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Research Article

Distance Spectra of Some Double Join Operations of Graphs

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In literature, several types of join operations of two graphs based on subdivision graph, Q-graph, R-graph, and total graph have been introduced, and their spectral properties have been studied. In this paper, we introduce a new double join operation based on (H_1, H_2)-merged subdivision graph. We compute the spectrum of a special block matrix and then use it to describe the distance spectra of some double join operations of graphs. At last, we give several families of distance equienergetic graphs of diameter 3.

1. Introduction

Let G be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. We denote the spectrum of the well-known adjacency matrix of G by $\mathrm{Spec}(G) = \{\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)\}$, where $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. The distance matrix of G is $\mathscr{D}(G) = (d_{ij})_{n\times n}$, where d_{ij} is the distance between the vertices v_i and v_j in G. The distance matrix was introduced in the year 1971 by Graham and Pollack [1] to study data communication problems. Studies on the eigenvalues of the distance matrix can be found in the survey article [2].

In spectral graph theory, graph products are used to construct special classes of graphs. Graph operations include complement, disjoint union, join, NEPS (particularly the Cartesian product, the direct product, the strong product, and the lexicographic product), corona product, edge (neighborhood) corona product, and subdivision (edge) vertex join. A survey on spectra of graphs resulting from various graph operations and products is done in [3]. Computation of distance spectra of some graph compositions can be found in [4–7]. The subdivision graph of G, denoted by $S(G)$, is obtained by inserting a new vertex into every edge of G. The graph $Q(G)$ is obtained from $S(G)$ by adding an edge between two new vertices whenever the corresponding edges are adjacent. The graph $R(G)$ is obtained by introducing a new vertex corresponding to every edge of G, and then joining the new vertex to the end vertices of the corresponding edge. The total graph of G, denoted by $T(G)$, is obtained from $R(G)$ by adding an edge between two new vertices whenever the corresponding edges are adjacent. See Figure 1 for examples. The line graph $L(G)$ of G is a graph with vertex set $E(G)$, and two vertices are adjacent if there are adjacent edges in G.

Let the new vertices of $S(G)$ be denoted by e_i, $i = 1, 2, \ldots, m$. Let H_1 and H_2 be two graphs with vertex set $V(H_1) = \{e_1, e_2, \ldots, e_m\}$ and $V(H_2) = \{v_1, v_2, \ldots, v_n\}$, respectively. The (H_1, H_2)-merged subdivision graph of G [8] is obtained by taking one copy of $S(G)$ and adding an edge between vertices v_i and v_j in $S(G)$ whenever $v_iv_j \in E(H_2)$, and also by adding an edge between the new vertices e_i and e_j if $e_ie_j \in E(H_1)$. It is denoted by $[S(G)]_{H_2}^{H_1}$. Note that $[S(G)]_{\overline{K_n}}^{\overline{K_m}} \cong S(G)$, $[S(G)]_{G}^{\overline{K_m}} \cong R(G)$, $[S(G)]_{\overline{K_n}}^{L(G)} \cong Q(G)$, and $[S(G)]_{G}^{L(G)} \cong T(G)$. In [8], the authors obtained the adjacency spectra and Laplacian spectra of $[S(G)]_{H_2}^{H_1}$ for some classes of graphs G, H_1, and H_2. Let $F \in \{S, R, Q, T\}$. The double join operation of $F(G)$ with graphs G_1 and G_2 [9]

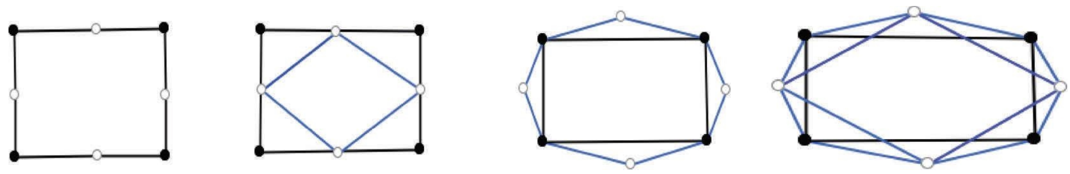

Figure 1: Graphs $S[C_4]$, $Q[C_4]$, $R[C_4]$, and $T[C_4]$.

is obtained by taking one copies of $F(G)$, G_1, and G_2, and joining each vertex of G in $F(G)$ with every vertices of G_1, and also by joining each new vertex of $F(G)$ with every vertices of G_2. It is denoted by $G^F \vee \{G_1^\bullet, G_2^\circ\}$. In [9], Tian, He, and Cui obtained the Laplacian spectrum of $G^F \vee$ when G is a regular graph. In analogous to the definition of double join operation of $F(G)$ ($F \in \{S, T, R, G\}$) with graphs G_1 and G_2, we define double join operation of (H_1, H_2)-merged subdivision graph of G with graphs G_1 and G_2 as follows.

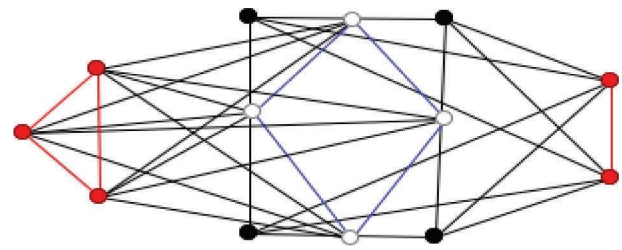

Figure 2: Graph $S[C_4]^{K_4}_{K_4} \vee \{C_3, K_2\}$.

Definition 1. The double join operation of (H_1, H_2)-merged subdivision graph $[S(G)]^{H_1}_{H_2}$ with the graphs G_1 and G_2 is the graph obtained by taking one copy of $[S(G)]^{H_1}_{H_2}$, G_1, and G_2, then joining each vertex of G in $[S(G)]^{H_1}_{H_2}$ with all the vertices of G_2, and also joining each new vertex of $[S(G)]^{H_1}_{H_2}$ with all the vertices of G_1. It is denoted by $[S(G)]^{H_1}_{H_2} \vee \{G^\circ{}_1, G_2^\bullet\}$ (see Figure 2).

This new join operation generalizes several known join operation based on subdivision graph (see [4, 9]). So, the new double join operation is interesting and thus makes sense to study its spectral properties.

This paper focuses on the distance spectrum of the double join graph $[S(G)]^{H_1}_{H_2} \vee \{G^\circ{}_1, G_2^\bullet\}$. Clearly the graph $[S(G)]^{H_1}_{H_2} \vee \{G^\circ{}_1, G_2^\bullet\}$ is a generalization of the double join graph $G^F \vee \{G_1^\bullet, G_2^\circ\}$. The distance energy of a graph G is denoted by $\mathscr{E}_D(G)$ and is defined to be the sum of all absolute values of the eigenvalues of the distance matrix $\mathscr{D}(G)$. In analogous to graph energy (ordinary energy of a graph), the concept of distance energy was put forward in the year 2008 by Indulal et al. [10]. Two graphs of the same order are distance equienergetic if their distance energies are same. In [11], Ramane et al. constructed a pair of distance equienergetic graphs of diameter 2 on $9 + n$ vertices for all $n \geq 1$. Some other constructions of distance equienergetic graphs can be found in [4, 5, 7].

Let $\mathscr{M}_{m\times n}(\mathbb{R})$ denote the set of all of real matrices of order $m \times n$ and let $\mathcal{S}_n(r)$ be the set of all real symmetric matrices of order n such that each of its row sum is a constant r. We denote by $J_{n\times m}$, the matrix of order $n \times m$ whose all entries are equal to 1. The column vector of order $n \times 1$, whose i th entry is 1 and all its other entries are 0, is denoted by $e_{i,n}$. Let $J'_{n\times m} = e_{1,n}e^T_{1,m}$ and let $\mathbf{1}_n$ be the column vector of size n whose all entries are equal to one. As usual, we denote by C_n the cycle graph, by K_n the complete graph, and by $\overline{G}$ the complement graph of G, each on n vertices. The paper is organized as follows. In Section 2, we compute the spectrum of a block matrix whose structure coincides with the distance matrix of $[S(G)]^{H_1}_{H_2} \vee \{G^\circ{}_1, G_2^\bullet\}$ in many cases. In Section 3, we give the distance spectra of the double join graph $[S(G)]^{H_1}_{H_2} \vee \{G^\circ{}_1, G_2^\bullet\}$ for some classes of graphs G, H_1, H_2, G_1, and G_2. As an application of our results, in Section 4, we give several families of distance equienergetic graphs of diameter 3.

2. Spectrum of a Partitioned Matrix

This section deals with finding the spectrum of a special blocked matrix given in Definition 1.

Definition 2. Let a, b, c, and d be real numbers. Let $A \in \mathcal{S}_m(a)$, $B \in \mathcal{S}_n(b)$, $C \in \mathcal{S}_p(c)$, and $D \in \mathcal{S}_q(d)$ with $m \geq n$. For real constants s, k, and l, define a partitioned matrix $\mathscr{P} = \mathscr{P}[A, B, C, D, M, s, k, l]$ as follows:

$$\mathscr{P} = \begin{bmatrix} A & M & sJ_{m\times p} & kJ_{m\times q} \\ M^T & B & kJ_{n\times p} & sJ_{n\times q} \\ sJ_{p\times m} & kJ_{p\times n} & C & lJ_{p\times q} \\ kJ_{q\times m} & sJ_{q\times n} & lJ_{q\times p} & D \end{bmatrix}, \tag{1}$$

where M is a real rectangular matrix of order $m \times n$ satisfying the following condition.

M has a singular value decomposition, $M = U_{m\times m}M'_{m\times n}V^T_{n\times n}$, with singular values $m_1(\neq 0), m_2, \ldots, m_n$ such that if $X_i := Ue_{i,m}$ $(i = 1, 2, \ldots, m)$ and $Y_j := Ve_{j,n}$ $(j = 1, 2, \ldots, n)$, then X_i's and Y_j's form a set of orthonormal eigenvectors of A and B, respectively. That is, $AX_i = a_i X_i$ and $BY_j = b_j Y_j$, where $a_i \mathrm{SA}\,(i = 1, 2, \ldots, m)$ and b_j $(j = 1, 2, \ldots, n)$ are the eigenvalues of A and B, respectively. Also, $M\mathbf{1}_n = t\mathbf{1}_m$ and $Y_1 = (1/\sqrt{n})\mathbf{1}_n$ for some scalar t.

The matrix $\mathscr{P}$ defined above is a real symmetric block square matrix of order $m + n + p + q$.

We need the following well-known Schur complement formula to describe the spectrum of the block matrix $\mathscr{P}$.

Lemma 3 (see [12]). *Let $A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ be a block matrix. Let P and S be square matrices.*

(1) If P is invertible, then $\det(A) = \det(P)\det(S - RP^{-1}Q)$

(2) If S is invertible, then $\det(A) = \det(S)\det(P - QS^{-1}R)$

Let the spectra of the matrices C and D as defined in Definition 2 be $\{c_1 = c, c_2, \ldots, c_p\}$ and $\{d_1 = d, d_2, \ldots, d_q\}$, respectively. The following theorem gives the spectrum of the block matrix $\mathscr{P}$.

Theorem 4. *Let $\mathscr{P}$ be a matrix as defined in Definition 2. Then, the spectrum of the matrix $\mathscr{P}$ consists of*

(i) c_i and d_j for $i = 2, 3, \ldots, p$ and $j = 2, 3, \ldots, q$

(ii) a_i for $i = n + 1, n + 2, \ldots, m$

(iii) $(1/2)((a_i + b_i) \pm \sqrt{(a_i - b_i)^2 + 4m_i^2})$ for $i = 2, 3, \ldots, n$

(iv) The four roots of the polynomial are as follows:

$$\begin{aligned} f(x) = {} & x^4 - (a + b + c + d)x^3 + \left((a + b)(c + d) + ab + cd - k^2(mq + np) - s^2(mp + nq) - l^2pq - m_1^2\right)x^2 \\ & + \left(-cd(a + b) - ab(c + d) + s^2(pm(b + d) + nq(a + c)) + k^2(np(a + d) + qm(b + c)) + l^2pq(a + b)\right. \\ & \left. - 2ks(lpq(m + n) + mt(p + q)) + m_1^2(c + d)\right)x \\ & + npqm\left(s^4 + k^4\right) - s^2(nacq + bdpm + 2lpqmt) - k^2(nadp + 2lpqmt + bcqm) \\ & - 2nk^2pqms^2 - l^2\left(abpq - m_1^2pq\right) + 2mkst(cq + dp) + 2kpqsl(na + mb) - cdm_1^2 + abcd \end{aligned} \tag{2}$$

Proof. Since C and D are regular real symmetric matrices, there exist orthogonal matrices P and Q such that $C = PC'P^T$ and $D = QD'Q^T$ where $C' = \mathrm{diag}(c_1 = c, c_2, \ldots, c_p)$, $D' = \mathrm{diag}(d_1 = d, d_2, \ldots, d_q)$, $Pe_{1,p} = (1/\sqrt{p})\mathbf{1}_p$, and $Qe_{1,q} = (1/\sqrt{q})\mathbf{1}_q$. By Definition 2, we get $M = UM'V^T$, $A = UA'U^T$, and $B = VB'V^T$, where $A' = \mathrm{diag}(a_1, a_2, \ldots, a_m)$ and $B' = \mathrm{diag}(b_1 = b, b_2, \ldots, b_n)$. Since $M\mathbf{1}_n = t\mathbf{1}_m$, we must have $M^T\mathbf{1}_m = t_1\mathbf{1}_n$ with $t_1 = (tm/n)$, and also since $Y_1 = (1/\sqrt{n})\mathbf{1}_n$, we get $X_1 = (MY_1/m_1) = (t/m_1\sqrt{n})\mathbf{1}_m$. Therefore, $M^TX_1 = m_1Y_1$ implies $m_1 = (t\sqrt{m}/\sqrt{n})$, and so $X_1 = (1/\sqrt{m})\mathbf{1}_m$.

Now,

$$\begin{aligned} \mathscr{P} &= \begin{bmatrix} A & M & sJ_{m\times p} & kJ_{m\times q} \\ M^T & B & kJ_{n\times p} & sJ_{n\times q} \\ sJ_{p\times m} & kJ_{p\times n} & C & lJ_{p\times q} \\ kJ_{q\times m} & sJ_{q\times n} & lJ_{q\times p} & D \end{bmatrix} \\ &= \begin{bmatrix} UA'U^T & UM'V^T & sJ_{m\times p} & kJ_{m\times q} \\ VM'^TU^T & VB'V^T & kJ_{n\times p} & sJ_{n\times q} \\ sJ_{p\times m} & kJ_{p\times n} & PC'P^T & lJ_{p\times q} \\ kJ_{q\times m} & sJ_{q\times n} & lJ_{q\times p} & QD'Q^T \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} U & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & Q \end{bmatrix} \begin{bmatrix} A' & M' & sU^T J_{m\times p} P & kU^T J_{m\times q} Q \\ M'^T & B' & kV^T J_{n\times p} P & sV^T J_{n\times q} Q \\ sP^T J_{p\times m} U & kP^T J_{p\times n} V & C' & lP^T J_{p\times q} Q \\ kQ^T J_{q\times m} U & sQ^T J_{q\times n} V & lQ^T J_{q\times p} P & D' \end{bmatrix} \begin{bmatrix} U^T & 0 & 0 & 0 \\ 0 & V^T & 0 & 0 \\ 0 & 0 & P^T & 0 \\ 0 & 0 & 0 & Q^T \end{bmatrix}$$

$$= \begin{bmatrix} U & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & Q \end{bmatrix} \begin{bmatrix} A' & M' & s\sqrt{mp}\, J'_{m\times p} & k\sqrt{mq}\, J'_{m\times q} \\ M'^T & B' & k\sqrt{np}\, J'_{n\times p} & s\sqrt{nq}\, J'_{n\times q} \\ s\sqrt{mp}\, J'_{p\times m} & k\sqrt{np}\, J'_{p\times n} & C' & l\sqrt{pq}\, J'_{p\times q} \\ k\sqrt{mq}\, J'_{q\times m} & s\sqrt{nq}\, J'_{q\times n} & l\sqrt{pq}\, J'_{q\times p} & D' \end{bmatrix} \begin{bmatrix} U^T & 0 & 0 & 0 \\ 0 & V^T & 0 & 0 \\ 0 & 0 & P^T & 0 \\ 0 & 0 & 0 & Q^T \end{bmatrix}. \quad (3)$$

Therefore,

$$\det(xI - \mathscr{P}) = \det \begin{bmatrix} xI - A' & -M' & -s\sqrt{mp}\, J'_{m\times p} & -k\sqrt{mq}\, J'_{m\times q} \\ -M'^T & xI - B' & -k\sqrt{np}\, J'_{n\times p} & -s\sqrt{nq}\, J'_{n\times q} \\ -s\sqrt{mp}\, J'_{p\times m} & -k\sqrt{np}\, J'_{p\times n} & xI - C' & -l\sqrt{pq}\, J'_{p\times q} \\ -k\sqrt{mq}\, J'_{q\times m} & -s\sqrt{nq}\, J'_{q\times n} & -l\sqrt{pq}\, J'_{q\times p} & xI - D' \end{bmatrix}. \quad (4)$$

Expanding $\det(xI - \mathscr{P})$ by Laplace's method [13] along the columns $(m+n+2), (m+n+3), \ldots, (m+n+p), (m+n+p+2), \ldots, (m+n+p+q)$, we get $\det(xI - \mathscr{P}) = \prod_{i=2}^{p}(x - c_i)\prod_{j=2}^{q}(x - d_i)$ $\det\begin{bmatrix} xI - A' & -M' & -s\sqrt{mp}\, e_{1,m} & -k\sqrt{mq}\, e_{1,m} \\ -M'^T & xI - B' & -k\sqrt{np}\, e_{1,n} & -s\sqrt{nq}\, e_{1,n} \\ -s\sqrt{mp}\, e_{1,m}^T & -k\sqrt{np}\, e_{1,n}^T & x - c & -l\sqrt{pq} \\ -k\sqrt{mq}\, e_{1,m}^T & -s\sqrt{nq}\, e_{1,n}^T & -l\sqrt{pq} & x - d \end{bmatrix}$.

Applying Lemma 3 to the determinant on the right side of the above equation, we have $\det(xI - \mathscr{P}) = p_0(x)\prod_{i=2}^{p}(x - c_i)$ $\prod_{i=2}^{q}(x - d_i)\det$ $\begin{bmatrix} xI - A' - (p_1(x)/p_0(x))J'_{m\times m} & -M' - (p_2(x)/p_0(x))J'_{m\times n} \\ -M'^T - (p_2(x)/p_0(x))J'_{n\times m} & xI - B' - (p_3(x)/p_0(x))J'_{n\times n} \end{bmatrix}$, where $p_0(x) = (x - c)(x - d) - l^2 pq$, $p_1(x) = m(s^2 p(x - d) + 2skpql + k^2 q(x - c))$, $p_2(x) = \sqrt{mn}(skp(x - d) + s^2 pql + k^2 pql + ksq(x - c))$ and $p_3(x) = n(k^2 p(x - d) + 2skpql + s^2 q(x - c))$.

Now, employing Lemma 3, we get

$$\det(xI - \mathscr{P}) = p_0(x)\left(x - a_1 - \frac{p_1(x)}{p_0(x)}\right)\prod_{i=2}^{p}(x - c_i)\prod_{i=2}^{q}(x - d_i)\prod_{i=2}^{m}(x - a_i)$$
$$\times \det\left[xI - B' - \frac{p_3(x)}{p_0(x)}J'_{n\times n} - \left(M'^T + \frac{p_2(x)}{p_0(x)}J'_{n\times m}\right)\left(xI - A' - \frac{p_1(x)}{p_0(x)}J'_{m\times m}\right)^{-1} \cdot \left(M' + \frac{p_2(x)}{p_0(x)}J'_{m\times n}\right)\right]. \quad (5)$$

Upon evaluating the determinant on the right hand side of the above equation, we get $\det(xI - \mathscr{P}) = f(x)\prod_{i=2}^{n}((x - a_i)(x - b_i) - m_i^2)\prod_{i=2}^{p}(x - c_i)$ $\prod_{i=2}^{q}(x - d_i)\prod_{i=n+1}^{m}(x - a_i)$. This completes the proof. □

Remark 5. The quotient matrix [14] of the partitioned matrix $\mathscr{P}$ is $\begin{bmatrix} a & t & sp & kq \\ tm/n & b & kp & sq \\ sm & kn & c & lq \\ km & sn & lp & d \end{bmatrix}$. It can be verified that the polynomial $f(x)$ defined in Theorem 4 is same the characteristic polynomial of the quotient matrix.

3. Distance Spectra of Some Double Join Graphs $S[G]_{H_2}^{H_1} \vee \{G^\circ{}_1, G_2^\bullet\}$

In this section, we give the distance spectra of $S[G]_{H_2}^{H_1} \vee \{G^\circ{}_1, G_2^\bullet\}$ under some conditions on graphs G, G_1, G_2, H_1, and H_2. The edge-vertex incidence matrix of G is denoted by $M(G)$ and is defined as the matrix of order $m \times n$ whose ij-th entry is 1 if v_j is an end vertex of the edge e_i.

Lemma 6 (see [15]). *Let G be an r-regular graph with n vertices and m edges. Then,*

(a) $M(G)M^T(G) = L(G) + 2I_m$ *and* $M^T(G)M(G) = A(G) + rI_n$

(b) $Spec(L(G)) = \{\lambda_1(G) + r - 2, \quad \lambda_2(G) + r - 2, \dots, \lambda_n(G) + r - 2, 0, 0, \dots, 0\}$

Remark 7. Let G be a regular graph on n vertices. If $M(G) = U \sum V^T$ is a singular value decomposition of $M(G)$, then the columns of the matrix U (resp., V) forms an orthonormal set of eigenvectors of the matrix $aJ_m + bI_m + cL(G)$ (resp., $J_n + bI_n + cA(G)$) for any constants a, b, and c. Also, we can assume that $Ve_{1,n} = 1_n/\sqrt{n}$. Further, it may be noted that $M(G)1_n = 21_m$.

The following theorem gives the distance spectrum of $[S(G)]\overline{K_m}/K_n \vee \{G^\circ{}_1, G_2^\bullet\}$ when G_1 and G_2 are regular graphs.

Theorem 8. *Let G be an r-regular graph with n vertices and m edges. Let G_1 be an r_1-regular graph of order p and let G_2 be an r_2-regular graph of order q, respectively. Then, the distance spectrum of the double join graph $[S(G)]_{K_n}^{\overline{K_m}} \vee \{G^\circ{}_1, G_2^\bullet\}$ consists of*

(1) $-(\lambda_i(G_1) + 2)$ *for* $i = 2, 3, \dots, p$ *and* $d_i = -(\lambda_i(G_2) + 2)$ *for* $i = 2, 3, \dots, q$

(2) -2 *with multiplicity m-n*

(3) $-2 \pm \sqrt{\lambda_i(G) + r}$ *for* $i = 2, 3, \dots, n$

(4) *The four eigenvalues of the matrix* $\begin{bmatrix} 2(m-1) & 3n-4 & p & 2q \\ 3m-2r & 2(n-1) & 2p & q \\ m & 2n & 2(p-1) - r_1 & 3q \\ 2m & n & 3p & 2(q-1) - r_2 \end{bmatrix}$

Proof. The distance matrix $\mathscr{D}$ of $[S(G)]_{K_n}^{\overline{K_m}} \vee \{G^\circ{}_1, G_2^\bullet\}$ is

$$\begin{bmatrix} 2(J_{m\times m} - I_m) & 3J_{m\times n} - 2M(G) & J_{m\times p} & 2J_{m\times q} \\ 3J_{n\times m} - 2M^T(G) & 2(J_{n\times n} - I_n) & 2J_{n\times p} & J_{n\times q} \\ J_{p\times m} & 2J_{p\times n} & 2\left(J_{p\times p} - I_p\right) - A(G_1) & 3J_{p\times q} \\ 2J_{q\times m} & J_{q\times n} & 3J_{q\times p} & 2\left(J_{q\times q} - I_q\right) - A(G_2) \end{bmatrix}. \tag{6}$$

Let $A = 2(J_{m\times m} - I_m)$, $B = 2(J_{n\times n} - I_n)$, $C = 2(J_{p\times p} - I_p) - A(G_1)$, $D = 2(J_{q\times q} - I_q) - A(G_2)$, $M = 3J_{m\times n} - 2M(G)$, $s = 1$, $k = 2$, and $l = 3$. Then, by Remark 7, $\mathscr{D} = \mathscr{P}[A, B, C, D, M, s, k, l]$. Further, the eigenvalues of A, B, C, and D are, respectively, as follows:

(1) $a_1 = 2m - 2$ and $a_i = -2$ for $i = 2, 3, \dots, m$

(2) $b_1 = 2n - 2$ and $b_i = -2$ for $i = 2, 3, \dots, n$

(3) $c_1 = 2p - r_1 - 2$ and $c_i = -(\lambda_i(G_1) + 2)$ for $i = 2, 3, \dots, p$

(4) $d_1 = 2q - r_2 - 2$ and $d_i = -(\lambda_i(G_2) + 2)$ for $i = 2, 3, \dots, q$

Plugging these values in Theorem 4, we obtain the required result. □

Theorem 9. *Let G be an r-regular triangle free graph with n vertices and m edges. Let G_1 be an r_1-regular graph of order p and let G_2 be an r_2-regular graph of order q, respectively. If H is an t-regular graph belonging to the set $\left\{\overline{K_m}, L(G), \overline{L(G)}\right\}$, then the distance spectrum of the double join graph $[S(G)]_G^H \vee \{G^\circ{}_1, G_2^\bullet\}$ consists of*

(1) $-(\lambda_i(G_1) + 2)$ *for* $i = 2, 3, \dots, p$ *and* $d_i = -(\lambda_i(G_2) + 2)$ *for* $i = 2, 3, \dots, q$

(2) $-(2 + \lambda_i(H))$ *for* $i = n + 1, n + 2, \dots, m$

(3) $1/2\left(l - k - 3 \pm \sqrt{(\lambda_i(G) + \lambda_i(H))^2 + 2\lambda_i(H) + 6\lambda_i(G) + 4r + 1}\right)$ *for* $i = 2, 3, \dots, n$

(4) *The four eigenvalues of the matrix* $\begin{bmatrix} 2(m-1) - t & 2(n-1) & p & 2q \\ 2m - r & n + r - 1 & 2p & q \\ m & 2n & 2(p-1) - r_1 & 3q \\ 2m & n & 3p & 2(q-1) - r_2 \end{bmatrix}$

Proof. The distance matrix $\mathscr{D}$ of $[S(G)]_G^H \vee \{G^\circ{}_1, G_2^\bullet\}$ is

$$\begin{bmatrix} 2(J_{m\times m}-I_m)-A(H) & 2J_{m\times n}-M(G) & J_{m\times p} & 2J_{m\times q} \\ 2J_{n\times m}-M^T(G) & A(G)+J_{n\times n}-I_n & 2J_{n\times p} & J_{n\times q} \\ J_{p\times m} & 2J_{p\times n} & 2\left(J_{p\times p}-I_p\right)-A(G_1) & 3J_{p\times q} \\ 2J_{q\times m} & J_{q\times n} & 3J_{q\times p} & 2\left(J_{q\times q}-I_q\right)-A(G_2) \end{bmatrix}. \quad (7)$$

Let $A=2(J_{m\times m}-I_m)-A(H)$, $B=A(G)+J_{n\times n}-I_n$, $C=2(J_{p\times p}-I_p)-A(G_1)$, $D=2(J_{q\times q}-I_q)-A(G_2)$, $M=2J_{m\times n}-M(G)$, $s=1$, $k=2$, and $l=3$. Then, by Remark 7, $\mathscr{D}=\mathscr{P}[A,B,C,D,M,s,k,l]$. Further, the eigenvalues of A, B, C, and D are, respectively, as follows:

(1) $a_1=2m-2-t$, $a_i=-(2+\lambda_i(H))$ for $i=2,3,\ldots,m$

(2) $b_1=n+r-1$, $b_i=\lambda_i(G)-1$ for $i=2,3,\ldots,n$

(3) $c_1=2p-r_1-2$, $c_i=-(\lambda_i(G_1)+2)$ for $i=2,3,\ldots,p$

(4) $d_1=2q-r_2-2$, $d_i=-(\lambda_i(G_2)+2)$ for $i=2,3,\ldots,q$

Plugging these values in Theorem 4, we obtain the required result. □

Theorem 10. *Let G be an r-regular graph with n vertices and m edges. Let H be an t-regular graph belonging to the set $\{\overline{K_n},K_n,G,\overline{G}\}$. If G_1 is an r_1-regular graph of order p and G_2 is an r_2-regular graph of order q, respectively, then the distance spectrum of the double join graph $[S(G)]_H^{K_m}\vee\{G^{\circ}{}_1,G_2^{\bullet}\}$ consists of*

(1) $-(\lambda_i(G_1)+2)$ for $i=2,3,\ldots,p$ and $d_i=-(\lambda_i(G_2)+2)$ for $i=2,3,\ldots,q$

(2) -1 with multiplicity m-n

(3) $1/2\left(-3-\lambda_i(H)\pm\sqrt{(\lambda_i(H)+1)^2+4\lambda_i(G)+4r}\right)$ for $i=2,3,\ldots,n$

(4) The four eigenvalues of the matrix

$$\begin{bmatrix} m-1 & 2(n-1) & p & 2q \\ 2m-r & 2(n-1)-t & 2p & q \\ m & 2n & 2(p-1)-r_1 & 3q \\ 2m & n & 3p & 2(q-1)-r_2 \end{bmatrix}$$

Proof. The distance matrix $\mathscr{D}$ of $[S(G)]_H^{K_m}\vee\{G^{\circ}{}_1,G_2^{\bullet}\}$ is

$$\begin{bmatrix} J_{m\times m}-I_m & 2J_{m\times n}-M(G) & J_{m\times p} & 2J_{m\times q} \\ 2J_{n\times m}-M^T(G) & 2(J_{n\times n}-I_n)-A(H) & 2J_{n\times p} & J_{n\times q} \\ J_{p\times m} & 2J_{p\times n} & 2\left(J_{p\times p}-I_p\right)-A(G_1) & 3J_{p\times q} \\ 2J_{q\times m} & J_{q\times n} & 3J_{q\times p} & 2\left(J_{q\times q}-I_q\right)-A(G_2) \end{bmatrix}. \quad (8)$$

Let $A=J_{m\times m}-I_m$, $B=2(J_{n\times n}-I_n)-A(H)$, $C=2(J_{p\times p}-I_p)-A(G_1)$, $D=2(J_{q\times q}-I_q)-A(G_2)$, $M=2J_{m\times n}-M(G)$, $s=1$, $k=2$, and $l=3$. Then, by Remark 7, $\mathscr{D}=\mathscr{P}[A,B,C,D,M,s,k,l]$. Further, the eigenvalues of A, B, C, and D are, respectively, as follows:

(1) $a_1=m-1$, $a_i=-1$ for $i=2,3,\ldots,m$

(2) $b_1=2n-2-t$, $b_i=-(\lambda_i(H)+2)$ for $i=2,3,\ldots,n$

(3) $c_1=2p-r_1-2$, $c_i=-(\lambda_i(G_1)+2)$ for $i=2,3,\ldots,p$

(4) $d_1=2q-r_2-2$, $d_i=-(\lambda_i(G_2)+2)$ for $i=2,3,\ldots,q$

Plugging these values in Theorem 4, we obtain the required result. □

Theorem 11. *Let G be an r-regular graph with n vertices and m edges. Let G_1 be an r_1-regular graph of order p and let G_2 be an r_2-regular graph of order q, respectively. If H is an t-regular graph belonging to the set $\left\{\overline{K_m},L(G),\overline{L(G)}\right\}$, then the distance spectrum of the double join graph $[S(G)]_{K_n}^{H}\vee\{G^{\circ}{}_1,G_2^{\bullet}\}$ consists of*

(1) $-(\lambda_i(G_1)+2)$ for $i=2,3,\ldots,p$ and $d_i=-(\lambda_i(G_2)+2)$ for $i=2,3,\ldots,q$

(2) $-(2+\lambda_i(H))$ for $i=n+1,n+2,\ldots,m$

(3) $1/2\left(-3-\lambda_i(H)\pm\sqrt{(\lambda_i(H)+1)^2+4\lambda_i(G)+4r}\right)$ for $i=2,3,\ldots,n$

(4) The four eigenvalues of the matrix

$$\begin{bmatrix} 2(m-1)-t & 2(n-1) & p & 2q \\ 2m-r & n-1 & 2p & q \\ m & 2n & 2(p-1)-r_1 & 3q \\ 2m & n & 3p & 2(q-1)-r_2 \end{bmatrix}$$

Proof. The distance matrix $\mathscr{D}$ of $[S(G)]^H_{K_n} \vee \{G^\circ{}_1, G^\bullet_2\}$ is

$$\begin{bmatrix} 2(J_{m\times m} - I_m) - A(H) & 2J_{m\times n} - M(G) & J_{m\times p} & 2J_{m\times q} \\ 2J_{n\times m} - M^T(G) & J_{n\times n} - I_n & 2J_{n\times p} & J_{n\times q} \\ J_{p\times m} & 2J_{p\times n} & 2\left(J_{p\times p} - I_p\right) - A(G_1) & 3J_{p\times q} \\ 2J_{q\times m} & J_{q\times n} & 3J_{q\times p} & 2\left(J_{q\times q} - I_q\right) - A(G_2) \end{bmatrix}, \tag{9}$$

where $M(G)$ is the edge-vertex incidence matrix of G. Let $A = 2(J_{m\times m} - I_m) - A(H)$, $B = A(G) + J_{n\times n} - I_n$, $C = 2(J_{p\times p} - I_p) - A(G_1)$, $D = 2(J_{q\times q} - I_q) - A(G_2)$, $M = 2J_{m\times n} - M(G)$, $s = 1$, $k = 2$, and $l = 3$. Then, by Remark 7, $\mathscr{D} = \mathscr{P}[A, B, C, D, M, s, k, l]$. Further, the eigenvalues of A, B, C, and D are, respectively, as follows:

(1) $a_1 = 2m - 2 - t, a_i = -(2 + \lambda_i(H))$ for $i = 2, 3, \ldots, m$

(2) $b_1 = n - 1$, $b_i = -1$ for $i = 2, 3, \ldots, n$

(3) $c_1 = 2p - r_1 - 2$, $c_i = -(\lambda_i(G_1) + 2)$ for $i = 2, 3, \ldots, p$

(4) $d_1 = 2q - r_2 - 2$, $d_i = -(\lambda_i(G_2) + 2)$ for $i = 2, 3, \ldots, q$

Plugging these values in Theorem 4, we obtain the required result. □

4. New Families of Distance Equienergetic Graphs of Diameter 3

In this section, we present families of distance equienergetic double join graphs of diameter 3 using the distance spectra of some double join graphs obtained in Section 3. Let $\mathscr{P}_n$ be the set of all partitions of the positive integer n into parts of size greater than or equal to 3, i.e., $\mathscr{P}_n = \{(n_1, n_2, \ldots, n_k) \mid k \geq 1,\ n_i \geq 3 \text{ for } i = 1, 2, \ldots, k \text{ and } \sum_{i=1}^k n_i = n\}$. Let $C(\mathscr{P}_n)$ denote a family of disjoint union of cycles given by $C(\mathscr{P}_n) = \left\{\cup_{i=1}^s C_{n_i} \mid (n_1, n_2, \ldots, n_k) \in \mathscr{P}_n\right\}$.

Classes of distance equienergetic double join graphs are presented in the following theorem. Note that the double join graph defined in Definition 1 is of diameter 3.

Theorem 12. *Let G_1 be a graph in $C(\mathscr{P}_p)$ and let G_2 be an r_2-regular graph of order q.*

(i) If G is an r-regular graph with n vertices and m edges, then the two classes of double join graphs $\{[S(G)]^{\overline{K_m}}_{K_n} \vee \{G^\circ{}_1, G^\bullet_2 | G_1 \in C(\mathscr{P}_p)\}$ and $\{[S(G)]^{\overline{K_m}}_{K_n} \vee \{G^\circ{}_1, G^\bullet_2 | G_2 \in C(\mathscr{P}_q)\}$ form two families of distance equienergetic graphs

(ii) If G is an r-regular triangle free graph with n vertices and m edges and also if $H \in \left\{\overline{K_m}, L(G), \overline{L(G)}\right\}$, then the two classes of double join graphs $\{[S(G)]^H_G \vee \{G^\circ{}_1, G^\bullet_2 | G_1 \in C(\mathscr{P}_p)\}$ and $\{[S(G)]^H_G \vee \{G^\circ{}_1, G^\bullet_2 | G_2 \in C(\mathscr{P}_q)\}$ form two families of distance equienergetic graphs

(iii) If G is an r-regular graph with n vertices and m edges and if $H \in \{\overline{K_n}, K_n, G, \overline{G}\}$, then the classes of double join graphs $\{[S(G)]^{K_m}_H \vee \{G^\circ{}_1, G^\bullet_2 | G_1 \in C(\mathscr{P}_p)\}$ and $\{[S(G)]^{K_m}_H \vee \{G^\circ{}_1, G^\bullet_2 | G_2 \in C(\mathscr{P}_q)\}$ form two families of distance equienergetic graphs

(iv) If G is an r-regular graph with n vertices and m edges and if $H \in \left\{\overline{K_m}, L(G), \overline{L(G)}\right\}$, then the two classes of double join graphs $\{[S(G)]^H_{K_n} \vee \{G^\circ{}_1, G^\bullet_2 | G_1 \in C(\mathscr{P}_p)\}$ and $\{[S(G)]^H_{K_n} \vee \{G^\circ{}_1, G^\bullet_2 | G_2 \in C(\mathscr{P}_q)\}$ form two families of distance equienergetic graphs

Proof. Let $G_1 = \cup_{i=1}^k C_{n_i}$ and $G_1' = \cup_{i=1}^{k'} C_{n_i'}$ be two graphs in $C(\mathscr{P}_p)$. Then, $\lambda_1(G_1) = \lambda_1(G_1') = 2$, $-2 \leq \lambda_i(G_1)$, and $-2 \leq \lambda_i(G_1')$ for $i = 1, 2, \ldots, p$. Let $\Gamma_1 = [S(G)]^{\overline{K_m}}_{K_n} \vee \{G^\circ{}_1, G^\bullet_2\}$ and $\Gamma_2 = [S(G)]^{\overline{K_m}}_{K_n} \vee \{G'^\circ{}_1, G^\bullet_2\}$. Then, from Theorem 8, we get

$$\begin{aligned} \mathscr{E}_D(\Gamma_1) - \mathscr{E}_D(\Gamma_2) &= \sum_{i=2}^p \left|\lambda_i(G_1) + 2\right| - \sum_{i=2}^p \left|\lambda_i(G_1') + 2\right| \\ &= \sum_{i=2}^p (\lambda_i(G_1) + 2) - \sum_{i=2}^p (\lambda_i(G_1') + 2). \end{aligned} \tag{10}$$

Since $\sum_{i=1}^p \lambda_i(G_1) = \sum_{i=1}^p \lambda_i(G_1') = 0$ and $\lambda_1(G_1) = \lambda_1(G_1') = 2$, the equation (10) simplifies to $\mathscr{E}_D(\Gamma_1) = \mathscr{E}_D(\Gamma_2)$. This completes the proof of (i). Similarly, rest of the proof follows. □

5. Conclusion

The new double join operation introduced in this paper generalizes several known join operation based on subdivision graph (see [4, 9]). So, the new double join operation is interesting and thus makes sense to study its spectral properties. The spectrum of the special block matrix computed here can also be used to describe various other spectra (like adjacency spectra, Laplacian spectra, signless Laplacian spectra, and Seidel spectra) of double join graphs.

Data Availability

No data were used to support this study.

Disclosure

The first version of this paper is archived in arXiv [16].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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