Hindawi
International Journal of Mathematics and Mathematical Sciences
Volume 2024, Article ID 4098771, 14 pages
https://doi.org/10.1155/2024/4098771

Research Article

New Weighted Burr XII Distribution: Statistical Properties, Applications, and Regression

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Received 10 December 2023; Revised 21 January 2024; Accepted 30 January 2024; Published 9 February 2024

Academic Editor: Niansheng Tang

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In this study, a three-parameter modification of the Burr XII distribution has been developed through the integration of the weighted version of the alpha power transformation family of distributions. This newly introduced model, termed the modified alpha power-transformed Burr XII distribution, exhibits the unique ability to effectively model decreasing, right-skewed, or unimodal densities. The paper systematically elucidates various statistical properties of the proposed distribution. The estimation of parameters was obtained using maximum likelihood estimation. The estimator has been evaluated for consistency through simulation studies. To gauge the practical applicability of the proposed distribution, two distinct datasets have been employed. Comparative analyses involving six alternative distributions unequivocally demonstrate that the modified alpha power-transformed Burr XII distribution provides a better fit. Additionally, a noteworthy extension is introduced in the form of a location-scale regression model known as the log-modified alpha power-transformed Burr XII model. This model is subsequently applied to a dataset related to stock market liquidity. The findings underscore the enhanced fitting capabilities of the proposed model in comparison to existing distributions, providing valuable insights for applications in financial modelling and analysis.

1. Introduction

The development of novel or new statistical models is a key area of study in the application of distribution theory. These distributions' usefulness has led to much research into their theory and the development of new distributions. The idea of generating new continuous distributions by modifying the existing distributions with one or more shape or scale parameters has gained attention in recent years. This parameter introduction has been shown to improve the ability of the developed distributions to fit varied real-life datasets with high degrees of skewness and kurtosis. Some of these newly developed distributions include the modified alpha power transformed Weibull [1], general two-parameter [2], truncated inverse power Ailamujia [3], half-logistic modified Kies exponential [4], truncated inverse power Lindley [5], Marshall–Olkin–Weibull–Burr XII [6], generalised unit half-logistic geometric [7], Chen Burr–Hatke exponential [8], modified XLindley [9], arctan power [10], harmonic mixture Fréchet [11], sine-Weibull geometric [12], bounded odd inverse Pareto exponential [13], new extended Chen [14], power XLindley [15], extended Poisson–Fréchet [16], exponentiated Fréchet loss [17], Gompertz–Makeham [18], and logistic exponential [19] distributions.

The authors of [20] introduced a new method by adding an additional parameter called the alpha power transformation (APT) family. The APT family has been used to develop several modified distributions, including the APT Fréchet [21], APT extended exponential distribution [22], APT inverse Lomax distribution [23], APT log-logistic distribution [24], APT inverse Lindley distribution [25], and APT Pareto distribution [26], among others. With the aim of improving the flexibility of the APT family of distributions, Alotaibi et al. [27] modified the APT family of distributions and obtained a new family of distributions called the modified alpha power transformed method (MAPT). According to Alotaibi et al. [27], the cumulative distribution function (CDF) and probability density function (PDF) of the MAPT are given as follows:

$$G_{\text{MAPT}}(x) = \frac{\alpha^{F(x)} - 1}{(\alpha - 1)\left(1 + \alpha - \alpha^{F(x)}\right)}, \quad \alpha > 0, \alpha \neq 1, \tag{1}$$

and

$$g_{\text{MAPT}}(x) = \frac{\alpha^{1+F(x)} \log(\alpha) f(x)}{(\alpha - 1)\left(1 + \alpha - \alpha^{F(x)}\right)^2}, \quad \alpha > 0, \alpha \neq 1. \tag{2}$$

This study proposes a modification of the Burr XII distribution using the MAPT proposed byAlotaibi et al. [27]. The Burr XII distribution introduced by Burr [28] is widely used in reliability analysis, actuarial studies, medicine, and agriculture. The PDF and CDF of the Burr XII distribution can be expressed, respectively, as follows:

$$f(x; \xi, \gamma) = \xi\gamma x^{\xi-1}\left(1 + x^{\xi}\right)^{-\gamma-1}, \quad x > 0, \xi > 0, \gamma > 0, \tag{3}$$

and

$$F(x; \xi, \gamma) = 1 - \left(1 + x^{\xi}\right)^{-\gamma}, \quad x > 0, \xi > 0, \gamma > 0. \tag{4}$$

We are motivated to contribute to the ongoing efforts to enhance the versatility of statistical distributions, thus providing researchers with a powerful tool to analyse and model diverse data scenarios effectively as no single distribution is omnibus. Specifically, our motivations for developing the modified alpha power transformed Burr XII (MAPTBXII) distribution are as follows:

(i) Develop an extension of the Burr XII distribution that provides a good parametric fit to data with complex traits

(ii) Propose a new Burr XII distribution with closed form CDF and tractable quantile function that facilitates easy generation of random observations for simulation experiments

(iii) Formulate a location-scale regression model using the proposed distribution

The subsequent sections of the paper are organised as follows. In Section 2, we develop the MAPTBXII distribution. Section 3 is devoted to deriving various statistical properties of the MAPTBXII distribution. The parameters of the MAPTBXII distribution are estimated through the maximum likelihood estimation method, as detailed in Section 4. Section 5 delves into the discussion of the MAPTBXII regression model. A comprehensive simulation study is presented in Section 6. Real-world applications of the MAPTBXII distribution on two datasets are presented in Section 7. Finally, the paper is concluded in Section 8.

2. Modified Alpha Power Transformed Burr XII Distribution

If a random variable X follows the MAPTBXII, then the PDF can be obtained by substituting equation (3) into equation (2), while the CDF can be obtained by substituting equation (4) into equation (1). The CDF and PDF of the MAPTBXII can then be expressed, respectively, as follows:

$$\begin{aligned} G_{\text{MAPT}}(x) &= \frac{\alpha^{1-\left(1+x^{\xi}\right)^{-\gamma}} - 1}{(\alpha - 1)\left(1 + \alpha - \alpha^{1-\left(1+x^{\xi}\right)^{-\gamma}}\right)}, \\ g_{\text{MAPT}}(x) &= \frac{\alpha^{2-\left(1+x^{\xi}\right)^{-\gamma}} \log(\alpha)\xi\gamma x^{\xi-1}\left(1 + x^{\xi}\right)^{-\gamma-1}}{(\alpha - 1)\left(1 + \alpha - \alpha^{1-\left(1+x^{\xi}\right)^{-\gamma}}\right)^2}, \end{aligned} \tag{5}$$

where $\alpha > 0$, $\xi > 0$, $\gamma > 0$, and $\alpha \neq 1$.

The hazard function of the MAPTBXII is obtained by finding the ratio of the PDF and complement of the CDF. The hazard function is given by the following equation:

$$h_{\text{MAPT}}(x) = \frac{\alpha^{1-\left(1+x^{\xi}\right)^{-\gamma}} \log(\alpha)\xi\gamma x^{\xi-1}\left(1 + x^{\xi}\right)^{-\gamma-1}}{\left(1 + \alpha - \alpha^{1-\left(1+x^{\xi}\right)^{-\gamma}}\right)\left(\alpha - \alpha^{1-\left(1+x^{\xi}\right)^{-\gamma}}\right)}. \tag{6}$$

The various shapes (decreasing, right-skewed, left-skewed, or unimodal) of the densities of the MAPTBXII distribution are shown in Figure 1.

The various shapes (decreasing or upside down bathtub) of the hazard function are displayed in Figure 2.

The PDF of the MAPTBXII distribution in linear form is given as follows:

$$g(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \eta_{ijm}\xi\gamma x^{\xi-1}\left(1 + x^{\xi}\right)^{-\gamma(m+1)-1}, \tag{7}$$

where $\eta_{ijm} = \left(\alpha(-1)^m (i+1)^{j+1} \quad (\log\alpha)^{j+1}\binom{j}{m}/(\alpha - 1)\right.$ $\left.(\alpha + 1)^{i+2} j!\right)$, $x > 0$, $\xi > 0, \gamma > 0$, and $\alpha > 0$.

The asymptotic nature of the CDF, PDF, and hazard function is as follows:

$G_{\text{MAPT}}(x) \longrightarrow 0$, as $x \longrightarrow 0$ and $G_{\text{MAPT}}(x) \longrightarrow 1$, as $x \longrightarrow \infty$

$g_{\text{MAPT}}(x) \longrightarrow 0$, as $x \longrightarrow 0$ and $g_{\text{MAPT}}(x) \longrightarrow 0$, as $x \longrightarrow \infty$

$h_{\text{MAPT}}(x) \longrightarrow 0$, as $x \longrightarrow 0$ and $h_{\text{MAPT}}(x) \longrightarrow \infty$, as $x \longrightarrow \infty$

The quantile function of MAPTBXII distribution is given as follows:

$$x_p = \left\{\left[1 - \frac{\log\left(p\left(\alpha^2 - 1\right) + 1/1 + p(\alpha - 1)\right)}{\log\alpha}\right]^{-1/\gamma} - 1\right\}^{1/\xi}, \tag{8}$$

where $p \in (0, 1)$ and $Q(p) = x_p$ is the quantile function.

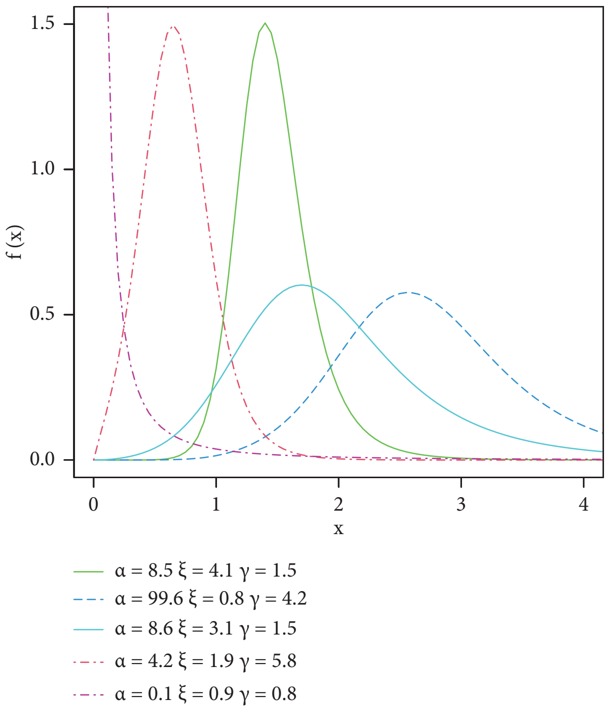

FIGURE 1: Densities of the MAPTBXII distribution.

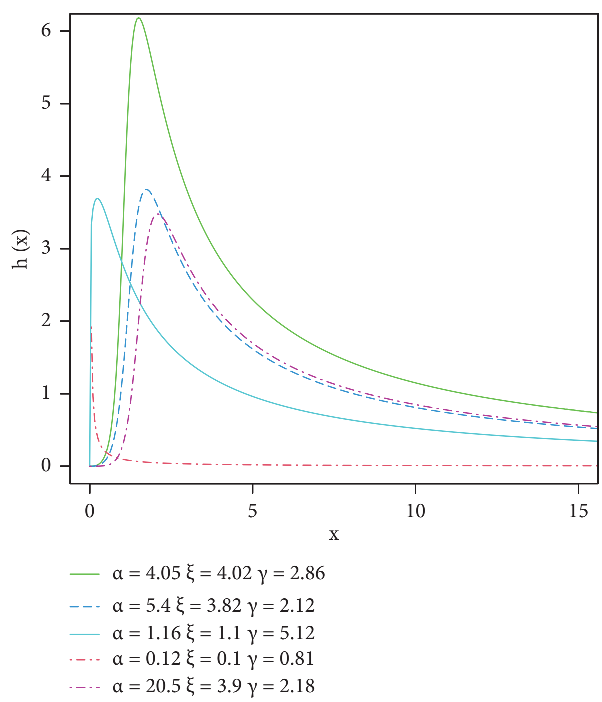

FIGURE 2: The hazard function plots of the MAPTBXII.

3. Statistical Properties

3.1. Moments. Moments are a crucial component of statistical theory, and they allow for the examination of many essential characteristics of any distribution.

Mathematically, the r^{th} moments of X are given by

$$\mu_r' = E\left(X^r\right) = \int_0^\infty x^r f(x)\mathrm{d}x. \tag{9}$$

We substitute equation (7) into equation (9), in order to obtain

$$\mu_r' = \xi\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \eta_{ijm} \int_0^\infty x^{r+\xi-1}\left(1+x^{\xi}\right)^{-\gamma(m+1)-1}\mathrm{d}x. \tag{10}$$

Letting $\nu = x^{\xi}$, which implies $x = \nu^{1/\xi}$ and $\mathrm{d}x = (1/\xi)\nu^{1/\xi-1}d\nu$, we obtain the following expression:

$$\mu_r' = \gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \eta_{ijm} \int_0^\infty \nu^{r/\xi}(1+\nu)^{-\gamma(m+1)-1}\mathrm{d}\nu. \tag{11}$$

Utilising the identity used in [29],

$$\mathscr{B}(c,d) = \int_0^\infty \nu^{c-1}(1+\nu)^{-(c+d)}\mathrm{d}\nu, c>0, d>0, \tag{12}$$

then gives

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \eta_{ijm}\gamma\mathscr{B}\left(\frac{r}{\xi}+1, \gamma(m+1)-\frac{r}{\xi}\right). \tag{13}$$

It follows that the r^{th} moment of the MAPTBXII distribution can be expressed as follows:

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \eta_{ijm}\gamma\mathscr{B}\left(\frac{r}{\xi}+1, \gamma(m+1)-\frac{r}{\xi}\right), \tag{14}$$

where $\mathscr{B}(\cdot,\cdot)$ is given as the beta function and $r = 1, 2, \ldots$.

Metrics such as variance (σ^2), coefficient of variation (CVAR), skewness (CSK), and kurtosis (CKUR) can be derived by utilizing moment-based computations. μ, σ^2, CVAR, CSK, and CKUR, respectively, are obtained using the following expression:

$$\begin{aligned}
\sigma^2 &= \mu_2' - (\mu)^2,\\
\text{CVAR} &= \frac{\sqrt{\mu_2' - (\mu)^2}}{\mu},\\
\text{CSK} &= \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{\sigma^3},\\
\text{CKUR} &= \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{\sigma^4}.
\end{aligned} \tag{15}$$

Table 1 displays the σ^2, CVAR, CSK, and CKUR of the MAPTBXII distribution, calculated using noncentral moments for specific parameter values. The results suggest that the MAPTBXII distribution exhibits a varying degree of skewness, ranging from highly skewed to moderately skewed. It is noteworthy that the skewness of the distribution may differ depending on the chosen parameter values. Some combinations lead to a positively skewed distribution, while others result in a negatively skewed distribution. This variability underscores the distribution's flexibility in capturing different skewness patterns based on the specified parameters. The MAPTBXII distribution is leptokurtic (CKUR > 3) and thus has heavier tails and a sharper peak.

3.2. Incomplete Moments. The incomplete moments can be used to calculate the Lorenz curve, the Bonferroni curve, the mean deviation, and the median deviation, among others.

Mathematically, the incomplete moment is given as follows:

$$m_r(z) = E\left(X^r \mid X \le z\right) = \int_0^z x^r f(x)\mathrm{dx}. \tag{16}$$

We substitute equation (7) into equation (16), in order to obtain

$$m_r(z) = \xi\gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \eta_{ijm} \int_0^z x^{r+\xi-1}\left(1+x^{\xi}\right)^{-\gamma(m+1)-1}\mathrm{dx}. \tag{17}$$

Letting $v = x^{\xi}$, which implies $x = v^{1/\xi}$ and $\mathrm{dx} = (1/\xi)v^{1/\xi-1}\mathrm{dv}$, we obtain the following expression:

$$m_r(z) = \gamma \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \eta_{ijm} \int_0^{z^{\xi}} v^{r/\xi}(1+v)^{-\gamma(m+1)-1}\mathrm{dv}. \tag{18}$$

Utilising the identity used in [29],

$$\mathscr{B}(c,d) = \int_0^{\infty} v^{c-1}(1+v)^{-(c+d)}\mathrm{dv}, c>0, d>0, \tag{19}$$

helps to obtain

$$m_r(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \eta_{ijm}\gamma\mathscr{B}\left(z^{\xi}\colon \frac{r}{\xi}+1, \gamma(m+1)-\frac{r}{\xi}\right). \tag{20}$$

It follows that the r^{th} incomplete moment of the MAPTBXII distribution is given by the following expression:

$$m_r(z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \eta_{ijm}\gamma\mathscr{B}\left(z^{\xi}\colon \frac{r}{\xi}+1, \gamma(m+1)-\frac{r}{\xi}\right), \tag{21}$$

where $\mathscr{B}(\cdot\colon \cdot,\cdot)$ is given as the incomplete beta function and $r = 1, 2, 3, \ldots$.

3.3. Moment Generating Function. The moment generating function (MGF) is used to determine the distribution's moments, if any. Mathematically,

$$M(t) = E\left(e^{tX}\right) = \sum_{r=0}^{\infty} \frac{t^r E\left(X^r\right)}{r!} = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'. \tag{22}$$

When we substitute equation (14) into equation (22), we obtain the MGF of the MAPTBXII distribution. Hence, the MGF of the MAPTBXII distribution is given by the following expression:

$$M(t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \eta_{ijm}\frac{t^r}{r!}\left[\gamma\mathscr{B}\left(\frac{r}{\xi}+1, \gamma(m+1)-\frac{r}{\xi}\right)\right]. \tag{23}$$

4. Estimation of Parameters

The maximum likelihood estimation (MLE) method is used as an estimator. The log-likelihood function is therefore given as

$$\begin{aligned}\ell(x,\alpha,\xi,\gamma) = &\sum_{i=1}^{n}\left(2-\left(1+x_i^{\xi}\right)^{-\gamma}\right)\log(\alpha) + n\log^2(\alpha) + n\log(\xi\gamma) + (\xi-1)\sum_{i=1}^{n}\log\ x_i \\ &- (\gamma+1)\sum_{i=1}^{n}\left(1+x_i^{\xi}\right) - n\log(\alpha-1) - 2\sum_{i=1}^{n}\log\left(1+\alpha-\alpha^{1-\left(1+x^{\xi}\right)^{-\gamma}}\right).\end{aligned} \tag{24}$$

By maximising equation (24), we obtain the parameter estimates.

5. Modified Alpha Power Transformed Burr XII Regression Model

Suppose X follows the MAPTBXII distribution and $Y = \log(\beta X)$. The PDF of Y can be derived by substituting $c = 1/\sigma$ and $\beta = \exp(\mu)$.

$$f(y) = \frac{\alpha^{2-(1+\exp((y-\mu)/\sigma))^{-\gamma}}((\log(\alpha)\gamma)/\sigma)\exp((y-\mu)/\sigma)(1+\exp((y-\mu)/\sigma))^{-\gamma-1}}{(\alpha-1)\left(1+\alpha-\alpha^{1-(1+\exp((y-\mu)/\sigma))^{-\gamma}}\right)^2}, \tag{25}$$

Table 1: First five moments varying parameter values.

r	$\alpha = 10.1,\ \xi = 10.9,\ \gamma = 5.8$	$\alpha = 9.5,\ \xi = 6.0,\ \gamma = 1.5$	$\alpha = 7.0,\ \xi = 2.1,\ \gamma = 9.9$	$\alpha = 15.2,\ \xi = 8.5,\ \gamma = 8.5$	$\alpha = 0.5,\ \xi = 5.5,\ \gamma = 1.05$
μ_1'	0.9693	1.4281	0.5907	0.9285	0.8634
μ_2'	0.9439	2.1384	0.3825	0.8666	0.8334
μ_3'	0.9230	3.3650	0.2662	0.8129	0.9209
μ_4'	0.9060	5.5974	0.1973	0.7655	1.2436
μ_5'	0.8926	9.9600	0.1553	0.7240	2.6232
σ^2	0.0043	0.0991	0.0335	0.0045	0.0879
CVAR	0.0673	0.2204	0.3099	0.0723	0.3435
CSK	−0.9618	0.9027	0.0936	−1.0168	1.8945
CKUR	5.9696	6.6343	3.5312	6.2421	16.0123

where $y \in R$, $\sigma > 0$ is the scale parameter, $\gamma > 0$ is the shape parameter, $\alpha > 0$, and $\mu \in \mathbb{R}$ is the location parameter.

Equation (25) is referred to as the log-MAPT Burr XII (LMAPTBXII) distribution. If $X \sim \mathrm{MAPTBXII}(\alpha, \xi, \gamma)$, then $Y = \log(\beta X) \sim \mathrm{LMAPTBXII}(\alpha, \xi, \gamma, \mu)$.

Figure 3 shows various shapes of the densities of the LMAPTBXII distribution.

The survival function of the LMAPTBXII distribution can be expressed as follows:

$$S(y) = 1 - \frac{\alpha^{1-(1+\exp((y-\mu)/\sigma))^{-\gamma}} - 1}{(\alpha - 1)\left(1 + \alpha - \alpha^{1-(1+\exp((y-\mu)/\sigma))^{-\gamma}}\right)}. \tag{26}$$

The proposed location-scale regression model is defined with the dependent variable y_j and predictor variables $z_j' = (1, z_{j1}, \ldots, z_{jp})$, where **1** is the intercept and expressed as follows:

$$y_j = Z_j'\boldsymbol{\beta} + \sigma W_j, \tag{27}$$

where $j = 1, 2, 3, \ldots, n$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3, \ldots \beta_p)'$ are the regression parameters, and W_j denotes the random error.

The log-likelihood function of the LMAPTBXII model is given by the following expression:

$$\begin{aligned}\ell = {} & \sum_{j=1}^{n}\left(2 - \left(1 + \exp\left(\frac{y_j - \mu_j}{\sigma}\right)\right)^{-\gamma}\right)\log(\alpha) + n\log^2(\alpha) + n\log\left(\frac{\gamma}{\sigma}\right) + \sum_{j=1}^{n}\left(\frac{y_j - \mu_j}{\sigma}\right) \\ & - (\gamma + 1)\sum_{j=1}^{n}\left(1 + \exp\left(\frac{y_j - \mu_j}{\sigma}\right)\right) - n\log(\alpha - 1) - 2\sum_{j=1}^{n}\log\left(1 + \alpha - \alpha^{1-\left(1+\exp\left(\left(y_j-\mu_j\right)/\sigma\right)\right)^{-\gamma}}\right).\end{aligned} \tag{28}$$

The parameter estimates of the model are obtained by maximising the log-likelihood function. An assessment of the Cox–Snell residuals to ascertain if they behave as a standard exponential distribution would help determine the adequacy of the model. We then diagnose the model using the goodness-of-fit measures (Cramér–von Mises, Anderson–Darling, and Kolmogorov–Smirnov) of the Cox–Snell residuals.

6. Monte Carlo Simulations

In this section, we ascertain the consistency of the estimators of the MAPTBXII distribution through a simulation study. The results were obtained using sample sizes of 50, 100, 200, 250, 300, 350, 500, and 600 with parameter values α = (6.85, 10.65, 1.65), c = (0.1, 6.10, 1.90), and k = (0.5, 10.5, 6.50), respectively. It can be observed that the average biases (AB) and root mean square error (RMSE) decrease as the sample size increases as shown in Tables 2.

7. Applications

In this section, we provide the applications of the MAPTBXII distribution using two uncensored datasets.

The precipitation (in inches) in the Minneapolis dataset was used in [30, 31]. The second dataset represents runoff amounts at Jug Bridge, Maryland, and was used by Makubate et al. [32]. The performance of the MAPTBXII distribution is compared with the Topp–Leone Burr-XII distribution (TLBXII) [33], Weibull Burr XII distribution (WBXII) [29], inverse Weibull distribution (INW) [34], exponentiated exponential Burr XII distribution (EEBXII) [35], Cauchy [36], and inverse Gompertz distribution (IGD) [37].

Analytical measures such as Kolmogorov–Smirnov (K-S) test, Anderson–Darling test (AD), and Cramér–von Mises test (CVM), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICc), and Bayesian Information Criterion (BIC) are considered in evaluating the

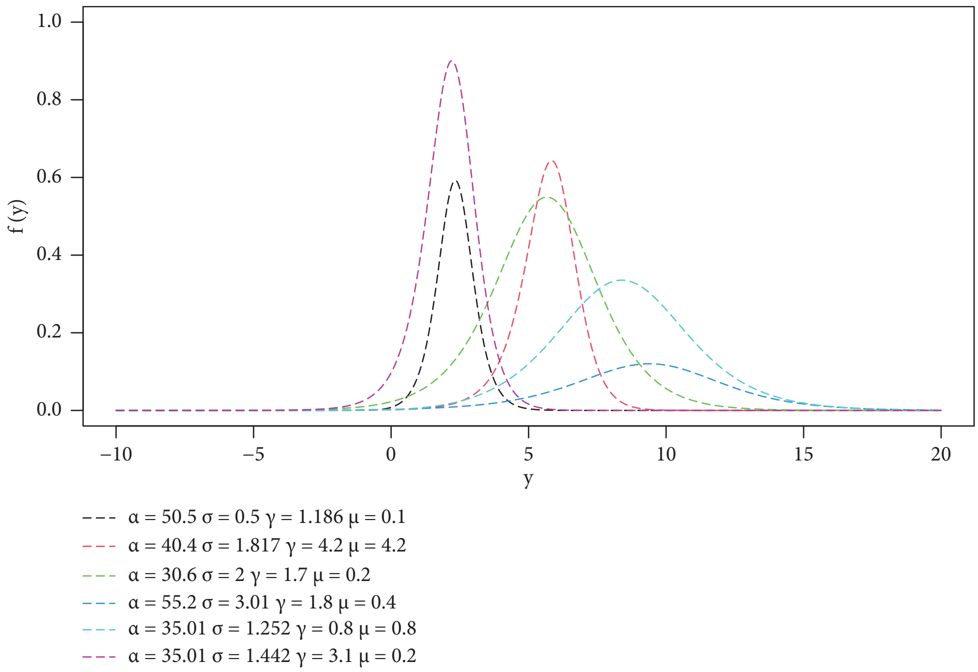

Figure 3: The PDF plot of the LMAPTBXII distribution.

goodness of fit of the proposed distribution and the other fitted models.

The maximum likelihood estimates and other goodness-of-fit statistics for the two datasets are presented in Tables 3 and 4.

The MLEs of the MAPTBXII model for datasets 1 and 2 are both unique and represent real maxima, as demonstrated by the profile log-likelihood plots shown in Figures 4 and 5.

The fitted densities are presented in Figures 6 and 7, whereas the fitted CDFs are shown in Figures 8 and 9. It can be observed from these results that the MAPTBXII provides a better fit to the two datasets than the other competing lifetime distributions.

7.1. Application of Modified Alpha Power Transformed Burr XII Regression Model. The LMAPTBXII model was employed in the analysis of a dataset related to stock market liquidity. Dataset can be retrieved from https://instruction.bus.wisc.edu/jfrees/jfreesbooks/Regression20Modeling/BookWebDec2010/data.html (accessed on 8 January 2023). The competing models are the log-harmonic mixture Burr XII (LHMBXII) distribution [31] and the log-Gumbel Burr XII (LGBXII) distribution [38]. The response variable y_j is the total number of shares that were traded on an exchange during a specific period (volume), while the covariate is the number of shares outstanding as of December 31, 1984, expressed in millions of shares (shares) (z_{j1}). The fitted model is given by the following expression:

$$y_j = \beta_0 + \beta_1 z_{j1}. \tag{29}$$

The response variable data are 16.221, 5.693, 11.965, 3.834, 13.235, 0.658, 13.794, 2.009, 27.600, 18.515, 23.466, 18.192, 28.163, 4.428, 4.912, 9.802, 15.513, 10.781, 11.149, 7.561, 14.969, 2.726, 17.275, 14.570, 11.967, 4.155, 3.072, 5.872, 17.744, 4.351, 16.995, 1.902, 19.727, 12.986, 2.823, 27.106, 46.832, 2.584, 18.056, 9.180, 7.999, 27.070, 8.914, 7.141, 28.574, 5.992, 2.118, 37.720, 13.080, 12.249, 19.874, 15.006, 4.503, 6.110, 12.238, 6.268, 5.810, 16.576, 33.565, 64.572, 3.000, 3.842, 16.242, 34.453, 21.718, 25.377, 8.317, 15.255, 4.971, 4.277, 2.473, 6.798, 10.848, 2.908, 3.695, 12.074, 37.923, 10.646, 19.624, 15.165, 4.054, 21.249, 8.726, 8.409, 34.073, 14.464, 8.903, 11.556, 5.067, 14.148, 4.712, 14.855, 6.180, 9.457, 6.170, 7.238, 9.877, 13.001, 12.967, 30.597, 15.134, 1.365, 39.273, 19.387, 15.505, 6.810, 18.001, 7.093, 3.791, 2.638, 18.960, 6.927, 19.436, 5.098, 16.257, 2.805, 10.961, 6.299, 5.971, 13.477, 11.716, 40.585, and 20.430.

The covariate's data are 81.141, 27.088, 189.680, 13.492, 72.600, 6.736, 107.743, 20.851, 220.776, 79.964, 97.225, 151.508, 141.309, 21.433, 27.309, 50.338, 137.269, 73.200, 46.544, 29.718, 38.301, 19.683, 92.775, 72.127, 83.009, 16.049, 13.689, 15.161, 176.210, 13.524, 39.761, 8.724, 190.192, 20.232, 26.879, 155.600, 172.200, 12.402, 46.600, 78.946, 46.870, 126.268, 67.662, 43.206, 454.876, 50.366, 20.402, 315.451, 70.039, 35.408, 108.496, 34.787, 280.173, 49.342, 46.713, 40.222, 55.816, 96.849, 256.478, 612.686, 36.720, 26.756, 49.942, 139.747, 182.845, 125.229, 52.777, 130.825, 41.947, 23.261, 27.168, 29.495, 87.213, 16.209, 30.065, 116.298, 407.704, 72.103, 189.167, 78.281, 9.121, 99.637, 46.101, 62.933, 87.766, 115.561, 21.716, 19.896, 29.574, 95.962, 22.525, 93.908, 30.959, 16.399, 44.673, 86.376, 38.180, 103.353, 84.354, 361.610, 59.854, 11.217, 300.702, 271.429,

Table 2: Simulation results.

n	Parameter	I: $\alpha=6.85,\ \xi=0.10,\ \gamma=0.50$		II: $\alpha=10.65,\ \xi=6.10,\ \gamma=10.50$		III: $\alpha=1.65,\ \xi=1.90,\ \gamma=6.50$	
		AB	RMSE	AB	RMSE	AB	RMSE
50	α	−1.5913	2.1356	1.1832	7.1186	0.8491	4.1517
	ξ	0.3191	0.7798	0.3079	1.1701	0.1043	1.5802
	γ	−0.2135	0.2571	−0.2841	1.0615	0.3718	3.3300
100	α	1.7718	2.5905	0.6330	7.9682	0.8389	3.9530
	ξ	−0.0076	0.0213	1.3033	3.1003	−0.0291	1.3054
	γ	0.1162	0.2155	−0.6349	1.7717	0.2448	2.4441
150	α	0.0324	1.2733	0.0324	1.2734	0.5711	3.2103
	ξ	0.0108	0.0468	0.1374	1.7029	−0.0034	1.1086
	γ	0.0220	0.1550	−0.1377	1.4706	0.1301	2.0427
200	α	−0.0261	0.7855	1.3514	7.8902	0.4383	2.5582
	ξ	0.0347	0.0729	1.3073	3.0935	−0.0452	0.9946
	γ	−0.0402	0.1925	−0.4977	1.8115	0.1107	1.7736
250	α	1.1190	1.8270	1.4778	8.2049	0.3693	2.1327
	ξ	0.0101	0.0439	0.6561	1.9062	−0.0679	0.9441
	γ	0.0385	0.1740	−0.3780	1.7054	0.0295	1.6772
300	α	0.3728	1.3912	0.4715	7.0629	0.2919	1.9030
	ξ	0.0195	0.0409	0.8568	2.2707	−0.0316	0.8298
	γ	−0.0357	0.1590	−0.3984	1.4509	0.0541	1.5455
350	α	0.4073	1.0639	−2.7935	6.6380	0.2249	1.5689
	ξ	0.0103	0.0446	1.8828	3.0989	−0.0242	0.7604
	γ	0.1590	0.0240	−0.9480	1.5690	0.0614	1.3707
500	α	0.4397	1.0547	−0.2573	5.7444	0.1156	1.1918
	ξ	0.0125	0.0306	0.6320	1.4910	−0.0049	0.6104
	γ	−0.0152	0.1335	−0.3102	1.2754	0.0049	1.1232
600	α	0.2089	1.0249	0.0530	5.2173	0.0989	0.7007
	ξ	0.0072	0.0179	0.4432	1.3874	−0.0523	0.5460
	γ	−0.0226	0.0839	−0.3068	1.1778	0.0829	1.0102

Table 3: MLEs and goodness-of-fit statistics for data 1.

Distribution	Parameter	Estimates (standard error)	LL	AIC	BIC	AICc	CVM	AD	KS
MAPTBXII	α ξ γ	3.5493 (3.42083) 1.9921 (0.4788) 1.7503 (0.6556)	−38.67247	83.34494	87.54853	84.26801	0.0179 (0.9988)	0.1289 (0.9996)	0.0695 (0.9987)
TLBXII	α c k	3.6562 (4.5116) 1.4767 (0.9438) 0.9366 (0.7727)	−39.3226	84.6453	88.8489	85.5683	0.0498 (0.8813)	0.3116 (0.9286)	0.1071 (0.8814)
WBXII	α c k λ	1.7914 (1.0932) 0.3592 (0.2740) 1.9329 (3.0939) 1.4312 (0.7448)	−38.1390	84.2779	89.8827	85.8779	0.0198 (0.9969)	0.1012 (0.9281)	0.0590 (0.9110)
INW	α c k	−1.2170 (0.6186) 3.5163 (0.9899) 0.0494 (0.0860)	−39.0394	84.0788	88.2824	85.0018	0.0198 (0.9561)	0.1909 (0.9320)	0.0901 (0.9190)
EEBXII	α β ξ γ	6.5384 (5.1732) 0.38550 (3.0890) 0.8343 (0.3642) 1.6775 (0.9982)	−38.5193	85.0386	90.6434	86.6386	0.0262 (0.9882)	0.1703 (0.9966)	0.0880 (0.9743)
Cauchy	α c	0.54696 (0.12694) 1.42381 (0.16552)	−45.42425	94.8485	97.6509	95.29293	0.0982 (0.5970)	0.7734 (0.4994)	0.1464 (0.5407)
IGD	α β	0.8155 (0.2265) 0.4408 (0.2389)	−44.7779	93.5558	96.3582	94.0002	0.2199 (0.2324)	1.4123 (0.1989)	0.1945 (0.2066)

Table 4: MLEs and goodness-of-fit statistics for data 2.

Distribution	Parameter	Estimates (standard error)	LL	AIC	BIC	AICc	CVM	AD	KS
MAPTBXII	α	0.6254 (0.5596)	−14.82152	35.6431	39.3000	36.7859	0.0134 (0.9999)	0.0988 (0.9999)	0.0658 (0.9999)
	ξ	2.6675 (0.6979)							
	γ	1.2310 (1.0149)							
TLBXII	α	3.0443 (7.3218)	−14.94482	35.8896	39.4463	36.6325	0.013647 (0.9999)	0.10498 (0.9891)	0.0681 (0.9998)
	c	1.3370 (1.3994)							
	k	1.6508 (1.6972)							
WBXII	α	4.0853 (5.7271)	−14.9478	37.8956	42.7711	39.8956	0.0197 (0.9978)	0.1288 (0.9967)	0.0899 (0.9574)
	c	0.2537 (1.0058)							
	k	3.0761 (11.5223)							
	λ	0.5274 (0.8721)							
INW	α	−0.9218 (0.4865)	−14.8267	35.6544	39.3100	36.7963	0.1929 (0.9789)	0.1009 (0.9620)	0.070126 (0.9567)
	c	4.4347 (1.5007)							
	k	0.1682 (0.3402)							
EEBXII	α	1.8269 (2.0841)	−14.7537	37.5073	42.3828	39.5073	0.0135 (0.9999)	0.1039 (0.9999)	0.0720 (0.9999)
	β	0.1951 (0.2949)							
	ξ	1.9949 (0.7099)							
	γ	10.3041 (14.3761)							
Cauchy	α	0.2466 (0.0655)	−19.8268	43.6537	46.0914	44.1991	0.1077 (0.5518)	0.1077 (0.5518)	0.1484 (0.6411)
	c	0.6602 (0.0766)							
IGD	α	0.3864 (0.1193)	−19.2401	42.4802	44.9179	43.0256	0.1704 (0.3346)	1.1093 (0.3041)	0.1726 (0.4457)
	β	0.2533 (0.1334)							

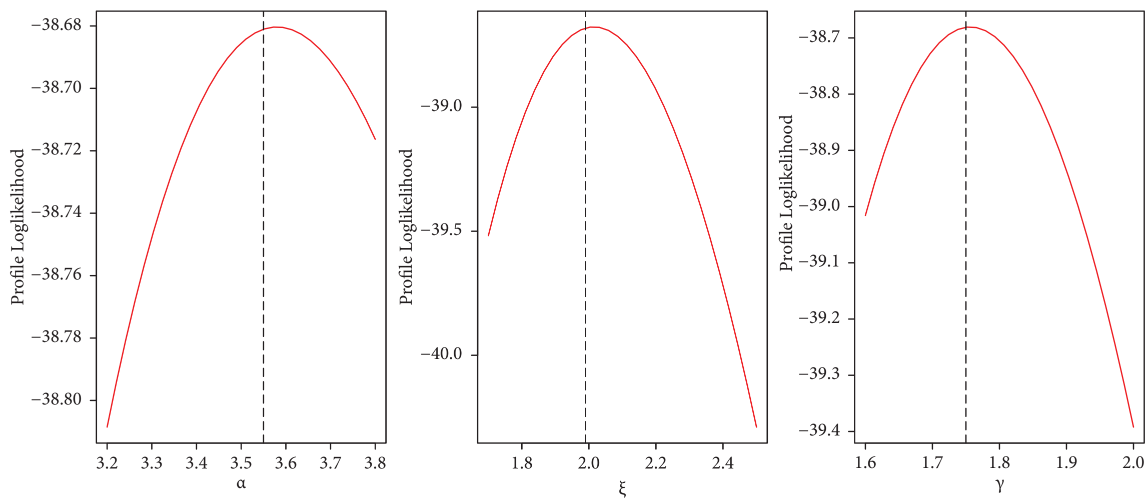

FIGURE 4: Profile log-likelihood of the MAPTBXII model for dataset 1.

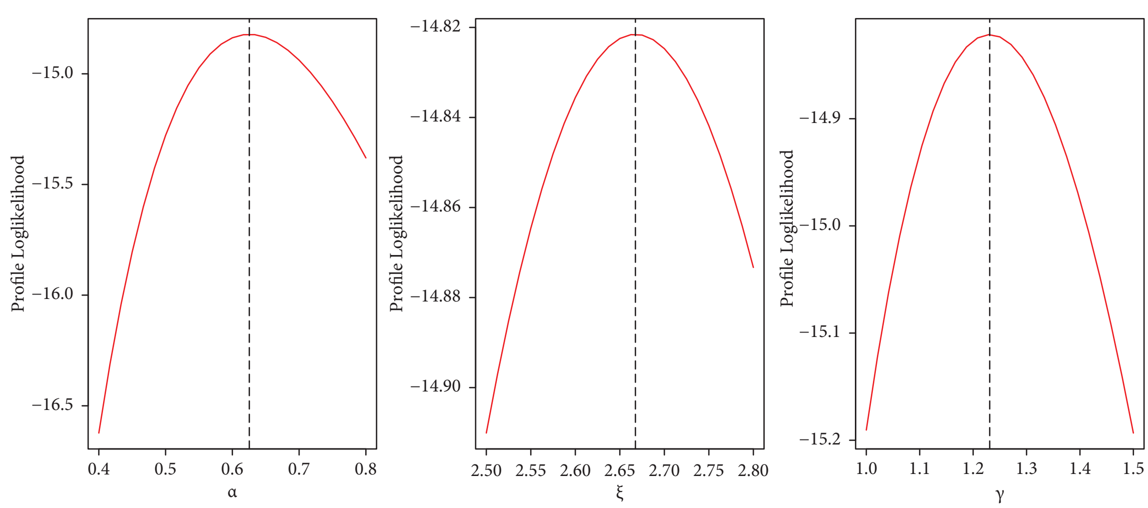

FIGURE 5: Profile log-likelihood of the MAPTBXII model for dataset 2.

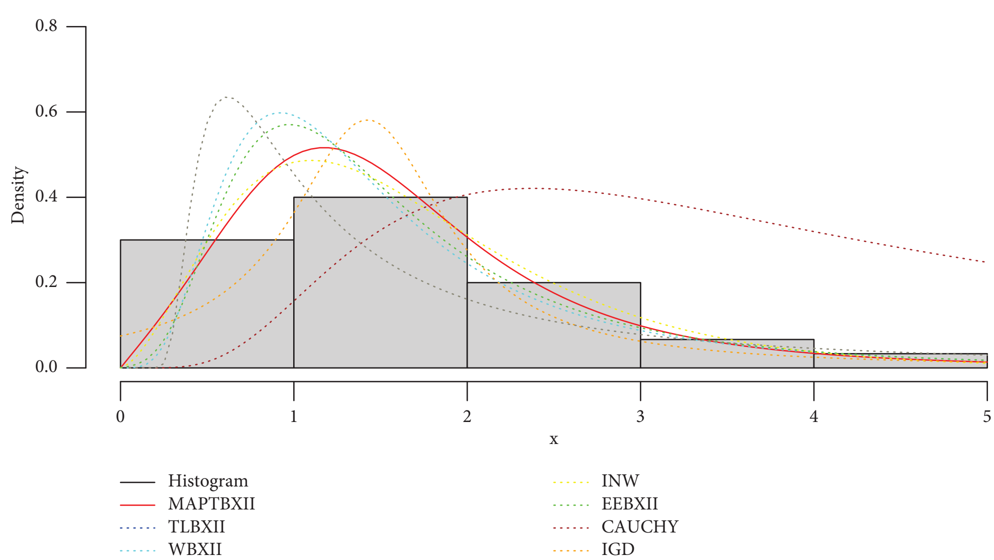

FIGURE 6: Fitted densities for data 1.

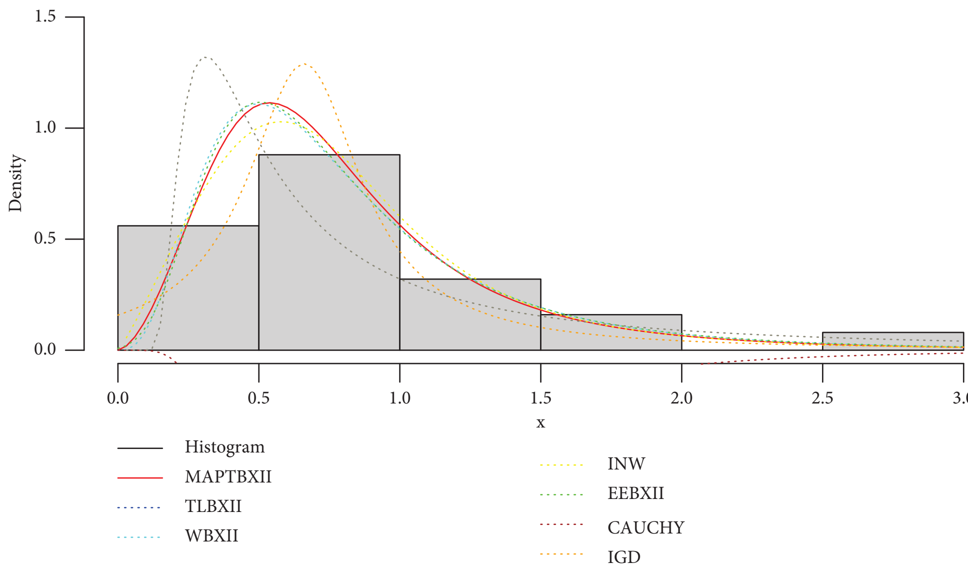

Figure 7: Fitted densities for data 2.

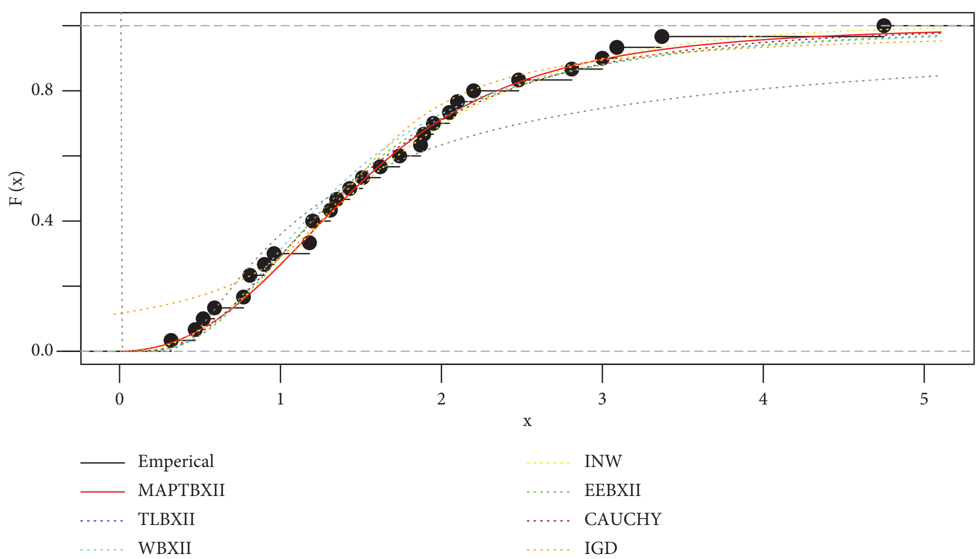

Figure 8: Fitted CDFs for data 1.

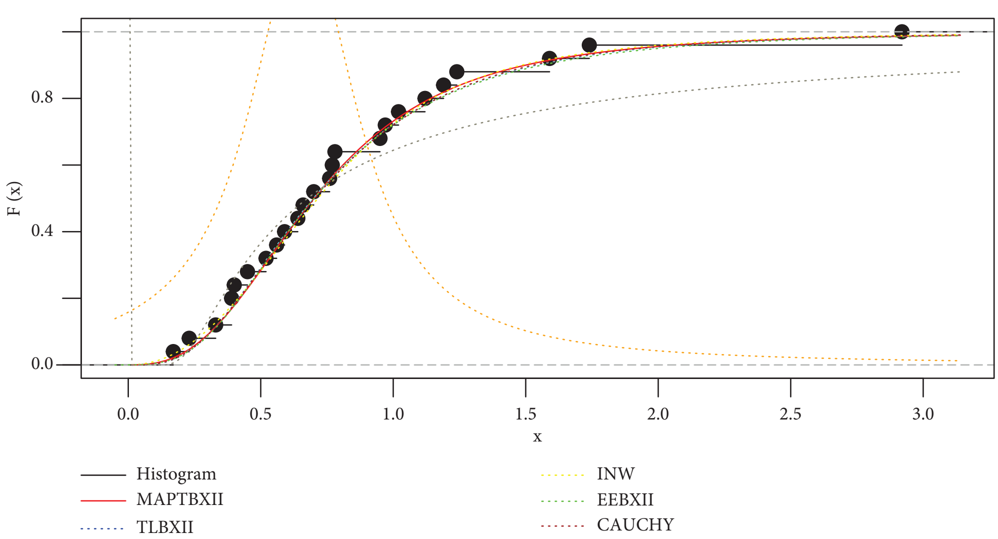

Figure 9: Fitted CDFs for data 2.

Table 5: Parameter estimates and selection criteria.

	Parameter	Estimates	p values	
LMAPTBXII	β_0	2.3975 (1.1464)	0.0365	
	β_1	0.0574 (0.0070)	2.2000×10^{-16}	
	α	0.8959 (0.3582)	0.0124	$\ell = -394.1776$
	k	0.2272 (0.0853)	0.0077	AIC = 798.3552
	σ	1.5234 (0.4471)	0.0007	BIC = 812.4162
LGBXII	β_0	−2.2341 (1.9840)	0.2601	
	β_1	0.0794 (0.0134)	3.2610×10^{-9}	
	β	5.7279 (2.2123)	0.0096	$\ell = -394.6443$
	k	8.2028 (5.7446)	0.1533	AIC = 801.2887
	σ	7.5979 (3.3597)	0.0237	BIC = 818.1618
	τ	4.2309 (1.8388)	0.0214	
LHMBXII	β_0	1.8535 (0.7882)	0.0187	
	β_1	0.0550 (0.0063)	2.2000×10^{-16}	
	θ	0.2648 (0.3633)	0.4661	$\ell = -394.5351$
	α	5.2756 (2.1225)	0.0129	AIC = 801.0701
	υ	0.0447 (0.0179)	0.0123	BIC = 817.9432
	σ	1.4440 (0.4401)	0.0010	

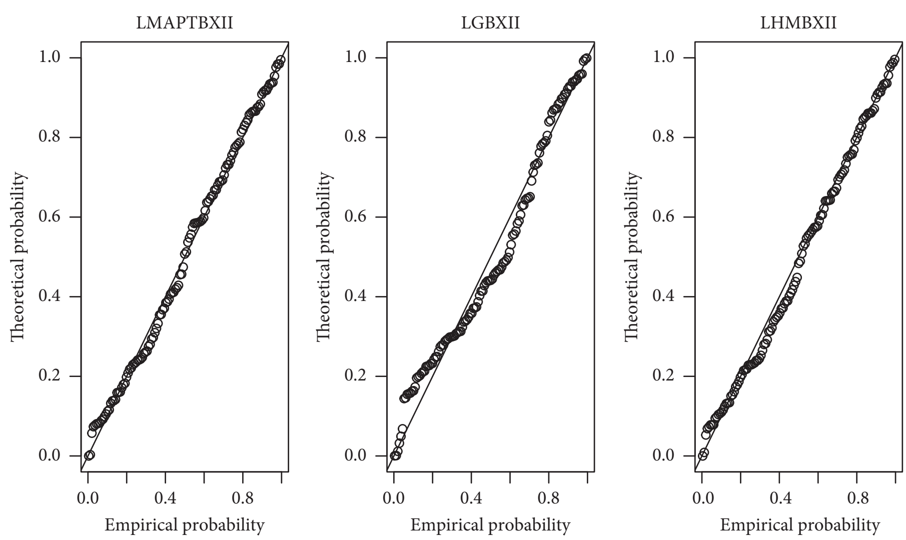

Figure 10: Cox–Snell residuals (P-P plots).

250.052, 53.415, 54.348, 32.016, 17.147, 19.276, 53.827, 24.616, 34.743, 30.653, 121.370, 13.610, 60.899, 25.436, 34.305, 139.916, 129.852, 783.051, and 95.704.

Table 5 presents the maximum likelihood estimates, standard errors (in parentheses), and p values for the fitted model. The model selection criteria clearly indicate that the LMAPTBXII model is the most suitable. Leveraging the parameter estimates derived from the LMAPTBXII model, we formulate the following equation:

$$\hat{y}_j = 2.3975 + 0.0574 z_{j1}. \tag{30}$$

We can deduce that the effect shares had on the total number of shares that were traded on an exchange during a specific period was positively significant.

To assess the appropriateness of the LMAPTBXII, LGBXII, and LHMBXII models, Cox–Snell residuals were generated. Upon examination of the probability-probability (P-P) plot illustrated in Figure 10, it is evident that the residuals of the LMAPTBXII model exhibit closer alignment to the diagonal line in comparison to those of the LGBXII and LHMBXII models. This observation signifies that the LMAPTBXII model provides a better fit to the data.

Table 6: Diagnostics results for residuals.

	KS statistic	CVM statistic	AD statistic
LMAPTBXII	0.0492 (0.9268)	0.0493 (0.8814)	0.4142 (0.8343)
LGBXII	0.0954 (0.2133)	0.2572 (0.1795)	1.5908 (0.1562)
LHMBXII	0.0576 (0.8088)	0.0742 (0.7267)	0.4960 (0.7506)

Table 6 summarises the diagnostics results (p values in parentheses), which affirm the conclusions made using the P-P plots.

8. Conclusion

In this study, we thoroughly investigated the tractability, performance, and flexibility of a novel three-parameter modified alpha power transformed Burr XII distribution. One notable feature of this model is the discovery of a closed-form expression for its quantile function, adding to its analytical convenience. The parameter estimation for the proposed model was carried out through the rigorous maximum likelihood estimation method. This approach ensures that the model parameters are optimised to best capture the characteristics of the data under consideration. To assess the performance and flexibility of the new model, a comprehensive simulation study was conducted, and the model's applicability was demonstrated through its application to two lifetime datasets. The robust results obtained from both the simulation study and the practical applications convincingly validate the enhanced flexibility of the proposed modified alpha power transformed Burr XII distribution. Building upon the original model, we introduced a logarithmic transformation to create a new log-modified alpha power transformed Burr XII model. Through thorough development and rigorous validation, we demonstrated the viability of this new model, offering an alternative perspective for researchers working with diverse data patterns. It is worth noting that the motivation behind introducing modifications to existing distributions, such as the Burr XII, is rooted in the need for increased flexibility. Recognising that no single distribution can perfectly fit all datasets, these modifications empower researchers to account for a broader range of data patterns. In this context, the modified alpha power transformed Burr XII stands out as a valuable addition, capable of accommodating decreasing, skewed, and near-symmetric datasets. In summary, our study makes a significant contribution to the continuous endeavour of improving the versatility of statistical distributions. As we continue to refine and expand our toolkit in statistical modelling, the MAPTBXII distribution stands out as a valuable addition, empowering researchers to address a wide range of data challenges with increased precision and adaptability. The modified alpha power transformed Burr XII distribution, along with its log-transformed counterpart, represents a valuable asset for researchers seeking a flexible and robust framework for data analysis and modelling.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest with regard to this article.

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