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Research Article

Primal Topologies on Finite-Dimensional Vector Spaces Induced by Matrices

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Given an matrix A, considered as a linear map $A\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$, then A induces a topological space structure on $\mathbb{R}^n$ which differs quite a lot from the usual one (induced by the Euclidean metric). This new topological structure on $\mathbb{R}^n$ has very interesting properties with a nice special geometric flavor, and it is a particular case of the so called "primal space," In particular, some algebraic information can be shown in a topological fashion and the other way around. If X is a non-empty set and $f\colon X \longrightarrow X$ is a map, there exists a topology τ_f induced on X by f, defined by $\tau_f = \{U \subset X\colon f^{-1}(U) \subset U\}$. The pair (X, τ_f) is called the primal space induced by f. In this paper, we investigate some characteristics of primal space structure induced on the vector space $\mathbb{R}^n$ by matrices; in particular, we describe geometrical properties of the respective spaces for the case.

1. Introduction

In this paper we present some elementary properties of a special topological space structure induced by matrices on the vector space $\mathbb{R}^n$. Such a structure is a particular case of a more general construction: If X is a non-empty set and $f\colon X \longrightarrow X$ is a map, the collection $\tau_f = \{U \subset X\colon f^{-1}(U) \subset U\}$ is a topology for X, which is named the primal topology induced by f, see Echi [1] and Echi and Turki [2]. The topological space (X, τ_f) is an example of the so called Alexandroff space, that is, a space where intersections of arbitrary collections of open sets is an open set. These spaces have, in general terms, a complicated structure, because in most of the cases they do not satisfy the axiom T_1. For general properties of functional Alexandroff spaces see Kukiela [3] and, Ayatollah Zadeh Shirazi and Golestani [4].

Examples of primal topologies with a description of their main properties are presented in Section 2 of this paper. Some of such spaces are part of active investigations. In particular, Example 5 shows how Vielma et al. [5] used a primal to topology on the set of the positive integers to obtain a topological version of the Collatz conjecture.

In our investigation we work with the primal topology generated by taking $X = \mathbb{R}^n$ and f a matrix (identified with a linear map on $\mathbb{R}^n$). Thus, we obtain the matrix primal space $(\mathbb{R}^n, \tau_A)$ and explore how its properties are derived from those of A, and the other way around.

In Section 3 we show some basic results which require the knowledge of elementary concepts from linear algebra and topology. In most of the cases those propositions have a nice geometric flavor, even though matrix primal spaces are quite different from Euclidean spaces. In Section 4 we consider the problem for the case of the vector space $\mathbb{R}^2$ and we conclude investigating properties of continuous maps on matrix primal spaces. A few of the propositions presented in this paper are suitable for generalizations which deserve to be investigated as, for example, in the context of vector spaces of infinite dimensions.

2. Primal Spaces

For an arbitrary topological space it is not true that arbitrary intersections of open sets is an open set, so those with this special property have been singled out. A topological space (X, τ) is said to be an Alexandroff space if any arbitrary intersection of open sets is an open set. General properties of Alexandroff spaces are presented in Kukiela [3]. There are some special examples of such structures, in particular the so, called "functional Alexandroff spaces," which are determined in the following way: let X be a non-empty set and $f\colon X \longrightarrow X$ a map, we consider the topology τ_f on X such that the closed sets are those $E \subset X$ such that $f(E) \subset E$. Some of the most important properties of functional Alexandroff spaces are found in Ayatollah Zadeh Shirazi and Golestani [4]. Another approach to these spaces by focusing on the open sets, was introduced by Echi [1] based on the fact that for any set $X \neq \varnothing$ and any map $f\colon X \longrightarrow X$, the collection $\tau_f = \{U \subset X\colon f^{-1}(U) \subset U\}$ is a topology for X.

The topology τ_f is called the primal topology induced by f and (X, τ_f) is said to be a primal space. Sometimes we use the notation X_f as an abbreviation for (X, τ_f), which is very convenient when we deal with more than one primal topology on the same set. Obviously the map $f\colon X_f \longrightarrow X_f$ is continuous.

Next, we describe the most elementary open and closed sets of primal spaces. Let us denote by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the set of non-negative integers and by f^r the r-fold composite $f \circ \cdots \circ f$, for $r > 0$ and f^0 the identity map. Given $p \in X$, the orbit of p, is defined by

$$\Omega(p) = \{y \in X\colon\ y = f^r(p)\ \text{for}\ r \in \mathbb{N}_0\}. \tag{1}$$

Thus $\Omega(p)$ is the smallest closed set containing p, that is $cl(p)$, the closure of $\{p\}$.

If there exists a number $r > 0$ such that $f^r(p) = p$ we say that $\Omega(p)$ is a periodic orbit and p itself is referred to as a periodic point. If $\Omega(p)$ contains just one element, then p is said to be a fixed point.

On the other hand, we define

$$\Gamma(p) = \{x \in X\colon\ p = f^r(x)\ \text{for}\ r \in \mathbb{N}_0\}, \tag{2}$$

which is the smallest open set containing p. We use $\Omega_f(p)$ and $\Gamma_f(p)$ whenever we want to emphasize that these are subsets of the space X_f.

Some general properties of primal spaces have been developed over the last few years, for example, Echi and Turki [2] characterize compactness with the following result.

Theorem 1. *Let (X, τ_f) a primal space. Then the following statements are equivalent:*

(1) The space (X, τ_f) is compact,

(2) The collection of periodic points is finite and for each $x \in X$, $f^r(x)$ is periodic, for some r.

In particular, this theorem implies that compact primal spaces have finitely many connected components each of them containing a periodic point.

In general, primal spaces are somehow elusive, especially in relation with separation axioms. As a matter of fact As a matter of, most of the cases the space is not even T_1 (Example 1). In order to illustrate this issue we introduce a few examples, the first two being the extreme cases.

Example 1. If $X \neq \varnothing$ and q is an element of X, consider the map $f\colon X \longrightarrow X$, such that $f(x) = q$ for all $x \in X$ (constant). Then for all $x \neq q$ we have that $\{x\} \in \tau_f$. Furthermore, the space (X, τ_f) satisfies the axiom T_0 but it is not a Hausdorff space.

Example 2. For any $X \neq \varnothing$ the primal topology induced by the identity map on X is the discrete topology. In this case, each $p \in X$ is a fixed point. This is the only case in which a primal space is T_1, because any set $\{x\}$ is closed if and only if $\Omega(x) = \{x\}$, and this is true if and only if the map that induces the topology is the identity.

Example 3. Let $s\colon \mathbb{N} \longrightarrow \mathbb{N}$ be the map defined by $s(x) = x + 1$. This space is suitable for a nice graphic representation:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow p \longrightarrow p+1 \longrightarrow \cdots, \tag{3}$$

where $p \longrightarrow p+1$ means $s(p) = p+1$.

Note that for all $p \in \mathbb{N}$ we have $\Gamma(p) \subset \Gamma(p+1)$ and the space $(\mathbb{N}, \tau_s)$ is connected, but it is not compact.

Example 4. Let $\mathbb{Z}_n$ denote the integers modulo n with its usual multiplication. For any $k \in \mathbb{Z}_n$, define $m_k\colon \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ by $m_k(x) = kx$. Then the space $(\mathbb{Z}_n, \tau_{m_k})$ is compact because $\mathbb{Z}_n$ is finite.

This space is suitable for a nice graphic representation with specific values of n and k. For example, for the space $(\mathbb{Z}_9, \tau_{m_3})$, where $\Gamma(0) = \mathbb{Z}_9$:

$$\begin{array}{ccccccccc} & & 1 & & & & 2 & & \\ & & \downarrow & & & & \downarrow & & \\ 4 & \longrightarrow & 3 & \longrightarrow & 0 & \longleftarrow & 6 & \longleftarrow & 5. \\ & & \uparrow & & & & \uparrow & & \\ & & 7 & & & & 8 & & \end{array} \tag{4}$$

We have a very different situation with the space $(\mathbb{Z}_9, \tau_{m_2})$, which has three connected components, namely $\Gamma(0)$, $\Gamma(1)$, and $\Gamma(3)$:

$$\begin{array}{cccccccc} & 1 & \longrightarrow & 2 & \longrightarrow & 4 & & \\ 0 & \uparrow & & & & \downarrow & & 3 \leftrightarrows 6. \\ & 5 & \longleftarrow & 7 & \longleftarrow & 8 & & \end{array} \tag{5}$$

It is easy to verify that $(\mathbb{Z}_9, \tau_{m_2})$ and $(\mathbb{Z}_9, \tau_{m_5})$ are homeomorphic.

The next example is associated to a very famous problem known as "the Collatz conjecture" which was set in the first half of the 20th century and still remains unsolved.

Example 5. Let $C\colon \mathbb{N} \longrightarrow \mathbb{N}$ the map defined by

$$C(x) = \begin{cases} \dfrac{x}{2}, & \text{if}\, x\ \text{is even}, \\ 3x+1, & \text{if}\, x\ \text{is odd}. \end{cases} \tag{6}$$

The Collatz conjecture states that for all $p \in \mathbb{N}$ there exists an $r \in \mathbb{N}$ such that $C^r(p) = 1$. This problem may be stated in terms of primal spaces as follows: For all $p \in \mathbb{N}$ we have that $1 \in \Omega(p)$. The next diagram shows part of $\Gamma(1)$ which is a connected component of $(\mathbb{N}, \tau_C)$:

$$\begin{tikzcd}
\cdots \arrow[r] & 6 \arrow[r] & 3 \arrow[d] & & & & \\
\cdots \arrow[r] & 20 \arrow[r] & 10 \arrow[r] & 5 \arrow[d] & & & \\
\cdots \arrow[r] & 64 \arrow[r] & 32 \arrow[r] & 16 \arrow[r] & 8 \arrow[r] & 4 \arrow[r] & 2 \arrow[d] \\
& & & & & & 1 \arrow[ul]
\end{tikzcd}. \tag{7}$$

Separation properties of the space $(\mathbb{N}, \tau_C)$ were determined by Vielma and Guale [6] by using some ideas introduced of Di Maio [7]. Furthermore, it is not known if the space $(\mathbb{N}, \tau_C)$ is connected; neither is known if it is compact. These facts set a very important problem because Vielma and Guale [6] used the result of Echi and Turki cited above (Theorem 1), to prove that the Collatz conjecture is equivalent to connectedness of the primal space $(\mathbb{N}, \tau_C)$. Some other properties of this space related to the conjecture are shown in [5].

3. Matrix Primal Spaces

In this section we explore the construction of primal topologies on the vector space $\mathbb{R}^n$ induced by linear maps. First, we establish some conventions of notation. Let $\mathscr{B}_n = \{e_1, e_2, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$, that is, $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 appears at the i-th position. If $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear map, we denote by μT the matrix associated to T with respect to the standard basis in $\mathbb{R}^n$. As usual, we identify T with $A = \mu T$, in particular if no confusion arises we use both τ_T and τ_A to denote the primal topology induced by T and the respective primal space $(\mathbb{R}^n, \tau_T) = (\mathbb{R}^n, \tau_A)$ is denoted by $\mathbb{R}^2_A$. Obviously, we have that 0 is always a fixed point. The next result provides information about $\Gamma(0)$ from invertibility of T.

Theorem 2. *If $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a non-invertible linear transformation, then $\Gamma(0)$ is a nontrivial invariant linear subspace of $\mathbb{R}^n_T$.*

Proof. If $x, y \in \Gamma(0)$, there exist $r, s \in \mathbb{N}$ such that $T^r(x) = T^s(y) = 0$. Then

$$T^{r+s}(x + y) = T^{r+s}(x) + T^{r+s}(y) = 0, \tag{8}$$

and $x + y \in \Gamma(0)$. On the other hand, if $a \in \mathbb{R}$ and $x \in \Gamma(0)$, there exists $k \in \mathbb{N}$ such that $T^k(x) = 0$. Thus

$$T^k(ax) = aT^k(x) = 0, \tag{9}$$

therefore $ax \in \Gamma(0)$.

If $x \in \Gamma(0)$ and $T^k(x) = 0$ for a number $k \in \mathbb{N}$, then $T^{k-1}(T(x)) = 0$. Thus $T(x) \in \Gamma(0)$, so $\Gamma(0)$ is closed and, by definition, this implies that $T(\Gamma(0)) \subset \Gamma(0)$. In other words words $\Gamma(0)$ is invariant under T.

Now, in the following two propositions we prove that invertibility of T is equivalent to some topological information about both kernel and image of T.

Theorem 3. *Let $T\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear map. Then the following propositions hold:*

(1) If T is invertible then $KerT$ is open in $\mathbb{R}^n_T$.

(2) If $KerT$ is open in $\mathbb{R}^n_T$ and $KerT \subset ImT$, then T is invertible.

Proof. Proposition (1) is obvious because $\mathrm{Ker}T = \{0\} = \Gamma(0) \in \tau_T$.

To prove the second suppose that there exists $x \in \mathbb{R}^n$, $x \neq 0$ such that $x \in \mathrm{Ker}T$. By assumption, $\mathrm{Ker}T \subset \mathrm{Im}T$, so there exists y such that $T(y) = x$. Then $y \in \Gamma(x) \subset \Gamma(0) = \mathrm{KerT}$, therefore $T(y) = 0$, which is a contradiction. □

Note that the hypothesis of openness of $\mathrm{Ker}T$ in the second part of Theorem [3] is not a sufficient condition. Let us consider, for example, $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $T(x_1, x_2) = (x_1, 0)$. Then $KerT = \{(0, x_2)\colon x_2 \in \mathbb{R}\} = \Gamma(0, 0)$ (is open), but $KerT \cap \mathrm{Im}T = \{(0, 0)\}$.

Theorem 4. *An $n \times n$ matrix A is invertible if and only if ImA is open in $\mathbb{R}^2_A$.*

Proof. Proof If A is invertible, then $\mathrm{Im}A = \mathbb{R}^n \in \tau_A$.

Now suppose that $\mathrm{Im}A \in \tau_A$. Let c_j be any column vector of A. Then $\Gamma(c_j) \subset \mathrm{Im}A$. But $Ae_j = c_j$, so $e_j \in \Gamma(c_j)$. Thus $\mathbb{R}^n \subset \mathrm{Im}A \subset \mathbb{R}^n$, that is $\mathrm{Im}A = \mathbb{R}^n$, and therefore A is invertible. □

4. Matrix Primal Topologies for the Plane

In this section we introduce a pretty long list of examples in order to illustrate in detail some specific facts about the primal space $\mathbb{R}^2_A$, for various matrices. Sometimes we make comparative references to the Euclidean topology ϵ for $\mathbb{R}^2$ (the one induced by the usual metric) and denote by $\mathbb{R}^2_\varepsilon$ the space $(\mathbb{R}^2, \epsilon)$. Let us begin with two non-invertible matrices.

Example 6. Let $\wp_1$ be the orthogonal projection of $\mathbb{R}^2$ on the x_1-axis:

$$\wp_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{10}$$

Note that the collection of singlegons (x_1, x_2) with $x_2 \neq 0$ and the vertical lines $\Gamma(k, 0)$, $k \in \mathbb{R}$ form a basis $\mathscr{B}$ for $\tau_{\wp_1}$. On the other hand, the collection $\mathscr{B}' = \{(0, x_2), (x_1, x_2) \colon x_1, x_2 \in \mathbb{R}\}$ is a basis for $\tau'_{\wp_1}$, the complement topology of $\tau_{\wp_1}$, that is the one whose elements are the $\tau_{\wp_1}$-closed sets.

Something similar happens with the orthogonal projection $\wp_2$ on the x_2-axis. Even in the much more general case, of the orthogonal projection $\wp_k$ on the x_k-axis in $\mathbb{R}^n$, $1 \leq k \leq n$, given by the matrix with entries.

$$a_{ij} = \begin{cases} 1, & \text{if } i = j = k, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Now, let us consider a very simple invertible matrix.

Example 7. Let ρ_i be the reflection through the x_i-axis:

$$\rho_i = \begin{pmatrix} (-1)^{i+1} & 0 \\ 0 & (-1)^i \end{pmatrix}. \tag{12}$$

Elements of the primal topology τ_{ρ_i} are subsets of $\mathbb{R}^2$ which are symmetric with respect to the x_i-axis, because $\rho_i = \rho_i^{-1}$, and therefore, all the open sets are also closed.

The next two examples illustrate the orbits for some general 2×2 diagonal matrices and describe them as sets in the Euclidean space $\mathbb{R}^2_\varepsilon$.

Example 8. For a positive number λ, let δ_λ be the matrix:

$$\delta_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}. \tag{13}$$

We have three different cases depending on λ.

Case 1. If $\lambda = 1$, then δ_1 is the identity and every $x \in \mathbb{R}^2$ is a fixed point.

Case 2. If $\lambda > 1$, there are not periodic orbits other than $\Omega(0)$. For each $x \in \mathbb{R}^2$, $x \neq 0$ we have that the sequence $\{\delta^n_\lambda(x)\}$ is divergent in the space $\mathbb{R}^2_\varepsilon$.

Case 3. If $\lambda < 1$, there are not periodic orbits other than $\Omega(0)$. For each $x \in \mathbb{R}^2$, $x \neq 0$ we have that the sequence $\{\delta^n_\lambda(x)\}$ converges to 0 in the space $\mathbb{R}^2_\varepsilon$.

Example 9. Given two positive numbers λ_1 and λ_2, consider the matrix:

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{14}$$

The description of the space $\mathbb{R}^2_A$ requires to consider a few cases.

Case 1. If $\lambda_1 = 1$ and $\lambda_2 < 1$, the only periodic orbits $\Omega(x)$ are those of points x with $x_2 = 0$ (in fact, fixed points). If $x_2 \neq 0$ then the sequence $\{A^n x\}$ converges to $(x_1, 0)$, in the sense of the standard space $\mathbb{R}^2_\varepsilon$.

Case 2. If $\lambda_1 = 1$ and $\lambda_2 > 1$, we also have that the only periodic orbits $\Omega(x)$ are those of points x with $x_2 = 0$. If $x_2 \neq 0$ then the sequence $\{A^n x\}$ diverges in the space $\mathbb{R}^2_\varepsilon$.

Case 3. If $\lambda_1, \lambda_2 > 1$, the only periodic orbit is $\Omega(0)$. In any other case, the sequence $\{A^n x\}$ diverges in $\mathbb{R}^2_\varepsilon$.

Case 4. If $\lambda_1, \lambda_2 < 1$, we also have that the only periodic orbit is $\Omega(0)$, but the sequence $\{A^n x\}$ converges to 0 in the space $\mathbb{R}^2_\varepsilon$.

Case 5. If $\lambda_1 > 1$ and $\lambda_2 < 1$, the only periodic orbit is $\Omega(0)$. In the space $\mathbb{R}^n_\varepsilon$, for $x_1 = 0$ the sequence $\{A^n x\}$ converges to 0, and it diverges if $x_1 \neq 0$.

Of course, we may consider a diagonal matrix as the one in Example 9 with no necessarily positive eigenvalues.

Example 10. Given the matrix

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{15}$$

It turns out that some families of sets which are either open or closed are symmetric with respect to the origin or the axes, particularly the circle S_r and the square K_r, $r > 0$:

$$S_r = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 + x_2^2 \leq r^2\}. \tag{16}$$

and

$$K_r = \{(x_1, x_2) \in \mathbb{R}^2 \colon |x_1| + |x_2| \leq r\}. \tag{17}$$

Case 1. If $|\lambda_1|, |\lambda_2| \geq 1$, we have that $\|Ax\| \geq \|x\|$, for all $x \in \mathbb{R}^2$. then S_r is open in $\mathbb{R}^2_A$ for any $r > 0$. On the other hand, if $x = (x_1, x_2) \in A^{-1}(K_r)$ there is a point $y = (y_1, y_2) \in K_r$ such that $A(x) = y$, that is

$$(y_1, y_2) = A(x_1, x_2) = (\lambda_1 x_1, \lambda_2 x_2). \tag{18}$$

Then

$$|x_1| + |x_2| \leq |\lambda_1 x_1| + |\lambda_2 x_2| = |y_1| + |y_2| \leq r, \tag{19}$$

and $A^{-1}(K) \subset K_r$, so K_r is open in $\mathbb{R}^2_A$.

Case 2. If $|\lambda_1|, |\lambda_2| \leq 1$, A is a contraction, $\|Ax\| \leq \|x\|$, for all $x \in \mathbb{R}^2$, and therefore S_r is closed in $\mathbb{R}^2_A$ for any $r > 0$.

In this case we have that K_r is also closed in $\mathbb{R}^2_A$ In fact, if $x = (x_1, x_2) \in K_r$, we have that $A(x_1, x_2) = (\lambda_1 x_1, \lambda_2 x_2) \in K$ and

$$|\lambda_1 x_1| + |\lambda_2 x_2| \leq |x_1| + |x_2| \leq r, \tag{20}$$

so $A(K) \subset K_r$. Similarly, we may proof that for all $r_1, r_2 \in \mathbb{R}$, the diamond shaped set

$$K_{r_1,r_2} = \{(x_1, x_2) \in \mathbb{R}^2\colon |r_2 x_1| + |r_1 x_2| \leq |r_1, r_2|\}, \tag{21}$$

is closed.

Let us note that all these cases may be deduced also from the fact that for any $a, b > 0$ the rectangle $R = [-a, a] \times [-b, b]$ is closed. for if $(x_1, x_2) \in R$ we have we have

$$|x_1 \lambda_1| \leq |x_1| \leq a \quad \text{and} \quad |x_2 \lambda_2| \leq |x_2| \leq b, \tag{22}$$

Thus $A(R) \subset R$.

Example 11. Let a be a positive number and let A be the upper triangular matrix

$$A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}. \tag{23}$$

For any point $x = (x_1, x_2)$ we have

$$\Gamma(x) = \{(x_1 - nax_2, x_2)\colon n \in \mathbb{Z}, n \geq 0\}. \tag{24}$$

Then the only periodic orbits in $\mathbb{R}^2_A$ are those of points on the x_1-axis.

Similarly, if b is a positive number and B is the matrix

$$B = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}. \tag{25}$$

then for any $x = (x_1, x_2)$ we have $\Gamma(x) = \{(x_1, -nbx_1 + x_2)\colon n \in \mathbb{Z}, n \geq 0\}$, and therefore, the only periodic orbits in $\mathbb{R}^2_B$ are those of points on the x_2-axis.

Example 12. For $\theta \in \mathbb{R}$, we consider the rotation A_θ given by

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{26}$$

Recall that the collection of such matrices is an abelian subgroup of the general linear group GL_2, because for any α and β we have

$$A_\alpha \cdot A_\beta = A_{\alpha+\beta}. \tag{27}$$

We denote by either $(\mathbb{R}^2, \tau_\theta)$ or $\mathbb{R}^2_\theta$ the space $\mathbb{R}^2$ with the primal topology τ_θ, and we call it a primal rotation space. If $\theta = r\pi$, with $r \in \mathbb{Q}$, $r \neq 0$, for each $x \in \mathbb{R}^2$ the orbit $\Omega(x)$ is finite, and it is both open and closed for $\Omega(x) = \Gamma(x)$.

Some more details about the space $\mathbb{R}^2_\theta$ may be given if we have specific information about r. Let us consider, for instance, that $p \in \mathbb{Z}, q \in \mathbb{N}$, $p \neq 0$, and $\theta = 2p\pi/q$. then $U \in \tau_\theta$ if and only if $A_\theta(U) = U$. Moreover for all $n \in \mathbb{N}$ we have

$$\Omega(x) \text{ in the space } \mathbb{R}^2_\theta = \Omega(x) \text{ in the space } \mathbb{R}^2_{2np\pi/q}, \tag{28}$$

and

$$\Gamma(x) \text{in the space } \mathbb{R}^2_\theta = \Gamma(x) \text{ in the space } \mathbb{R}^2_{2np\pi/q}, \tag{29}$$

Therefore the spaces $\mathbb{R}^2_\theta$ and $\mathbb{R}^2_{2pn\pi/q}$ are homeomorphic. Let us note that for all x

$$\Omega(x) = \Gamma(x) = \{x, Ax, A^2x, \ldots, A^{q-1}x\}, \tag{30}$$

and therefore for all $x \in \mathbb{R}^2_\theta$, $x \neq 0$, $\Omega(x)$ is the set of vertices of a regular polygon with q vertices and center in the origin, with a vertex in x.

5. Continuity

In this section we study some facts about continuity of functions defined on matrix primal spaces which are deduce from some general propositions. We begin with an example which shows that even in very simple cases things may not be as one might expect.

Example 13. Let $\iota\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the map defined by $\iota(x) = x$ for all x, but seen as a function $\iota\colon \mathbb{R}^2_{\pi/2} \longrightarrow \mathbb{R}^2_\pi$. Then ι is not continuous, for instance, $\Gamma_\pi(e_1) = \{e_1, -e_1\} \notin \tau_{\pi/2}$, because nontrivial open sets in $\mathbb{R}^2_{\pi/2}$ contain at least 4 elements.

Before addressing some general problems, we deal with a few particular results. Thus the, following lemmas offer a hint about hypothesis on the matrices A and B in order to deduce the continuity of some linear maps $f\colon \mathbb{R}^n_A \longrightarrow \mathbb{R}^n_B$.

Lemma 1. *Let X be a set, $f, g\colon X \longrightarrow X$ two maps. Let τ_f and τ_g be the respective primal topologies induced by f and g. If $h\colon X_f \longrightarrow X_g$ is a function such that $h \circ f = g \circ h$, then for all $x, y \in X$ if $y \in \Omega_f(x)$ we have that $h(y) \in \Omega_g(h(x))$.*

Proof. Suppose that $x \in X$ and $y \in \Omega_f(x)$. then there exists a $k \in \mathbb{N}_0$ such that $f^k(x) = y$. Then

$$h(y) = h \circ f^k(x) = h \circ f \circ f^{k-1}(x) = g \circ h \circ f^{k-1}(x) = g \circ h \circ f \circ f^{k-2}(x) = g^2 \circ h \circ f^{k-3}(x) = g^k \circ h(x). \quad (31)$$

Thus $y \in \Omega_g(h(x))$. □

Lemma 2. *Let X_f, X_g be primal spaces and $h\colon X_f \longrightarrow X_g$ a homeomorphism. Then $h(\Omega_f(x)) = \Omega_g(h(x))$.*

Proof. This is a direct consequence of the fact that para any set $A \subset X$, since h is a homemorphism, we have that $h(cl(A)) = cl(h(A))$. □

Corollary 1. *Let X_f, X_g be primal spaces and let be $h\colon X_f \longrightarrow X_g$ a homeomorphism. Then we have that for all $x, y \in X$ $x \in \Omega_f(y)$ if and only if $h(x) \in \Omega_g(h(y))$.*

Now we return to the problem of continuity of functions in the context of matrix primal spaces.

Theorem 5. *Let A, B two square matrices. If A and B are similar matrices then the primal spaces $\mathbb{R}^n_A$ and $\mathbb{R}^n_B$ are homeomorphic. Furthermore, if $A = P^{-1}BP$ then $P\colon \mathbb{R}^n_A \longrightarrow \mathbb{R}^n_B$ is a homeomorphism.*

Proof. By hypothesis, $P\colon \mathbb{R}^n_A \longrightarrow \mathbb{R}^n_B$ is invertible. If $V \in \tau_B$, then

$$A^{-1}\left(P^{-1}(V)\right) = (PA)^{-1}(V) = (BP)^{-1}(V) = P^{-1}\left(B^{-1}(V)\right) \subset P^{-1}(V), \quad (32)$$

therefore $P^{-1}(V) \in \tau_A$, so P is continuous.

Let F be a closed set in $\mathbb{R}^n_A$, that is $A(F) \subset F$. Now we have

$$B(P(F)) = (BP)(F) = (PA)(F) = P(A(F)) \subset P(F). \quad (33)$$

Then P is a closed map, which means that P^{-1} is continuous and therefore P is a homeomorphism. □ □

Theorem 6. *If A and B are two $n \times n$ matrices which commute with each other, then the maps $A\colon \mathbb{R}^n_B \longrightarrow \mathbb{R}^n_B$ and $B\colon \mathbb{R}^n_A \longrightarrow \mathbb{R}^n_A$ are both continuous.*

Proof. Of course, it is enough to prove that $A\colon \mathbb{R}^n_B \longrightarrow \mathbb{R}^n_B$ is continuous.

Let $V \in \tau_B$ and $x \in B^{-1}(A^{-1}(V))$, then $AB(x) \in V$ and $B^{-1}AB(x) \in B^{-1}(V)$. But A and B commute, thus $A(x) \in B^{-1}(V) \subset V$, and therefore $x \in A^{-1}(V)$, which means that $A^{-1}(V) \in \tau_B$. □

Let us note that the hypothesis of commutativity of the matrices A and B in Theorem 6 cannot be dropped. In fact, if we consider A and B as in Example 11 with $a = b = 1$, that is

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad (34)$$

then $AB \neq BA$ and $\Gamma_B(0,1) = \{(0,1)\} \in \tau_B$, but $A^{-1}(\Gamma_B(0,1)) = \{(-1,1)\} \notin \tau_B$ because $(1,2) \in \Gamma_B(-1,1)$.

Moreover, invertibility of the matrices A and B in Theorem 6 is not a necessary condition: the orthogonal projections (Example 6) $\wp_j\colon \mathbb{R}^2_{P_i} \longrightarrow \mathbb{R}^2_{P_i}$, with $i, j \in \{1,2\}$, $i \neq j$ are continuous and $\wp_1\wp_2 = \wp_2\wp_1$.

Lemma 3. *Let A and B be two invertible $n \times n$ matrices. If $\tau_B \subset \tau_{AB^{-1}}$, then $\tau_B \subset \tau_A$.*

Proof. Suppose that $V \in \tau_B$. Then $V \in \tau_{AB^{-1}}$, which means that

$$\begin{aligned} \left(AB^{-1}\right)^{-1}(V) &\subset V, \\ BA^{-1}(V) &\subset V, \\ A^{-1}(V) \subset B^{-1}(V) &\subset V, \end{aligned} \quad (35)$$

because $V \in \tau_B$, in other words, $V \in \tau_A$. □

Now we obtain the following result, which somehow fixes the problem set in Example 13 related to the function $\iota\colon \mathbb{R}^n_A \longrightarrow \mathbb{R}^n_B$ defined by $\iota(x) = x$.

Corollary 2. *Let A and B be two invertible $n \times n$ matrices. If $\tau_B \subset \tau_{AB^{-1}}$ then the map $\iota\colon \mathbb{R}^n_A \longrightarrow \mathbb{R}^n_B$ is continuous.*

Theorem 7. *Let A and B be two invertible $n \times n$ matrices which commute with each other. If $\tau_A \subset \tau_{BA^{-1}}$ and $\tau_B \subset \tau_{AB^{-1}}$, then the spaces $\mathbb{R}^n_A$ and $\mathbb{R}^n_B$ are homeomorphic.*

Proof. Let us consider the composite $\varphi = A^\circ\iota\colon \mathbb{R}^n_A \longrightarrow \mathbb{R}^n_B$:

$$\mathbb{R}^n_A \xrightarrow{\iota} \mathbb{R}^n_B \xrightarrow{A} \mathbb{R}^n_B. \quad (36)$$

Obviously, φ is bijective. By Corollary 2 we have that ι is continuous and, by Theorem 6, we have that A is continuous, therefore φ is continuous.

On the other hand, we have $\tau_A \subset \tau_{BA^{-1}}$ then Theorem [6] implies that A^{-1} is continuous, and so is φ^{-1}. □

Data Availability

The theoretical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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