear term can be approximated by an infinite series of polynomials expressed in a specific form.

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n), \tag{13}$$

where A_n are the Adomian polynomials [26] defined as $A_n = 1/n!\,[d^n/dp^n N(\sum_{i=0}^{\infty} p^i u_i)]_{p=0}$, where $n = 0, 1, 2, \ldots$

Now,

$$\begin{aligned} A_0 &= u_0 u_{0x}, \\ A_1 &= u_1 u_{0x} + u_0 u_{1x}, \\ A_2 &= u_2 u_{0x} + u_1 u_{1x} + u_0 u_{2x}, \\ &\text{.,} \\ A_n &= u_n u_{0x} + u_{n-1} u_{1x} + u_{n-2} u_{2x} + \ldots + u_0 u_{nx} = \left(\sum_{m=s}^{0} \sum_{n=0}^{s} \left(u_i^{j-1+m}\right)\left(u_{i+1}^{j-1+n}\right) - \left(u_{i-1}^{j-1+n}\right) \right). \end{aligned} \tag{14}$$

The Adomian decomposition method's stability was examined in the research conducted by Azim and Hosseini [27]. Furthermore, the analysis of its convergence was established by Ahmed [28] and Zhang [19].

3. Methodology

In this section, the finite difference method involves solving the DSW system by substituting the derivatives in (1) and (2) with central finite difference derivatives as given in (7) and (9). This transformation converts the system into a set of regular algebraic equations that can be solved using standard methods to determine the values of certain variables. On the other hand, the hybrid method, which is the EFD-AP method, relies on replacing the nonlinear component in the finite differences with central derivatives. After this substitution, we simplify the system, resulting in a set of algebraic equations that are solved to obtain the values of u and v.

3.1. Mathematical Formulation of the EFD Method for the DSW System. When applying the EFD method, solutions appear outside the region and, therefore, this case is dealt with in the second equation of the system in the following manner.

Divide the rectangle $R = \{(x,t)\colon a \le x \le b, 0 \le t \le c\}$ into $n-1$ and $m-1$ of rectangles length of each side ($\Delta x = H$) and ($\Delta t = k$). We start from $t = t_0 = 0$, where the solution values are calculated from the initial conditions as follows:

$$\begin{aligned} u(x,0) &= p \sec h^2(x), \\ v(x,0) &= q \sec h(x). \end{aligned} \tag{15}$$

Then, the calculation of approximations is explained for $u(x,t)$ and $v(x,t)$, and the grid points of the other rows $\left\{u(x_i,t_j), v(x_i,t_j), i = 1,2,\ldots,n-1, j = 1,2,\ldots,m\right\}$ and the solution method will be as follows.

To calculate the approximate value of the solution $u(x_i,t_j)$, we use (3) and the central finite difference (7), and substituting them into (1), we get

$$\frac{u_i^{j+1} - u_i^j}{k} + p\, v(i,j)\, \frac{v_{i+1}^j - v_{i-1}^j}{2H} = 0. \tag{16}$$

Let $\alpha = k/2H$ and by rearranging the above equation, we get

$$u_i^{j+1} = u_i^j - \alpha\, p v(i,j)\left(v_i^{j+1} - v_i^{j-1}\right). \tag{17}$$

This equation computes the value of the solution u at level $j+1$ from the known values of level j at $i = 1,2,\ldots,n-1$ from (1), and to calculate approximate values for the solution $v(x_i,t_j)$ when $i = 1$, we use forward difference equations (3), (4), and (6), and substituting them into (2), we get

$$\frac{v_i^{j+1} - v_i^j}{k} + q\left(\frac{-5v_i^j + 18v_{i+1}^j - 24v_{i+2}^j + 14v_{i+3}^j - 3v_{i+4}^j}{2H^3}\right) + \mathrm{sv}(i,j)\left(\frac{-3u_i^j + 4u_{i+1}^j - u_{i+2}^j}{2H}\right) + \mathrm{ru}(i,j)\left(\frac{-3v_i^j + 4v_{i+1}^j - v_{i+2}^j}{2H}\right) = 0. \tag{18}$$

Suppose that $\beta = k/2H^3, \mu = k/2H$ and by rearranging the above equation, we get

$$v_i^{j+1} = v_i^j - q\mu\left(-5v_i^j + 18v_{i+1}^j - 24v_{i+2}^j + 14v_{i+3}^j - 3v_{i+4}^j\right) - \beta s\, v(i,j)\left(-3u_i^j + 4u_{i+1}^j - u_{i+2}^j\right) - r\alpha\, u(i,j)\left(-|\,3v_i^j + 4v_{i+1}^j - v_{i+2}^j\right). \tag{19}$$

Equation (19) denotes the solution v at point x_1 on the grid.

To determine the remaining approximate values $v(x_i, t_j)$, where i is in the range of $2, 3, \ldots, n-2$ within the same row, we substitute the central (7) and (9) into (2) as follows:

$$\frac{v_i^{j+1} - v_i^j}{k} + q\left(\frac{v_{i+2}^j - 2v_{i+1}^j + 2v_{i-1}^j - v_{i-2}^j}{2H^3}\right) + \mathrm{sv}(i,j)\left(\frac{u_{i+1}^j - u_{i-1}^j}{2H}\right) + \mathrm{ru}(i,j)\left(\frac{v_{i+1}^j - v_{i-1}^j}{2H}\right) = 0. \tag{20}$$

By compensating for values of α, β, and μ and rearranging equation (20), we get

$$v_i^{j+1} = v_i^j - q\mu\left(v_{i+2}^j - 2v_{i+1}^j + 2v_{i-1}^j - v_{i-2}^j\right) - \beta\mathrm{sv}(i,j)\left(u_{i+1}^j - u_{i-1}^j\right) - \alpha\mathrm{ru}(i,j)\left(v_{i+1}^j - v_{i-1}^j\right). \tag{21}$$

Equation (21) computes the value of the solution v at level $j+1$ using the known values of level j at $i = 3, 4, \ldots, n-2$.

Subsequently, we can calculate the values of v at level $j+1$ when $i = n-1$.

Similarly, by substituting the backward equations (10) and (12) into (2) and rearranging, we obtain

$$v_i^{j+1} = v_i^j - q\mu\left(5u_i^j - 18u_{i+1}^j + 24u_{i+2}^j - 14u_{i+3}^j + 3u_{i+4}^j\right) - \beta s\, v(i,j)\left(3u_i^j - 4u_{i+1}^j + u_{i+2}^j\right) - r\alpha\, u(i,j)\left(3v_i^j - 4v_{i+1}^j + v_{i+2}^j\right). \tag{22}$$

This equation calculates the solution v at the point x_{n-1} of the grid.

Upon examining the central numerical approximation of the third derivative as defined in (22), we observe that the two approximate solutions computed at grid node locations include values for points situated beyond the boundaries of the solution domain and these values remain unknown. Furthermore, the boundary conditions cannot be applied to these points as they do not fall within the boundary constraints. It is important to highlight that the local truncation error for these points is significant that the local truncation error for the method is $(i, j) = O(k) + O(H^2)$; the method is stable and its stability condition is $0 \le \alpha \le 1/2$ (Algorithm 1).

3.2. Mathematical Formulation of the EFD-AP Method. To calculate the approximate value of the solution $u(x_i, t_j)$, we substitute the central finite difference (7) into (1), and in the second step, we substitute the Adomian polynomial $\sum_{n=0}^{\infty}$An into (22), i.e., $u_i^{j+1} - u_i^j/k + p\sum_{n=0}^{\infty} An = 0$, where

$$\begin{aligned} A_0 &= v_0 v_{0x}, \\ A_1 &= v_1 v_{0x} + v_0 v_{1x}, \\ A_2 &= v_2 v_{0x} + v_1 v_{1x} + v_0 v_{2x}, \\ &\cdots \\ A_n &= v_n v_{0x} + v_{n-1} v_{1x} + v_{n-2} v_{2x} + \ldots + v_0 v_{nx}. \end{aligned} \tag{23}$$

Now, by replacing the above derivative by the finite differences formula

$$\begin{aligned} A_0 &= v_i^{j-1}\left(\frac{v_{i+1}^{j-1} - v_{i-1}^{j-1}}{2H}\right), \\ A_1 &= v_i^j\left(\frac{v_{i+1}^{j-1} - v_{i-1}^{j-1}}{2H}\right) + v_i^{j-1}\left(\frac{v_{i+1}^j - v_{i-1}^j}{2H}\right), \\ A_2 &= v_i^{j+1}\left(\frac{v_{i+1}^{j-1} - v_{i-1}^{j-1}}{2H}\right) + v_i^j\left(\frac{v_{i+1}^j - v_{i-1}^j}{2H}\right) + v_i^{j-1}\left(\frac{v_{i+1}^{j+1} - v_{i-1}^{j+1}}{2H}\right), \\ &\cdots \\ A_n &= \left(\sum_{m=s}^{0}\sum_{n=0}^{s}\left(v_i^{j-1+m}\right)\left(v_{i+1}^{j-1+n}\right) - \left(v_{i-1}^{j-1+n}\right)\right), \end{aligned} \tag{24}$$

Input: the values of a, b, h, and confections $p, q, r,$ and s, number of space fragmentation n, and for time m.
Step 1: evaluate $H = (b-a)/n$ space step size and $k = t/m$ time step size.
Step 2: find the values of $\alpha = T/2\mathrm{H}$, $\beta = T/H^2$, and $\mu = k/2H^2$
Step 3: compute the initial condition and boundary conditions
$u(x,0) = u_0(x)$, $v(x,o) = v_0(x)$, $a < x < b$
$u(a,t) = u_1(t)$, $u(b,t) = u_2(t)$
$v(a,t) = v_1(t)$, $(b,t) = v_2(t)$, $0 < t < c$.
Step 4: compute the numerical solution u_i^{j+1} from
$u_i^{j+1} = u_i^j - \alpha\, pv(i,j)(v_i^{j+1} - v_i^{j-1})$, where $j = 1, 2, \ldots, m-1$, and $i = 1, 2, \ldots, n-1$
Step 5: compute the numerical solution v_i^{j+1} from
$v_i^{j+1} = v_i^j - q\mu(-5v_i^j + 18v_{i+1}^j - 24v_{i+2}^j + 14v_{i+3}^j - 3v_{i+4}^j) - \beta s\, v(i,j)(-3u_i^j + 4u_{i+1}^j - u_{i+2}^j) - r\alpha\, u(i,j)(-3v_i^j + 4v_{i+1}^j - v_{i+2}^j)$,
where $i = 1$ and for $i = n-1$ compute the numerical solution v_i^{j+1} from
$v_i^{j+1} = v_i^j - q\mu(5u_i^j - 18u_{i+1}^j + 24u_{i+2}^j - 14u_{i+3}^j + 3u_{i+4}^j) - \beta s\, v(i,j)(3u_i^j - 4u_{i+1}^j + u_{i+2}^j) - r\alpha\, u(i,j)(3v_i^j - 4v_{i+1}^j + v_{i+2}^j)$
Step 6: compute the numerical solution v_i^{j+1} for $i = 2, 3, \ldots, n-2$ from $v_i^{j+1} = v_i^j - q\mu(v_{i+2}^j - 2v_{i+1}^j + 2v_{i-1}^j - v_{i-2}^j) - \beta s v(i,j)(u_{i+1}^j - u_{i-1}^j) - \alpha r u(i,j)(v_{i+1}^j - v_{i-1}^j)$
Step 7: print the numerical solutions $u(x,t)$ and $v(x,t)$.

Algorithm 1: Algorithm of the EFD method.

and by compensating equation (24) in (1), we obtain

$$u_i^{j+1} = u_i^j - \frac{pk}{2H}\left(\sum_{m=s}^{0}\sum_{n=0}^{s}\left(v_i^{j-1+m}\right)\left(v_{i+1}^{j-1+n}\right) - \left(v_{i-1}^{j-1+n}\right)\right). \tag{25}$$

This equation computes the value of the solution u at level $j+1$ from the known values of level j at $i = 1, 2, \ldots, n-1$ from (1).

To calculate approximate values for the solution $v(x_i, t_j)$ when $i = 1$, we use forward difference equations (3), (4), and (6), and substituting them into (2), we get

$$\frac{\partial u}{\partial t} + q\frac{\partial^3 v}{\partial x3} + r\sum_{n=0}^{\infty} An + s\sum_{n=0}^{\infty} Bn = 0, \tag{26}$$

where $\sum_{n=0}^{\infty}$An and $\sum_{n=0}^{\infty}$Bn are Adomian polynomials as follows:

$$\begin{aligned}
A_0 &= u_0 v_{0x},\\
A_1 &= u_1 v_{0x} + u_0 v_{1x},\\
A_2 &= u_2 v_{0x} + u_1 v_{1x} + u_0 v_{2x},\\
&.,\\
A_n &= u_n v_{0x} + u_{n-1} v_{1x} + u_{n-2} v_{2x} + \ldots + u_0 v_{nx},\\
B_0 &= v_0 u_{0x},\\
B_1 &= v_1 u_{0x} + v_0 u_{1x},\\
B_2 &= v_2 u_{0x} + v_1 u_{1x} + v_0 u_{2x},\\
&.,\\
B_n &= v_n u_{0x} + v_{n-1} u_{1x} + v_{n-2} u_{2x} + \ldots + v_0 u_{nx}.
\end{aligned} \tag{27}$$

Now, by replacing the above derivative by the finite differences formula

$$\begin{aligned}
A_0 &= u_i^{j-1}\left(\frac{v_{i+1}^{j-1} - v_{i-1}^{j-1}}{2H}\right),\\
A_1 &= u_i^{j}\left(\frac{v_{i+1}^{j-1} - v_{i-1}^{j-1}}{2H}\right) + u_i^{j-1}\left(\frac{v_{i+1}^{j} - v_{i-1}^{j}}{2H}\right),\\
A_2 &= u_i^{j+1}\left(\frac{v_{i+1}^{j-1} - v_{i-1}^{j-1}}{2H}\right) + u_i^{j}\left(\frac{v_{i+1}^{j} - v_{i-1}^{j}}{2H}\right) + u_i^{j-1}\left(\frac{v_{i+1}^{j+1} - v_{i-1}^{j+1}}{2H}\right),\\
B_0 &= v_i^{j-1}\left(\frac{u_{i+1}^{j-1} - u_{i-1}^{j-1}}{2H}\right),\\
B_1 &= v_i^{j}\left(\frac{u_{i+1}^{j-1} - u_{i-1}^{j-1}}{2H}\right) + v_i^{j-1}\left(\frac{u_{i+1}^{j} - u_{i-1}^{j}}{2H}\right),
\end{aligned} \tag{28}$$

$B_2 = v_i^{j+1}(u_{i+1}^{j-1} - u_{i-1}^{j-1}/2H) + v_i^j \ (u_{i+1}^j - u_{i-1}^j/2H) + v_i^{j-1}(u_{i+1}^{j+1} - u_{i-1}^{j+1}/2H)$, and so on, we obtain

$$v_i^{j+1} = v_i^j - q\mu\left(-5v_i^j + 18v_{i+1}^j - 24v_{i+2}^j + 14v_{i+3}^j - 3v_{i+4}^j\right) - \beta s \sum_{m=s}^{0}\sum_{n=0}^{s}\left(v_i^{j-1+m}\right)\left(\left(u_{i+1}^{j-1+n}\right) - u_{i-1}^{j-1+n}\right) - \alpha r \sum_{m=s}^{0}\sum_{n=0}^{s}\left(u_i^{j-1+m}\right)\left(\left(v_{i+1}^{j-1+n}\right) - v_{i-1}^{j-1+n}\right), \tag{29}$$

where $\mu = k/2H^3, \beta = k/2H$.

To determine the remaining approximate values $v(x_i, t_j)$, where i is in the range of $2, 3, \ldots, n-2$ within the same level, we substitute the central equations into equation (2) as follows:

$$v_i^{j+1} = v_i^j - q\mu\left(v_{i+2}^j - 2v_{i+1}^j + 2v_{i-1}^j - v_{i-2}^j\right) - \beta s\left(\sum_{m=s}^{0}\sum_{n=0}^{s}\left(v_i^{j-1+m}\right)\left(u_{i+1}^{j-1+n}\right) - \left(u_{i-1}^{j-1+n}\right)\right) - \beta r\left(\sum_{m=s}^{0}\sum_{n=0}^{s}\left(u_i^{j-1+m}\right)\left(v_{i+1}^{j-1+n}\right) - \left(v_{i-1}^{j-1+n}\right)\right). \tag{30}$$

Subsequently, we can calculate the values of v at the level $j+1$ when $i = n-1$.

Similarly, by substituting the backward equations (10) and (12) into (2) and rearranging, we obtain

$$v_i^{j+1} = v_i^j - q\mu\left(5v_i^j - 18v_{i+1}^j + 24v_{i+2}^j - 14v_{i+3}^j + 3v_{i+4}^j\right) - \beta s\left(\sum_{m=s}^{0}\sum_{n=0}^{s}\left(v_i^{j-1+m}\right)\left(u_{i+1}^{j-1+n}\right) - \left(u_{i-1}^{j-1+n}\right)\right) - \alpha r\left(\sum_{m=s}^{0}\sum_{n=0}^{s}\left(u_i^{j-1+m}\right)\left(v_{i+1}^{j-1+n}\right) - \left(v_{i-1}^{j-1+n}\right)\right). \tag{31}$$

By expressing nonlinearities through Adomian polynomials, we can enhance the precision in approximating nonlinear components. This has the potential to result in a more precise overall solution, particularly when extending the polynomial expansion to higher degrees. In addition, augmenting the number of terms in the Adomian polynomial can boost the efficiency of the method proposed. However, choosing the optimal number of terms in the Adomian expansion to strike a balance between accuracy and computational efficiency can be a challenging task. Similar to other numerical methods, errors have the potential to propagate and amplify, especially if not applied correctly.

3.2.1. Convergence Analysis of the EFD-AP Method. Consider the approximate solution for the DSW system from (25) and the exact solutions of this system $\tilde{U}, \tilde{V}$ and all the necessary derivatives of $\tilde{U}, \tilde{V}$ exist and from (25), we have

$$\acute{\mathrm{U}}_i^{j+1} = \tilde{U}_i^j - \frac{pk}{2H}\left(\sum_{m=s}^{0}\sum_{n=0}^{s}\left(\tilde{V}_i^{j-1+m}\right)\left(\tilde{V}_{i+1}^{j-1+n}\right) - \left(\tilde{V}_{i-1}^{j-1+n}\right)\right) + \mathrm{kT}_{i,j}, \tag{32}$$

where $T_{i,j}$ is the local truncation error.

Input: the values of a, b, h, and confections $p, q, r,$ and s, number of space fragmentation n, and for time m.
Step 1: evaluate $H = (b-a)/n$ space step size and $k = t/m$ time step size.
Step 2: find the values of $\alpha = T/2\mathrm{H}$, $\beta = T/H^2$, and $\mu = k/2H^2$
Step 3: compute the initial condition and boundary conditions
$u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$, $a < x < b$
$u(a,t) = u_1(t)$, $u(b,t) = u_2(t)$
$v(a,t) = v_1(t)$, $(b,t) = v_2(t)$, $0 < t < c$.
Step 4: calculate the Adomian polynomials
$\sum_{n=0}^{\infty}\mathrm{An} = \sum_{m=s}^{0}\sum_{n=0}^{s}(v_i^{j-1+m})(v_{i+1}^{j-1+n} - v_{i-1}^{j-1+n})$
$\sum_{n=0}^{\infty}\mathrm{Bn} = (\sum_{m=s}^{0}\sum_{n=0}^{s}(u_i^{j-1+m})(v_{i+1}^{j-1+n} - v_{i-1}^{j-1+n}))$
Step 5: compute the numerical solution u_i^{j+1} from
$u_i^{j+1} = u_i^j - pk/2H(\sum_{m=s}^{0}\sum_{n=0}^{s}(v_i^{j-1+m})(v_{i+1}^{j-1+n}) - v_{i-1}^{j-1+n})$.
where $j = 1, 2, \ldots, m-1$, and $i = 1, 2, \ldots, n-1$
compute the numerical solution v_i^{j+1} at $i = 1$ from
$v_i^{j+1} = v_i^j - q\mu(-5v_i^j + 18v_{i+1}^j - 24v_{i+2}^j + 14v_{i+3}^j - 3v_{i+4}^j) - \beta s(\sum_{m=s}^{0}\sum_{n=0}^{s}(v_i^{j-1+m})(u_{i+1}^{j-1+n}) - u_{i-1}^{j-1+n}) - \alpha r(\sum_{m=s}^{0}\sum_{n=0}^{s}(u_i^{j-1+m})(v_{i+1}^{j-1+n}) - v_{i-1}^{j-1+n})$
and the numerical solution v_i^{j+1} at $i = n-1$ from
$v_i^{j+1} = v_i^j - q\mu(5v_i^j - 18v_{i+1}^j + 24v_{i+2}^j - 14v_{i+3}^j + 3v_{i+4}^j) - \beta s(\sum_{m=s}^{0}\sum_{n=0}^{s}(v_i^{j-1+m})(u_{i+1}^{j-1+n}) - u_{i-1}^{j-1+n}) - \alpha r(\sum_{m=s}^{0}\sum_{n=0}^{s}(u_i^{j-1+m})(v_{i+1}^{j-1+n}) - v_{i-1}^{j-1+n})$.
Step 6: when $i = 2, 3, \ldots, n-2$, compute the numerical solution v_i^{j+1} from $v_i^{j+1} = v_i^j - q\mu(v_{i+2}^j - 2v_{i+1}^j + 2v_{i-1}^j - v_{i-2}^j) - \beta s(\sum_{m=s}^{0}\sum_{n=0}^{s}(v_i^{j-1+m})(u_{i+1}^{j-1+n}) - u_{i-1}^{j-1+n}) - \beta r(\sum_{m=s}^{0}\sum_{n=0}^{s}(u_i^{j-1+m})(v_{i+1}^{j-1+n}) - v_{i-1}^{j-1+n})$.
Step 7: print the numerical solutions $u(x, t)$ and $v(x, t)$.

Algorithm 2: Algorithm of the EFD-AP method.

Table 1: Comparing the EFD and EFD-AP methods with the analytical solution for the DSW system for u.

x	u_{Exact}	u_{EFD}	ABSE	$u_{\text{EFD-AP}}$	ABSE
−20	0.000000000000000	0.000000000000000	0.000000$e+00$	0.000000000000000	0.000000$e+00$
−18	0.000000000000002	0.000000000000002	0.000000$e+00$	0.000000000000003	0.000000$e+00$
−16	0.000000000000130	0.000000000000112	1.700000$e-14$	0.000000000000152	1.000000$e-15$
−14	0.000000000007070	0.000000000006119	9.520000$e-13$	0.000000000008297	6.600000$e-14$
−12	0.000000000386035	0.000000000334082	5.195200$e-11$	0.000000000453014	3.607000$e-12$
−10	0.000000021076791	0.000000018240133	2.836658$e-09$	0.000000024733709	1.969470$e-10$
−8	0.000001150753581	0.000000995445776	1.553078$e-07$	0.000001350414470	1.075295$e-08$
−6	0.000062828370830	0.000053705269772	9.123101$e-06$	0.000073729242103	5.870768$e-07$
−4	0.003428388362795	0.002899003871216	5.293845$e-04$	0.004022830562664	3.201161$e-05$
−2	0.181578470291215	0.181311969465876	2.665008$e-04$	0.211951898242369	1.627998$e-03$
0	2.980881623890990	2.999776943549420	1.889532$e-02$	3.000000000312190	4.799980$e-05$
2	0.247183304832390	0.242768208620821	4.415096$e-03$	0.211953050552054	1.640126$e-03$
4	0.004720311275506	0.005191658272445	4.713470$e-04$	0.004022873546090	3.226855$e-05$
6	0.000086522352175	0.000096544463815	1.002211$e-05$	0.000073730044055	5.917936$e-07$
8	0.000001584734592	0.000001782921624	1.981870$e-07$	0.000001350429164	1.083933$e-08$
10	0.000000029025434	0.000000032665115	3.639681$e-09$	0.000000024733978	1.985290$e-10$
12	0.000000000531619	0.000000000598286	6.666600$e-11$	0.000000000453019	3.636000$e-12$
14	0.000000000009737	0.000000000010958	1.221000$e-12$	0.000000000008297	6.700000$e-14$
16	0.000000000000178	0.000000000000201	2.200000$e-14$	0.000000000000152	1.000000$e-15$
18	0.000000000000003	0.000000000000004	0.000000$e+00$	0.000000000000003	0.000000 + 00
20	0.000000000000000	0.000000000000000	0.000000$e+00$	0.000000000000000	0.000000$e+00$
MSE			**3.807658e − 04**		**4.533085844e − 05**

Bold values represent the overall mean for MSE.

Table 2: Comparing the EFD and EFD-AP methods with the analytical solution for the DSW system for v.

x	v_{Exact}	v_{EFD}	ABSE	$v_{\text{EFD-AP}}$	ABSE
−20	0.000000007610738	0.000000006675452	$9.35286e-10$	0.000000008162226	$0.000000e+00$
−18	0.000000056236173	0.000000049325291	$6.91088e-09$	0.000000060311146	$3.000000e-15$
−16	0.000000415532237	0.000000364467339	$5.10649e-08$	0.000000445642439	$1.520000e-13$
−14	0.000003070391012	0.000002693069248	$3.77322e-07$	0.000003292876984	$8.297000e-12$
−12	0.000022687291435	0.000019899092759	$2.78820e-06$	0.000024331252426	$4.530140e-10$
−10	0.000167637668860	0.000146977305345	$2.06604e-05$	0.000179784855851	$2.473371e-08$
−8	0.001238684022889	0.001074570778155	$1.64113e-04$	0.001328416707133	$1.350414e-06$
−6	0.009152658690634	0.007973830914374	$1.17883e-03$	0.009817959806057	$7.372924e-05$
−4	0.067610535301286	0.068261515182232	$6.50980e-04$	0.072999276292701	$4.022831e-03$
−2	0.492041285925229	0.524515277312650	$3.24740e-02$	0.532436818143197	$2.119519e-01$
0	1.993617022362784	1.999494403179591	$5.87738e-03$	1.999999762175650	$3.000000e+00$
2	0.574088616658195	0.539384921903072	$3.47037e-02$	0.530772248534432	$2.119531e-01$
4	0.079333147973226	0.078209018431132	$1.12413e-03$	0.073476928282824	$4.022874e-03$
6	0.010740723263978	0.011851152313279	$1.11043e-03$	0.010011775130338	$7.373004e-05$
8	0.001453609114015	0.001612040439639	$1.58431e-04$	0.001355290064193	$1.350429e-06$
10	0.000196724626664	0.000219748058095	$2.30234e-05$	0.000183423393176	$2.473398e-08$
12	0.000026623783132	0.000029747622897	$3.12384e-06$	0.000024823678933	$4.530190e-10$
14	0.000003603137231	0.000004025923061	$4.22786e-07$	0.000003359519675	$8.297000e-12$
16	0.000000487631598	0.000000544849488	$5.72179e-08$	0.000000454661547	$1.520000e-13$
18	0.000000065993760	0.000000073737360	$7.74360e-09$	0.000000061531749	$3.000000e-15$
20	0.000000008931284	0.000000009979266	$1.04798e-09$	0.000000008327417	$0.000000e+00$
MSE			**1.0495530e − 03**		**8.93920866e − 05**

Bold values represent the overall mean for MSE.

Defining the function $\gamma_{i,j+1} = u_i^{j+1} - \acute{\mathrm{U}}_i^{j+1}$ and then subtracting (25) from (32), we get

$$\gamma_i^{j+1} = \gamma_i^j - r\left(\sum_{m=s}^{0}\sum_{n=0}^{s}\left(\gamma_i^{'j-1+m}\right)\left(\gamma_{i+1}^{'j-1+n}\right) - \left(\gamma_{i-1}^{'j-1+n}\right)\right) + \mathrm{kT}_{i,j}. \tag{33}$$

The initial and boundary conditions are $\gamma_j^0 = 0, \gamma_0^j = 0$, and $\gamma_{n+1}^j = 0$.

Define the function $\epsilon_{j=}\max_{i=0,n+1}|\gamma_i^j|$, and $\epsilon_{j=}^{'}\max_{i=0,n+1}|\gamma_i^{'j}|$

$$\left|\gamma_i^{j+1}\right| \le \left|\gamma_i^j\right| + |r|.\left|\sum_{m=s}^{0}\sum_{n=0}^{s}\left(\gamma_i^{'j-1+m}\right)\left(\gamma_{i+1}^{'j-1+n}\right) - \left(\gamma_{i-1}^{j-1+n}\right)\right| + \mathrm{KT}, \tag{34}$$

where $T = \max_{i,j}.\ |T_{i,j}|$, then

$$\left|\epsilon_{j+1}\right| \le \left|\epsilon_j\right| + |r|.\left|\sum_{m=s}^{0}\sum_{n=0}^{s}\left(\epsilon_{j-1+m}^{'}\right)\left(\left(\epsilon_{j-1+n}^{'}\right) - \left(\epsilon_{j-1+n}^{'}\right)\right)\right| + \mathrm{KT},$$
$$\left|\epsilon_{j+1}\right| \le \left|\epsilon_j\right| + \mathrm{KT}. \tag{35}$$

This is trivially true for the cases $i = 0$ and $i = n + 1$.

$$\epsilon_{j+1} \le \epsilon_j + \mathrm{KT}, \tag{36}$$

and from the initial condition $\epsilon_0 = 0$ and

$$\begin{aligned}\epsilon_1 &\le \mathrm{KT},\\ \epsilon_2 &\le 2\mathrm{KT},\end{aligned} \tag{37}$$

by induction, we can show that $\epsilon_j \le j\mathrm{KT}$ for all $j \ge 0$.

Since $t_j = t_{0+}jK$, then $t_j = jK$ and

$$\left|u_i^j - \acute{\mathrm{U}}_i^j\right| \le \epsilon_j \le j\mathrm{KT} = t_jT. \tag{38}$$

We have convergence if keeping x_i, t_j fixed and letting $i \longrightarrow \infty, j \longrightarrow \infty, h \longrightarrow 0, k \longrightarrow 0$; we note that since $T = O(k) + O(h^2)$, then $T \longrightarrow 0$ as, $h \longrightarrow 0, k \longrightarrow 0$ and therefore $|u_i^j - \acute{\mathrm{U}}_i^j| \longrightarrow 0$, which gives the convergence.

By the same approach, we have the approximate solution from (30); similarly, if we take the approximate solution from (29) and (31), we obtain

$$\begin{aligned}\tilde{\mathrm{V}}_i^{j+1} &= \tilde{\mathrm{V}}_i^j - R\mu\left(\tilde{\mathrm{V}}_{i+2}^j - 2\tilde{\mathrm{V}}_{i+1}^j + 2\tilde{\mathrm{V}}_{i-1}^j - \tilde{\mathrm{V}}_{i-2}^j\right) - \beta s\left(\sum_{m=s}^{0}\sum_{n=0}^{s}\tilde{\mathrm{V}}_i^{j-1+m}\acute{\mathrm{U}}_{i+1}^{j-1+n} - \left(\acute{\mathrm{U}}_{i-1}^{j-1+n}\right)\right)\\ &\quad + \alpha r\left(\sum_{m=s}^{0}\sum_{n=0}^{s}\acute{\mathrm{U}}_i^{j-1+m}\left(\tilde{\mathrm{V}}_{i+1}^{j-1+n} - \tilde{\mathrm{V}}_{i-1}^{j-1+n}\right)\right) + \mathrm{kT}_{i,j}.\end{aligned} \tag{39}$$

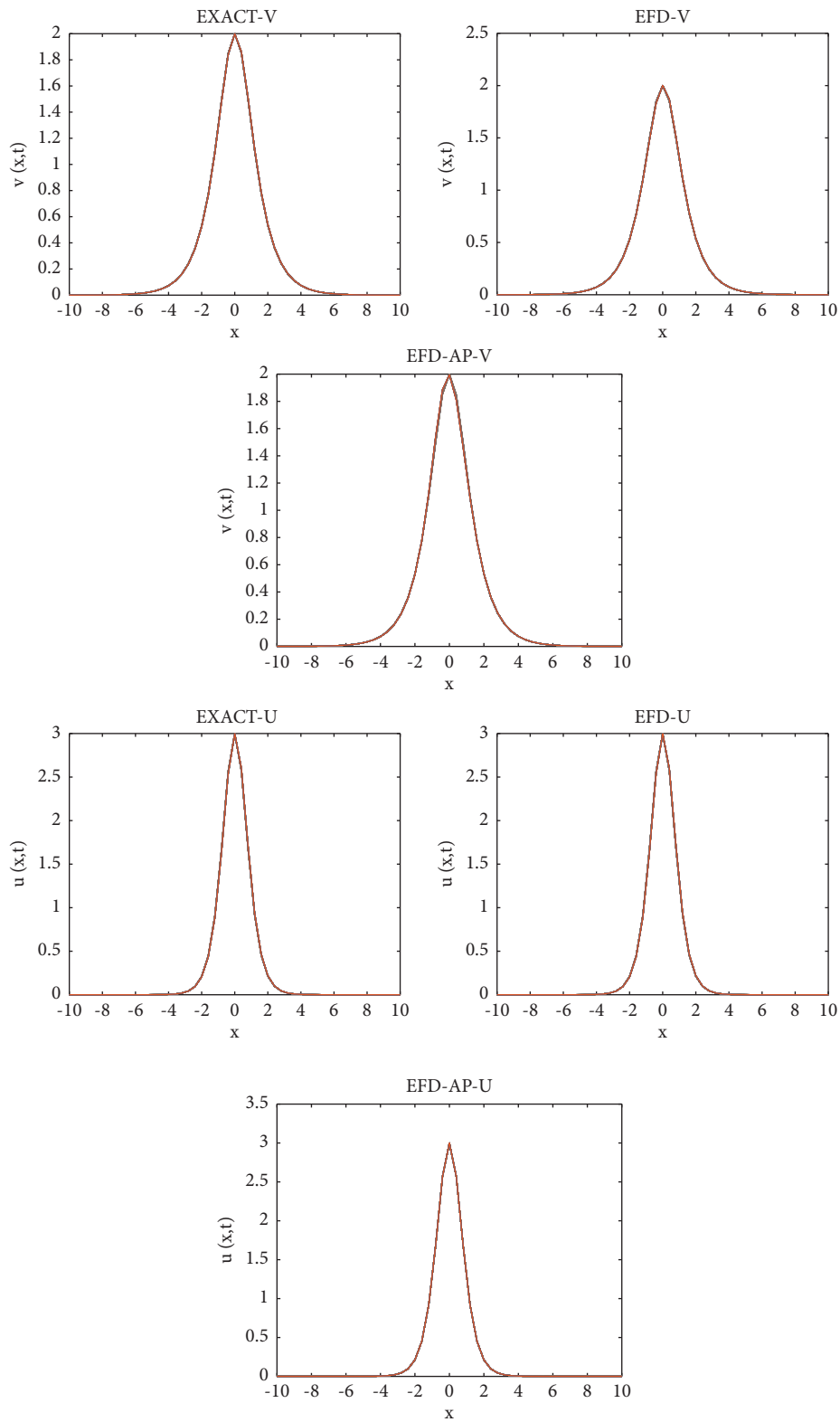

Figure 1: 2D solutions of the exact, EFD, and EFD-AP methods for each of u and v at $a = 0$ and $c = 2$.

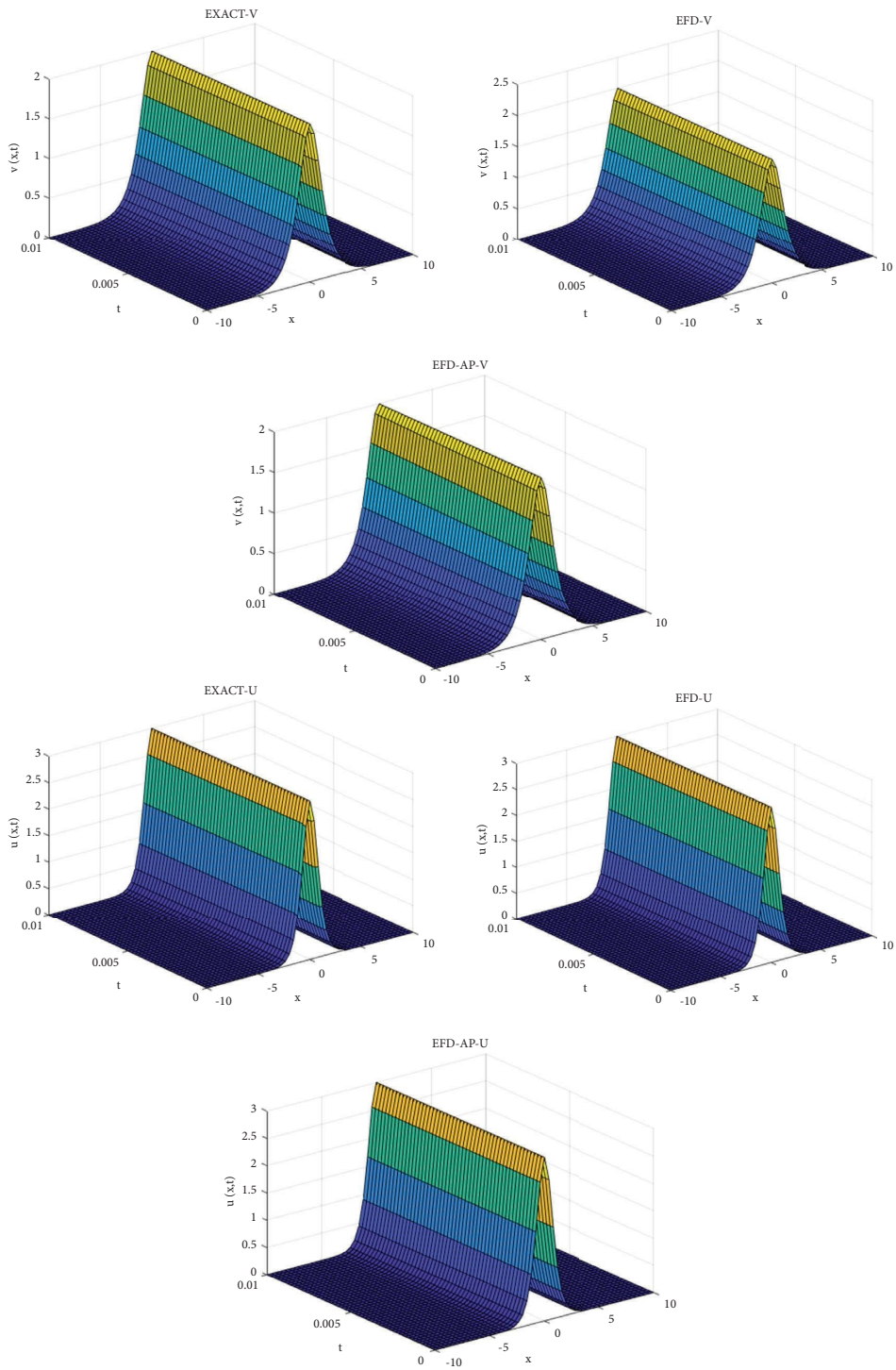

Figure 2: 3D solutions of the exact, EFD, and EFD-AP methods for each of u and v at $a = 0$ and $c = 2$.

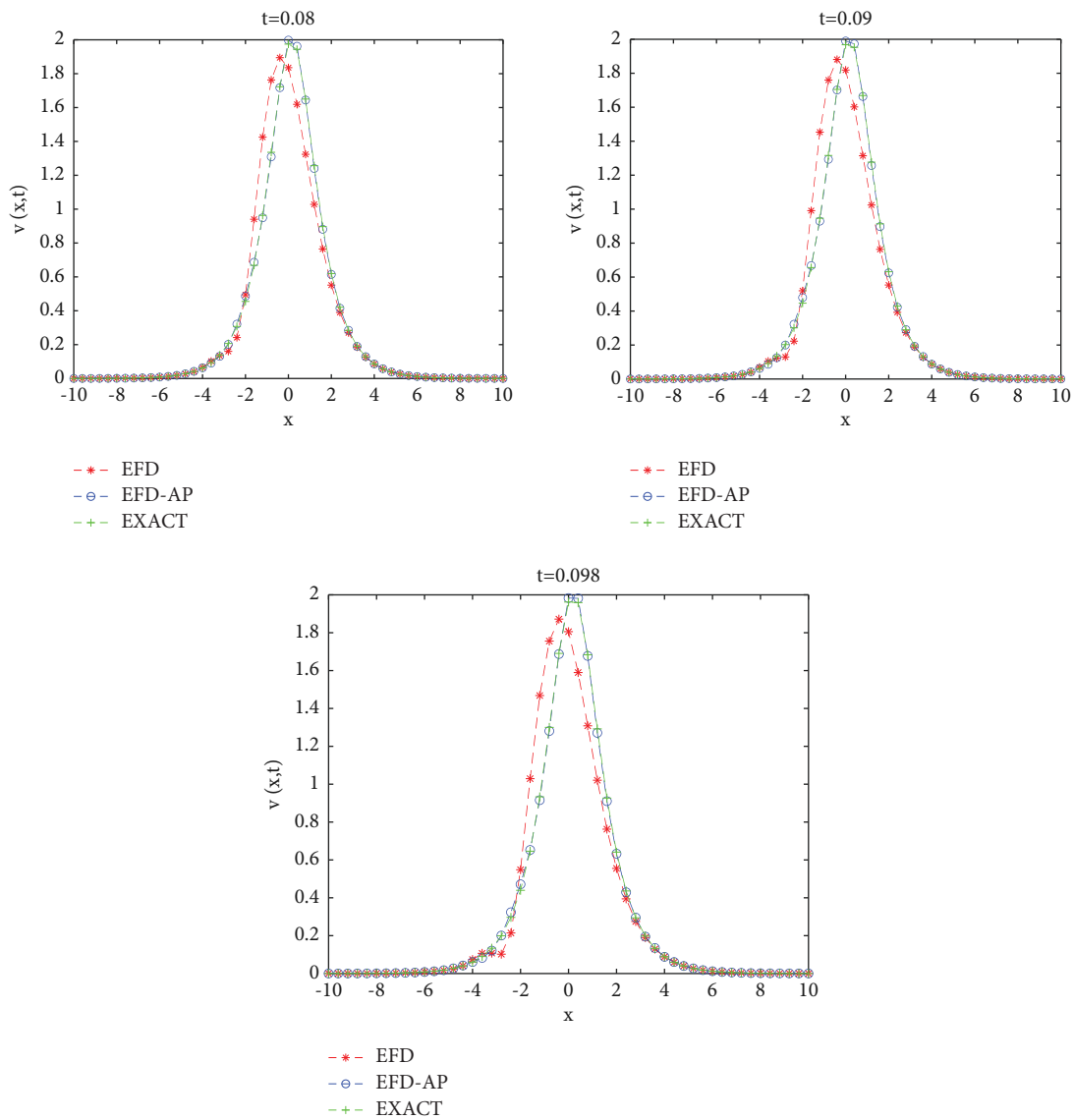

Figure 3: Comparison EFD, EFD-AP, and exact solutions for u with different times.

By defining the function $\gamma_{i,j+1} = v_i^{j+1} - \tilde{V}_i^{j+1}$, we get

$$\begin{aligned}\gamma_i^{j+1} = \gamma_i^j - R\mu\left(\gamma_{i+2}^j - 2\gamma_{i+1}^j + 2\gamma_{i-1}^j - \gamma_{i-2}^j\right) - \beta s\left(\sum_{m=s}^{0}\sum_{n=0}^{s}\gamma_i^{j-1+m}\left(\gamma'^{\,j-1+n}_{i+1} - \gamma'^{\,j-1+n}_{i-1}\right)\right)\\ + \alpha r\left(\sum_{m=s}^{0}\sum_{n=0}^{s}\gamma'^{\,j-1+m}_{i}\left(\gamma_{i+1}^{j-1+n} - \gamma_{i-1}^{j-1+n}\right)\right) + \mathrm{kT}_{i,j}.\end{aligned} \tag{40}$$

In these three cases, we have $|v_i^j - \tilde{V}_i^j| \longrightarrow 0$, which gives the convergence.

It is worth mentioning that the proposed method is stable and its stability condition is $0 \le R \le 1/2$ (Algorithm 2).

4. Numerical Application

The analytic solution of the DSW system given by [2, 29] is as follows:

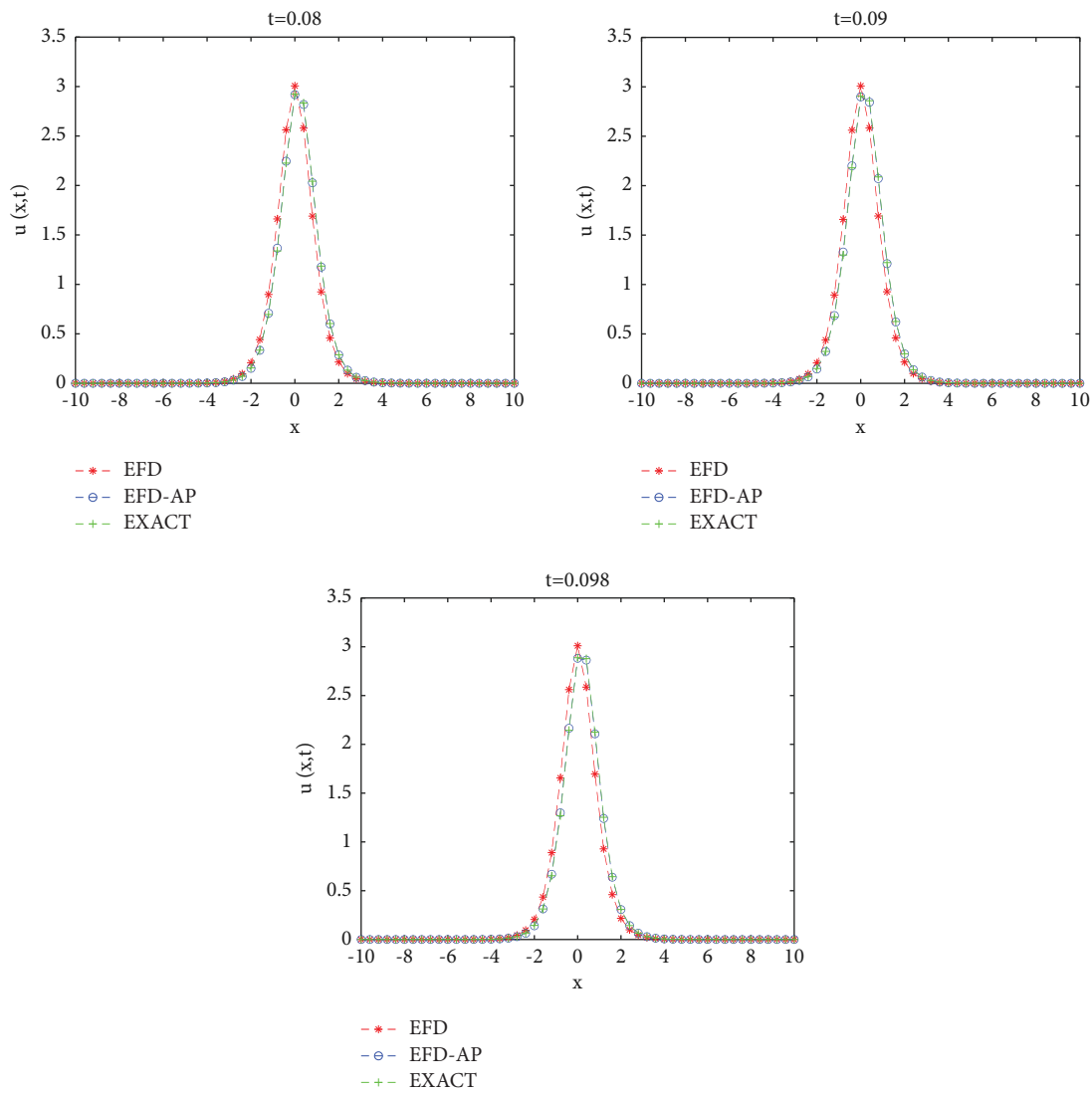

Figure 4: Comparison EFD, EFD-AP, and exact solutions for v with different times.

$$u(x,t) = \frac{3c}{2}\sec h^2\left(\sqrt{\frac{c}{2}}\,(x - ct - a)\right), \tag{41}$$

$$v(x,t) = c\sec h\left(\sqrt{\left(\frac{c}{2}\right)}\,(x - ct - a)\right). \tag{42}$$

To assess the accuracy and convergence of the numerical solution in comparison to the exact solution (as outlined in (41) and (42)), we will employ two error criteria, namely, the absolute error (ABSE) and the mean square error (MSE). These error metrics enable us to quantitatively evaluate how closely the numerical solution aligns with the exact solution. The application of these error measures yields valuable insights into the effectiveness and reliability of the numerical method being used.

$$\begin{aligned} \text{ABSE} &= \left|u_i(x,t) - u(x,t)\right|, \\ \text{MSE} &= \frac{\sqrt{\sum_{i=0}^{n}\left(u_i(x,t) - u(x,t)\right)^2}}{n}. \end{aligned} \tag{43}$$

When equations (1) and (2) are solved by equations (3), (4), (6), (7), (9), (10), and (12), the results are shown in Tables 1 and 2, all computations we fixed $a = 0$, $c = 2$, $t = 0.01$, $H = 0.4$, $k = 0.002$, and $s = 5$ are the number of iterations by using MATLAB R2022a.

Comparing Tables 1 and 2, it becomes evident that the MSE values obtained through the proposed technique surpass those achieved by the explicit approach. This implies that better results can be achieved using the proposed method. In addition, Figures 1 and 2 depict the exact

solution alongside the numerical solutions generated by the EFD and EFD-AP methods for both u and v. These visual representations correspond to specific parameter values, namely, $a=0$, $c=2$, $t=0.01$, $H=0.4$, $k=0.002$, and $s=5$, where "s" denotes the number of iterations in 2D and 3D, respectively.

These results underscore the efficiency of the EFD-AP method in adeptly managing the nonlinearity within the equation, leading to accurate and efficient solutions. In addition, the comparison of outcomes derived from the EFD-AP and EFD methods against the exact solution further reinforces this point.

5. Conclusion

In this study, we introduce a novel approach called EFD-AP, which integrates the finite difference method with the Adomian polynomial to address the nonlinearity within the DSW system. This innovative method offers a fresh perspective on handling the nonlinear aspects of the equations, resulting in improved accuracy and efficiency of the solutions. By combining the strengths of these two methods, our goal is to overcome the limitations of conventional numerical techniques, achieve more precise approximations for the DSW system, and address situations where points fall outside the solution region. The finite difference method provides a robust framework for discretizing the equations in space, while the Adomian polynomial offers a systematic and efficient method for solving the resulting nonlinear subproblems. We conducted a convergence analysis of the proposed method and found that EFD-AP exhibits greater stability compared to the EFD method, as shown in Figure 3, as evidenced by the results presented in Tables 1 and 2 and in Figures 1–4. The EFD-AP approach offers enhanced computational efficiency, higher precision, reduced mean square error, improved stability with larger time step sizes, and faster convergence. This approach also proves to be highly efficient and suitable for approximating solutions. In the future, this method can be extended to tackle nonlinear fractional systems and nonlinear fuzzy systems. In addition, there is a potential to develop new techniques by combining the homotopy method with the finite difference method.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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