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Research Article

Properties of a Linear Operator Involving Lambert Series and Rabotnov Function

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This work is an attempt to apply Lambert series in the theory of univalent functions. We first consider the Hadamard product of Rabotnov function and Lambert series with coefficients derived from the arithmetic function $\sigma(n)$ to introduce a normalized linear operator $\mathscr{J}\mathbb{R}_{\alpha,\beta}(z)$. We then acquire sufficient conditions for $\mathscr{J}\mathbb{R}_{\alpha,\beta}(z)$ to be univalent, starlike and convex, respectively. Furthermore, we discuss the inclusion results in some special classes, namely, spiral-like and convex spiral-like subclasses. In addition, we extend the findings by incorporating two Robin's inequalities, one of which is analogous to the Riemann hypothesis.

1. Introduction

A series introduced by Johann Heinrich Lambert, commonly known as Lambert series is expressed as follows:

$$S(x) = \sum_{n=1}^{\infty} a_n \frac{x^n}{1 - x^n}. \tag{1}$$

It is a type of series that is well-known in both number theory and analytic function theory. Lambert (see [1]) considered it in the context of the convergence of power series. Lambert series given by (1) converges either everywhere except at $x = \pm 1$ when $\sum_1^{\infty} a_n$ converges, or at every x such that $\sum_1^{\infty} a_n x^n$ converges.

In number theory (see [2–5]), Lambert series is used for certain problems due to its connection to the well-known arithmetic functions such as

$$\sum_{n=1}^{\infty} \sigma_0(n) x^n = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n}, \tag{2}$$

where $\sigma_0(n) = d(n)$ is the number of positive divisors of n.

$$\sum_{n=1}^{\infty} \sigma_\alpha(n) x^n = \sum_{n=1}^{\infty} \frac{n^\alpha x^n}{1 - x^n}, \tag{3}$$

where $\sigma_\alpha(n)$ is the higher-order sum of divisors function of n.

We restrict our attention to the series given by (3). In particular, when $\alpha = 1$, we write $\sigma_1(n) = \sigma(n)$, here $\sigma(n)$ is the sum of divisors function that appears in one of the elementary equivalent statements to the well-known Riemann Hypothesis.

We distinguish at the outset between Lambert series and Lambert W function that appears naturally in the solution of a wide range of problems in science and engineering [6].

In 1984, Nicolas and Robin [7] proved that

$$\sigma(n) < e^{\gamma} n \log\log n + \frac{0.6483n}{\log\log n}, \quad n \geq 3. \tag{4}$$

Moreover, he proved that Riemann hypothesis is equivalent to

$$\sigma(n) < e^{\gamma} n \log\log n, \quad n > 5040, \tag{5}$$

where $\gamma = 0.7721\cdots$, is the Euler–Mascheroni constant.

This article makes no attempt to prove or refute Robin's inequality (5) or the Riemann hypothesis. For more details, we refer the interested readers to read the articles listed in references [8–13].

Let $\mathscr{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \tag{6}$$

and $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of univalent (or one-to-one) functions on $\mathbb{D}$. The importance of the coefficients given by the power series in (6) emerged in the early stage of the theory of univalent functions. Earlier in 1916, Bieberbach [14] proved that the second coefficient $|a_2| \leq 2$, with equality holding if and only if f is a rotation of the Köebe's function: $f(z) = z + \sum_{n=2}^{\infty} n z^n$.

In the same work, Bieberbach conjectured the general coefficient bound that $|a_n| \leq n, n \geq 2$, while the equality holds if and only if f is a rotation of the Köebe's function. This conjecture came to be known as the famed Bieberbach Conjecture and resisted a rigorous proof for about seven decades until Louis de Branges proved it in 1985 [15], and the result came to be known as de Branges's Theorem. Geometrically this amounts to shrinking or expanding the domain $\mathbb{D}$, and possibly rotating $\mathbb{D}$ but does not disturb the univalence of the function. Later on, new concepts were introduced in the theory of univalent functions including, but not limited to, starlike, convex, spiral-like, and uniformly starlike (convex).

In fact, the study on introducing new subclasses of analytic functions goes on by means of various applications, such as fractional calculus, quantum calculus, or by involving some special functions such as Mittag–Leffler function and Faber polynomial functions, see for details [16–24]. The most common concern in such a study is the inclusion conditions. Alternatively, it means that for a given new subclass (say) $\mathscr{H}$, seek a set of useful conditions on the sequence $\{a_n\}$ that are both necessary and sufficient for $f(z)$ to be a member of $\mathscr{H}$.

The Rabotnov function defined as follows (see [25]):

$$R_{\alpha,\beta}(z) = z^{\alpha} \sum_{n=0}^{\infty} \frac{\beta^n}{\Gamma((n+1)(\alpha+1))} z^{n(\alpha+1)}, \quad \alpha, \beta, z \in \mathbb{C}. \tag{7}$$

Clearly, $R_{0,\beta}(z) = \sum_{n=0}^{\infty} \beta^n / n! z^n = e^{\beta z}$.

Rabotnov function is the particular case of the familiar Mittag–Leffler function widely used in the solution of fractional order integral equations or fractional order differential equations. The relation between the Rabotnov function and Mittag–Leffler function can be written as follows:

$$R_{\alpha,\beta}(z) = z^{\alpha} E_{\alpha+1,\alpha+1}\left(\beta z^{\alpha+1}\right), \tag{8}$$

where $E_{\alpha+1,\alpha+1}$ is the two parameters Mittag–Leffler function. Several properties of Mittag–Leffler function and generalized Mittag–Leffler function can be found in [26–30].

It is clear that the Rabotnov function does not belong to the family $\mathscr{A}$. Thus, it is natural to consider the following normalization of Rabotnov function:

$$\mathbb{R}_{\alpha,\beta}(z) := z^{1/\alpha+1} \Gamma(\alpha+1) R_{\alpha,\beta}\left(z^{1/\alpha+1}\right) = z + \sum_{n=2}^{\infty} \frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))} z^n. \tag{9}$$

Geometric properties including starlikeness, convexity and close-to-convexity for the normalized Rabotnov function were recently studies in [31].

In this study, we aim to utilize the two remarkable special functions the Lambert series and the Rabotnov function. The first is for its connection to some well-known arithmetic functions among which the sum of divisors function that we consider here, and we use its relevant inequalities (4) and (5) when applicable. The second is the Rabotnov function that can be viewed as a generalization of the exponential function since $R_{0,0}(z) = e^z$ and a special case of the two-parameters Mittag–Leffler function, because $R_{0,1}(z) = E_{1,1}(z)$. Thus, the findings of this study apply to e^z and $E_{1,1}(z)$. Furthermore, one can obtain additional conclusions if the sum of divisors function $\sigma(n)$ in the Lambert series is replaced by the higher-order sum of divisors function $\sigma_{\alpha}(n)$.

In general, applying the Lambert series that has not yet been considered in the theory of univalent functions leads to introducing new subclasses of analytic functions and investigating the relevant topics such as the coefficients bound, the distortion theorems, Hankel determinants, subordination properties, and Fekete–Szegö inequalities.

Here, we recall the definition of Hadamard product (convolution). For a given function $f \in \mathscr{A}$ of the form (6) and $g \in \mathscr{A}$ of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in \mathbb{D}, \tag{10}$$

then the convolution $(*)$ of the two functions f and g becomes

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}. \tag{11}$$

Subsequently, we utilize the Lambert series whose coefficients are the sum of divisors function $\sigma(n)$. The mathematical form is as follows:

$$\begin{aligned}\mathscr{L}(z) &= \sum_{n=1}^{\infty} \frac{n z^n}{1 - z^n} \\ &= \sum_{n=1}^{\infty} \sigma(n) z^n \\ &= z + \sum_{n=2}^{\infty} \sigma(n) z^n, \quad z \in \mathbb{D}.\end{aligned} \tag{12}$$

We define the linear operator $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)\colon \mathcal{A} \longrightarrow \mathcal{A}$ as follows:

$$\mathcal{J}\mathbb{R}_{\alpha,\beta}(z) \coloneqq \left(\mathbb{R}_{\alpha,\beta} * \mathscr{L}\right)(z) = z + \sum_{n=2}^{\infty} \frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))} \sigma(n) z^n, \quad \in \mathbb{D}. \tag{13}$$

Now, for short hand we denote the coefficient of $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ by

$$a_n = \frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))} \sigma(n). \tag{14}$$

From Robin's inequalities, we obtain.

Remark 1. Unconditionally, from Robin's inequality (3). For $n \geq 3$

$$a_n < \frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))} \left\{ e^{\gamma} n \log\log n + \frac{0.6483 n}{\log\log n} \right\}. \tag{15}$$

Remark 2. If Riemann hypothesis (4) holds true, then for $n > 5040$

$$a_n < \frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))} \left\{ e^{\gamma} n \log\log n \right\}. \tag{16}$$

Next, we provide sufficient conditions for the operator (10) to be starlike, convex, and closed-to-convex, respectively. We also evoke the consequence of Robin's inequalities or Riemann hypothesis in each derived result and vice versa.

Firstly, we recall some relevant definitions and Lemmas that we consider in this study.

Definition 3. Function $f \in \mathcal{A}$ of the form (6) is said to be starlike or $f \in \mathcal{S}^*$, if

$$\operatorname{Re}\left(\frac{z f'(z)}{f(z)} \right) > 0, \quad z \in \mathbb{D}. \tag{17}$$

Definition 4. Function $f \in \mathcal{A}$ of the form (6) is said to be convex or $f \in \mathcal{C}$, if

$$\operatorname{Re}\left(\frac{z f''(z)}{f'(z)} + 1 \right) > 0, \quad z \in \mathbb{D}. \tag{18}$$

Definition 5. Function $f \in \mathcal{A}$ of the form (6) is said to be closed-to-convex or $f \in \mathcal{K}$, if

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)} \right) > 0, \quad g \in \mathcal{C}, z \in \mathbb{D}. \tag{19}$$

The above definitions have been investigated in different studies; see for example [32–35]. Moreover, Noshiro–Warschawski [36, 37] provided the following inclusion result: $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$.

Lemma 6 (see [38]). *Function $f \in \mathcal{A}$ of the form (6) is univalent in $\mathbb{D}$, if*

$$1 \geq 2a_2 \geq \ldots \geq 2a_n \geq \ldots \geq 0, \tag{20}$$

or

$$1 \leq 2a_2 \leq \ldots \leq 2a_n \leq \ldots \leq 2. \tag{21}$$

Furthermore, f is closed-to-convex with respect to the convex function $-\operatorname{Log}(1 - z)$.

Lemma 7 (see [39]). *Function $f \in \mathcal{A}$ of the form (6) is starlike in $\mathbb{D}$ if $a_n \geq 0$, $\{na_n\}$ and $\{na_n - (n+1)a_{n+1}\}$ both are nonincreasing.*

2. Geometric Properties of the Linear Operator $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$

Theorem 8. *The operator $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ defined in (10) is close-to-convex with respect to $-\log(1 - z)$ and therefore univalent in $\mathbb{D}$ if for every consecutive natural numbers n and $n+1$, with $\alpha \geq 0$ and $\beta > 0$.*

$$(\alpha+1)\sigma(n) \geq 2\beta\sigma(n+1) \tag{22}$$

Proof. We utilize Lemma 6. First, we need to prove by induction that

$$(\alpha+1)^{n-1}(n-1)!\Gamma(\alpha+1)\leq\Gamma(n(\alpha+1)),\quad n=1,2,\ldots. \tag{23}$$

For $n=1$, (11) obviously holds true. Assuming (11) is true for $n-1$, we conclude

$$\begin{aligned}(\alpha+1)^n n!\Gamma(\alpha+1)&=(\alpha+1)n(\alpha+1)^{n-1}(n-1)!\Gamma(\alpha+1)\\&\leq(\alpha+1)n\Gamma(n(\alpha+1))\\&=\Gamma((\alpha+1)n+1)\\&\leq\Gamma((\alpha+1)(n+1)).\end{aligned} \tag{24}$$

Recall

$$a_n=\frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))}\sigma(n),\quad n\geq 2, a_1=1. \tag{25}$$

From (23) when $n=2$, we obtain

$$\frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(2(\alpha+1))}\leq 1. \tag{26}$$

Using condition (11)

$$2a_2=\frac{2\beta\sigma(2)\Gamma(\alpha+1)}{\Gamma(2(\alpha+1))}\leq\frac{2\beta\sigma(2)}{\alpha+1}\leq 1. \tag{27}$$

To verify that the first condition of Lemma 6 holds, we need to show that the sequence $\{na_n\}$ is decreasing:

$$\begin{aligned}na_n-(n+1)a_{n+1}&=\frac{n\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))}\sigma(n)-\frac{(n+1)\beta^n\Gamma(\alpha+1)}{\Gamma((n+1)(\alpha+1))}\sigma(n+1)\\&\geq\frac{n\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))}\sigma(n)-\frac{(n+1)\beta^n\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}\sigma(n+1)\\&=\frac{n^2(\alpha+1)\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}\sigma(n)-\frac{(n+1)\beta^n\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}\sigma(n+1)\\&=\frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}X(n),\end{aligned} \tag{28}$$

where $X(n)=n^2(\alpha+1)\sigma(n)-(n+1)\beta\sigma(n+1)$ considering $n^2\geq n+1, n>1$, we receive

$$\begin{aligned}X(n)&=n^2(\alpha+1)\sigma(n)-(n+1)\beta\sigma(n+1)\\&\geq(n+1)(\alpha+1)\sigma(n)-(n+1)\beta\sigma(n+1)\\&\geq(n+1)((\alpha+1)\sigma(n)-\beta\sigma(n+1))\\&\geq(n+1)((\alpha+1)\sigma(n)-2\beta\sigma(n+1))\geq 0.\end{aligned} \tag{29}$$

From Theorem 8. Using the fact that the coefficients of a univalent function satisfy the inequality $a_n\leq n$, we derive the following results. □

Corollary 9. *If the conditions of Theorem 8 hold true, then*

$$\sigma(n)\leq\frac{n\Gamma(n(\alpha+1))}{\beta^{n-1}\Gamma(\alpha+1)},\quad n\geq 2. \tag{30}$$

Corollary 10. *If the conditions of Theorem 8 hold true and* $(\Gamma(n(\alpha+1)))/(\beta^{n-1}\Gamma(\alpha+1))<e^{\gamma}\log\ \log n, n>5040$ *then Riemann hypothesis holds true.*

Theorem 11. *The operator* $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ *defined in (10) is starlike in* $\mathbb{D}$ *if for every consecutive natural numbers n and* $n+1$, *with* $\alpha\geq 0$ *and* $\beta>0$. $(\alpha+1)\sigma(n)\geq 2\beta\sigma(n+1)$

Proof. In here, we use Lemma 7. We omit the proof that the sequence $\{na_n\}$ is nonincreasing since it is similar to the one in Theorem 8. Therefore, we need to show that $\{na_n-(n+1)a_{n+1}\}$ is also nonincreasing. For the sake of simplicity, we let $b_n=na_n-(n+1)a_{n+1}$ and we have

$$\begin{aligned}b_n-b_{n+1}&=na_n-2(n+1)a_{n+1}+(n+2)a_{n+2}\\&=\frac{n\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))}\sigma(n)-\frac{2(n+1)\beta^n\Gamma(\alpha+1)}{\Gamma((n+1)(\alpha+1))}\sigma(n+1)+\frac{(n+2)\beta^{n+1}\Gamma(\alpha+1)}{\Gamma((n+2)(\alpha+1))}\sigma(n+2)\end{aligned}$$

$$\begin{aligned}
&\geq \frac{n\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))}\sigma(n) - \frac{2(n+1)\beta^{n}\Gamma(\alpha+1)}{\Gamma((n+1)(\alpha+1))}\sigma(n+1)\\
&\geq \frac{n\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))}\sigma(n) - \frac{2(n+1)\beta^{n}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}\sigma(n+1)\\
&= \frac{n^2(\alpha+1)\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}\sigma(n) - \frac{2(n+1)\beta^{n}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}\sigma(n+1)\\
&= \frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1)+1)}Y(n),
\end{aligned} \tag{31}$$

where $Y(n) = n^2(\alpha+1)\sigma(n) - 2(n+1)\beta\sigma(n+1)$. We use the fact that $n^2 \geq n+1, n>1$,

$$\begin{aligned}
Y(n) &= n^2(\alpha+1)\sigma(n) - 2(n+1)\beta\sigma(n+1)\\
&\geq (n+1)(\alpha+1)\sigma(n) - 2(n+1)\beta\sigma(n+1)\\
&\geq (n+1)((\alpha+1)\sigma(n) - 2\beta\sigma(n+1)) \geq 0.
\end{aligned} \tag{32}$$

□

Theorem 12. *The operator $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ defined in (10) is convex in $\mathbb{D}$ if for every consecutive natural numbers n and $n+1$, with $\alpha \geq 0$ and $\beta > 0$.*

$$\sum_{n=2}^{\infty} \frac{n^2\beta^{n-1}\sigma(n)}{(\alpha+1)^{n-1}(n-1)!} < 1. \tag{33}$$

Proof. Let

$$p(z) = 1 + \frac{z\mathcal{J}\mathbb{R}''_{\alpha,\beta}(z)}{\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}, \quad z \in \mathbb{D}, \tag{34}$$

$p(z)$ is analytic in $\mathbb{D}$ and $p(0) = 1$. We need to prove that $|p(z) - 1| < 1$,

$$\begin{aligned}
\left|z\mathcal{J}\mathbb{R}''_{\alpha,\beta}(z)\right| &= \left|\sum_{n=2}^{\infty} \frac{n(n-1)\beta^{n-1}\Gamma(\alpha+1)\sigma(n)}{\Gamma(n(\alpha+1))}z^{n-1}\right|\\
&< \sum_{n=2}^{\infty} \frac{n(n-1)\beta^{n-1}\Gamma(\alpha+1)\sigma(n)}{\Gamma(n(\alpha+1))}\\
&\leq \sum_{n=2}^{\infty} \frac{n(n-1)\beta^{n-1}\sigma(n)}{(\alpha+1)^{n-1}(n-1)!}\\
\left|\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)\right| &= \left|1 + \sum_{n=2}^{\infty} \frac{n\beta^{n-1}\Gamma(\alpha+1)\sigma(n)}{\Gamma(n(\alpha+1))}z^{n-1}\right|\\
&> 1 - \sum_{n=2}^{\infty} \frac{n\beta^{n-1}\Gamma(\alpha+1)\sigma(n)}{\Gamma(n(\alpha+1))}\\
&\geq 1 - \sum_{n=2}^{\infty} \frac{n\beta^{n-1}\sigma(n)}{(\alpha+1)^{n-1}(n-1)!}.
\end{aligned} \tag{35}$$

From equations (13) and (14), we obtain

$$\left|\frac{z\mathcal{J}\mathbb{R}''_{\alpha,\beta}(z)}{\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}\right| < \frac{\sum_{n=2}^{\infty} n(n-1)\beta^{n-1}\sigma(n)/(\alpha+1)^{n-1}(n-1)!}{1 - \sum_{n=2}^{\infty} n\beta^{n-1}\sigma(n)/(\alpha+1)^{n-1}(n-1)!} < 1. \tag{36}$$

□

Since the coefficients of a univalent function satisfy the inequality $a_n \leq n$, we derive the following results.

Corollary 13. *If the conditions of Theorem 12 hold true, then*

$$\sigma(n) \leq \frac{\Gamma(n(\alpha+1))}{\beta^{n-1}\Gamma(\alpha+1)}, \quad n \geq 2. \tag{37}$$

Corollary 14. *If the conditions of Theorem 12 hold true, and $(\Gamma(n(\alpha+1)))/(\beta^{n-1}\Gamma(\alpha+1)) < e^{\gamma} n \log\log n, n > 5040$ then Riemann hypothesis holds true.*

3. Inclusion Results of $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ in Spiral-Like and Convex Spiral-Like Subclasses

Definition 15. Function $f \in \mathcal{A}$ of the form (6) is said to spiral-like if and only if it meets the following conditions $\Re e(e^{i\xi}(z'f(z))/(f(z))) > 0, z \in \mathbb{D}, \xi \in \mathbb{C}$ but $|\xi| < \pi/2$.

Note that a function $f \in \mathcal{A}$ is said to be a convex spiral-like if $z'f(z)$ is spiral-like.

Next, we utilize the linear operator $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ to introduce the following two subclasses of spiral-like functions and convex spiral-like functions.

Definition 16. For the real values of $\rho, \gamma, \lambda, \xi$ such that $0 \leq \{\rho, \gamma\} < 1$ $\alpha \geq 0$, $\beta > 0$ and $|\xi| < \pi/2$, a subclass $S_{\eta,\lambda}(\xi, \gamma, \rho)$, named as spiral-starlike function is defined by

$$\mathcal{S}_{\eta,\lambda}(\xi,\gamma,\rho) := \left\{ \mathcal{Re}\left(e^{i\xi} \frac{z\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}{(1-\rho)\mathcal{J}\mathbb{R}_{\alpha,\beta}(z) + \rho\mathcal{J}z\mathbb{R}'_{\alpha,\beta}(z)} \right) > \gamma\cos\xi, \quad z \in \mathbb{D} \right\}. \tag{38}$$

Definition 17. For the real values of $\rho, \gamma, \lambda, \xi$ such that $0 \leq \{\rho, \gamma\} < 1$ with $\alpha \geq 0$, $\beta > 0$ and $|\xi| < \pi/2$, a subclass $\mathcal{K}_{\eta,\lambda}(\xi, \gamma, \rho)$, named as spiral-convex function is defined by

$$\mathcal{K}_{\eta,\lambda}(\xi,\gamma,\rho) := \left\{ \mathcal{Re}\left(e^{i\xi} \frac{z\mathcal{J}\mathbb{R}''_{\alpha,\beta}(z) + \mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}{\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z) + \rho\mathcal{J}z\mathbb{R}''_{\alpha,\beta}(z)} \right) > \gamma\cos\xi, \quad z \in \mathbb{D} \right\}. \tag{39}$$

Next, we obtain sufficient conditions for function $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ given by (13) to be a member of the classes $S_{\eta,\lambda}(\xi, \gamma, \rho)$ and $K_{\eta,\lambda}(\xi, \gamma, \rho)$, respectively. These conditions are presented in the following theorems.

Theorem 18. *The operator $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ is in the class $\mathcal{S}_{\eta,\lambda}(\xi, \gamma, \rho)$, if*

$$\left| \frac{z\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}{(1-\rho)\mathcal{J}\mathbb{R}_{\alpha,\beta}(z) + \rho\mathcal{J}z\mathbb{R}'_{\alpha,\beta}(z)} - 1 \right| < 1 - \sigma$$

$$|\xi| \leq \cos^{-1}\left(\frac{1-\sigma}{1-\xi}\right), \tag{40}$$

where $0 \leq \{\rho, \gamma\} < 1$, $\alpha \geq 0$, $\beta > 0$ and $|\xi| < \pi/2$.

Proof. Using the inequality that if $|(z\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z))/((1-\rho)\mathcal{J}\mathbb{R}_{\alpha,\beta}(z) + \rho\mathcal{J}z\mathbb{R}'_{\alpha,\beta}(z)) - 1| < 1 - \sigma$, then for $w(z) \in \Omega$: The Class of analytic functions with positive real part, we can write

$$\frac{z\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}{(1-\rho)\mathcal{J}\mathbb{R}_{\alpha,\beta}(z) + \rho\mathcal{J}z\mathbb{R}'_{\alpha,\beta}(z)} = 1 + (1-\sigma)\mathrm{w}(z). \tag{41}$$

□

Thus, for spiral properties, we have

$$\begin{aligned} \mathcal{Re}\left(e^{i\xi} \frac{z\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}{(1-\rho)\mathcal{J}\mathbb{R}_{\alpha,\beta}(z) + \rho\mathcal{J}z\mathbb{R}'_{\alpha,\beta}(z)} \right) &= \mathcal{Re}\left\{ e^{i\xi} [1 + (1-\sigma)] \right\} \\ &= \cos\xi + (1-\sigma)\mathcal{R}\left\{ e^{i\xi} w(z) \right\} \\ &\geq \cos\xi - (1-\sigma)\left| e^{i\xi} w(z) \right| \\ &> \cos\xi - (1-\sigma) \\ &\geq \lambda\cos\xi. \end{aligned} \tag{42}$$

As long $|\xi| \leq \cos^{-1}((1-\sigma)/(1-\xi))$, the function f belongs to the class $\mathcal{S}_{\eta,\lambda}(\xi, \gamma, \rho)$.

Observed that by putting $\sigma = 1 - (1-\lambda)$ $\cos\xi$ in Theorem 18, we deduce a corollary as follows.

Corollary 19. $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z) \in \mathcal{S}_{\eta,\lambda}(\xi, \gamma, \rho)$ *if the following inequality holds*

$$\left| \frac{z\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}{(1-\rho)\mathcal{J}\mathbb{R}_{\alpha,\beta}(z) + \rho\mathcal{J}z\mathbb{R}'_{\alpha,\beta}(z)} - 1 \right| < (1-\lambda)\cos\xi. \tag{43}$$

Theorem 20. *The operator $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ belongs to the class $S_{\eta,\lambda}(\xi, \gamma, \rho)$, if*

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))}\sigma(n)\le 1-\gamma, \tag{44}$$

where $0\le\{\rho,\gamma\}<1$, $\alpha\ge 0$, $\beta>0$ *and* $|\xi|<\pi/2$.

Proof. By Corollary 14, it is enough to show that the expressions in (17) are satisfied. Considering the left-hand side (L.H.S.).

$$\begin{aligned}\text{L.H.S} &= \left|\frac{z\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}{(1-\rho)\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)+\rho\mathcal{J}z\mathbb{R}'_{\alpha,\beta}(z)}-1\right|\\ &= \left|\frac{\sum_{n=2}^{\infty}(1-\rho)(n-1)\beta^{n-1}\Gamma(\alpha+1)/\Gamma(n(\alpha+1))\sigma(n)z^n}{1+\sum_{n=2}^{\infty}(\rho n-\rho+1)\beta^{n-1}\Gamma(\alpha+1)/\Gamma(n(\alpha+1))\sigma(n)z^n}\right|\\ &< \frac{\sum_{n=2}^{\infty}(1-\rho)(n-1)\beta^{n-1}\Gamma(\alpha+1)/\Gamma(n(\alpha+1))\sigma(n)}{1-\sum_{n=2}^{\infty}(1+n\rho-\rho)\beta^{n-1}\Gamma(\alpha+1)/\Gamma(n(\alpha+1))\sigma(n)}.\end{aligned} \tag{45}$$

The last expression is bounded above by $(1-\gamma)\cos\xi$, if

$$\begin{aligned}&\sum_{n=2}^{\infty}(1-\rho)(n-1)\frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))}\sigma(n)\\ &\quad<(1-\gamma)\cos\xi\left\{1-\sum_{n=2}^{\infty}(1+n\rho-\rho)\frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))}\sigma(n)\right\}.\end{aligned} \tag{46}$$

After some simple work the equivalent form we received from above expression is

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))}\sigma(n)\le 1-\gamma. \tag{47}$$

The last expression proves the assertion of Theorem 20. □

Theorem 21. *The operator* $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ *is in the class* $\mathcal{K}_{\eta,\lambda}(\xi,\gamma,\rho)$, *if*

$$\left|\frac{z\mathcal{J}\mathbb{R}''_{\alpha,\beta}(z)+\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}{\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)+\rho\mathcal{J}z\mathbb{R}''_{\alpha,\beta}(z)}-1\right|<1-\sigma$$

$$|\xi|\le\cos^{-1}\left(\frac{1-\sigma}{1-\xi}\right)\text{hold,}\tag{48}$$

where $0\le\{\rho,\gamma\}<1,\ \alpha\ge0,\ \beta>0,$ *and* $|\xi|<\pi/2$.

Proof. Using the inequality that if $|(z\mathcal{J}\mathbb{R}''_{\alpha,\beta}(z)+\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z))/(\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)+\rho\mathcal{J}z\mathbb{R}''_{\alpha,\beta}(z))-1|<1-\sigma$, then for $w(z)\in\Omega$, we can write

$$\begin{aligned}&\frac{z\mathcal{J}\mathbb{R}''_{\alpha,\beta}(z)+\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}{\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)+\rho\mathcal{J}z\mathbb{R}''_{\alpha,\beta}(z)}-1=1+(1-\sigma)w(z),\\
&\mathcal{R}e\left(e^{i\xi}\,\frac{z\mathcal{J}\mathbb{R}''_{\alpha,\beta}(z)+\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}{\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)+\rho\mathcal{J}z\mathbb{R}''_{\alpha,\beta}(z)}-1\right)=\mathcal{R}e\left\{e^{i\xi}[1+(1-\sigma)]\right\}\\
&=\cos\xi+(1-\sigma)\mathcal{R}\left\{e^{i\xi}w(z)\right\}\\
&\ge\cos\xi-(1-\sigma)\left|e^{i\xi}w(z)\right|\\
&>\cos\xi-(1-\sigma)\\
&\ge\lambda\cos\xi,\end{aligned}\tag{49}$$

As long $|\xi|\le\cos^{-1}(1-\sigma/1-\xi)$, the function $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ belongs to the class $\mathcal{K}_{\eta,\lambda}(\xi,\gamma,\rho)$. □

Observed that by putting $\sigma=1-(1-\lambda)\cos\xi$ in Theorem 21, we deduce a corollary as follows.

Corollary 22. $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)\in\mathcal{K}_{\eta,\lambda}(\xi,\gamma,\rho)$ *if the following inequality holds*

$$\left|\frac{z\mathcal{J}\mathbb{R}''_{\alpha,\beta}(z)+\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}{\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)+\rho\mathcal{J}z\mathbb{R}''_{\alpha,\beta}(z)}-1\right|<(1-\lambda)\cos\xi.\tag{50}$$

Theorem 23. *The operator* $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ *belongs to the class* $\mathcal{K}_{\eta,\lambda}(\xi,\gamma,\rho)$, *if*

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]n\frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))}\sigma(n)\le1-\gamma,\tag{51}$$

where $0\le\{\rho,\gamma\}<1,\ \alpha\ge0,\ \beta>0$ *and* $|\xi|<\pi/2$.

Proof. By Corollary 22, it is enough to show that the expressions in (18) are satisfied; then, by considering the left-hand side of (6), we obtain

$$\begin{aligned}\left|\frac{z\mathcal{J}\mathbb{R}''_{\alpha,\beta}(z)+\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)}{\mathcal{J}\mathbb{R}'_{\alpha,\beta}(z)+\rho\mathcal{J}z\mathbb{R}''_{\alpha,\beta}(z)}-1\right|&=\left|\frac{\sum_{n=2}^{\infty}(1-\rho)(n-1)n\beta^{n-1}\Gamma(\alpha+1)/\Gamma(n(\alpha+1))\sigma(n)z^n}{1+\sum_{n=2}^{\infty}n(\rho n-\rho+1)\beta^{n-1}\Gamma(\alpha+1)/\Gamma(n(\alpha+1))\sigma(n)z^n}\right|\\
&<\frac{\sum_{n=2}^{\infty}(1-\rho)(n-1)n\beta^{n-1}\Gamma(\alpha+1)/\Gamma(n(\alpha+1))\sigma(n)}{1-\sum_{n=2}^{\infty}(1+n\rho-\rho)n\beta^{n-1}\Gamma(\alpha+1)/\Gamma(n(\alpha+1))\sigma(n)}.\end{aligned}\tag{52}$$

The last expression is bounded above by $(1-\gamma)\cos\xi$, if

$$\sum_{n=2}^{\infty}(1-\rho)(n-1)\frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))}\sigma(n) < (1-\gamma)\cos\xi\left\{1-\sum_{n=2}^{\infty}(1+n\rho-\rho)\frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))}\sigma(n)\right\}. \quad (53)$$

After some simple calculation, the above inequality becomes:

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{\beta^{n-1}\Gamma(\alpha+1)}{\Gamma(n(\alpha+1))}\sigma(n)\leq 1-\gamma. \quad (54)$$

The last expression proves the proclamation in Theorem 23. □

4. Conclusions

By means of Lambert series whose coefficients are the sum of divisors function, we considered the normalized linear operator $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ by applying the convolution with Rabotnov function. We provided sufficient conditions for $\mathcal{J}\mathbb{R}_{\alpha,\beta}(z)$ to be univalent, starlike, and convex, respectively. When applicable, we expand the derived results by applying two Robin's inequalities (3) and (4).

Data Availability

All data generated or analyzed during the study are included in the article.

Disclosure

A preprint has previously been published [39].

Conflicts of Interest

The author declares that there are no conflicts of interest.

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