Research Article

New Weighted Burr XII Distribution: Statistical Properties, Applications, and Regression

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In this study, a three-parameter modification of the Burr XII distribution has been developed through the integration of the weighted version of the alpha power transformation family of distributions. This newly introduced model, termed the modified alpha power-transformed Burr XII distribution, exhibits the unique ability to effectively model decreasing, right-skewed, or unimodal densities. The paper systematically elucidates various statistical properties of the proposed distribution. The estimation of parameters was obtained using maximum likelihood estimation. The estimator has been evaluated for consistency through simulation studies. To gauge the practical applicability of the proposed distribution, two distinct datasets have been employed. Comparative analyses involving six alternative distributions unequivocally demonstrate that the modified alpha power-transformed Burr XII distribution provides a better fit. Additionally, a noteworthy extension is introduced in the form of a location-scale regression model known as the log-modified alpha power-transformed Burr XII model. This model is subsequently applied to a dataset related to stock market liquidity. The findings underscore the enhanced fitting capabilities of the proposed model in comparison to existing distributions, providing valuable insights for applications in financial modeling and analysis.

1. Introduction

The development of novel or new statistical models is a key area of study in the application of distribution theory. These distributions’ usefulness has led to much research into their theory and the development of new distributions. The idea of generating new continuous distributions by modifying the existing distributions with one or more shape or scale parameters has gained attention in recent years. This parameter introduction has been shown to improve the ability of the developed distributions to fit varied real-life datasets with high degrees of skewness and kurtosis. Some of these newly developed distributions include the modified alpha power transformed Weibull [1], general two-parameter [2], truncated inverse power Ailamujia [3], half-logistic modified Kies exponential [4], truncated inverse power Lindley [5], Marshall–Olkin–Weibull–Burr XII [6], generalised unit half-logistic geometric [7], Chen Burr–Hatke exponential [8], modified XLindley [9], arctan power [10], harmonic mixture Fréchet [11], sine-Weibull geometric [12], bounded odd inverse Pareto exponential [13], new extended Chen [14], power XLindley [15], extended Poisson–Fréchet [16], exponentiated Fréchet loss [17], Gompertz–Makeham [18], and logistic exponential [19] distributions.

The authors of [20] introduced a new method by adding an additional parameter called the alpha power transformation (APT) family. The APT family has been used to develop several modified distributions, including the APT Fréchet [21], APT extended exponential distribution [22], APT inverse Lomax distribution [23], APT log-logistic
distribution [24], APT inverse Lindley distribution [25], and APT Pareto distribution [26], among others. With the aim of improving the flexibility of the APT family of distributions, Alotaibi et al. [27] modified the APT family of distributions and obtained a new family of distributions called the modified alpha power transformed method (MAPT). According to Alotaibi et al. [27], the cumulative distribution function (CDF) and probability density function (PDF) of the MAPT are given as follows:

\[ G_{\text{MAPT}}(x) = \frac{\alpha^{F(x)} - 1}{(\alpha - 1)(1 + \alpha - \alpha^{F(x)})}, \quad \alpha > 0, \alpha \neq 1, \]

(1)

and

\[ g_{\text{MAPT}}(x) = \frac{\alpha^{1+F(x)} \log(\alpha) f(x)}{(\alpha - 1)(1 + \alpha - \alpha^{F(x)})^2}, \quad \alpha > 0, \alpha \neq 1. \]

(2)

This study proposes a modification of the Burr XII distribution using the MAPT proposed by Alotaibi et al. [27]. The Burr XII distribution introduced by Burr [28] is widely used in reliability analysis, actuarial studies, medicine, and agriculture. The PDF and CDF of the Burr XII distribution can be expressed, respectively, as follows:

\[ f(x; \xi, \gamma) = \xi \gamma x^{\xi-1}(1 + x^\gamma)^{-\gamma-1}, \quad x > 0, \xi > 0, \gamma > 0, \]

(3)

and

\[ F(x; \xi, \gamma) = 1 - (1 + x^\gamma)^{-\gamma}, \quad x > 0, \xi > 0, \gamma > 0. \]

(4)

We are motivated to contribute to the ongoing efforts to enhance the versatility of statistical distributions, thus providing researchers with a powerful tool to analyse and model diverse data scenarios effectively as no single distribution is omnibus. Specifically, our motivations for developing the modified alpha power transformed Burr XII (MAPTBXII) distribution are as follows:

(i) Develop an extension of the Burr XII distribution that provides a good parametric fit to data with complex traits

(ii) Propose a new Burr XII distribution with closed form CDF and tractable quantile function that facilitates easy generation of random observations for simulation experiments

(iii) Formulate a location-scale regression model using the proposed distribution

The subsequent sections of the paper are organised as follows. In Section 2, we develop the MAPTBXII distribution. Section 3 is devoted to deriving various statistical properties of the MAPTBXII distribution. The parameters of the MAPTBXII distribution are estimated through the maximum likelihood estimation method, as detailed in Section 4. Section 5 delves into the discussion of the MAPTBXII regression model. A comprehensive simulation study is presented in Section 6. Real-world applications of the MAPTBXII distribution on two datasets are presented in Section 7. Finally, the paper is concluded in Section 8.

2. Modified Alpha Power Transformed Burr XII Distribution

If a random variable \( X \) follows the MAPTBXII, then the PDF can be obtained by substituting equation (3) into equation (2), while the CDF can be obtained by substituting equation (4) into equation (1). The CDF and PDF of the MAPTBXII can then be expressed, respectively, as follows:

\[ G_{\text{MAPT}}(x) = \frac{\alpha^{1-(1+x^\gamma)^{-\gamma}} - 1}{(\alpha - 1)(1 + \alpha - \alpha^{1-(1+x^\gamma)^{-\gamma}})}, \]

(5)

\[ g_{\text{MAPT}}(x) = \frac{\alpha^{2-(1+x^\gamma)^{-\gamma}} \log(\alpha) \xi x^{\xi-1}(1 + x^\gamma)^{-\gamma-1}}{(\alpha - 1)(1 + \alpha - \alpha^{1-(1+x^\gamma)^{-\gamma}})^2}, \]

(6)

where \( \alpha > 0, \xi > 0, \gamma > 0, \) and \( \alpha \neq 1. \)

The hazard function of the MAPTBXII is obtained by finding the ratio of the PDF and complement of the CDF. The hazard function is given by the following equation:

\[ h_{\text{MAPT}}(x) = \frac{\alpha^{1-(1+x^\gamma)^{-\gamma}} \log(\alpha) \xi x^{\xi-1}(1 + x^\gamma)^{-\gamma-1}}{(\alpha - 1)(1 + \alpha - \alpha^{1-(1+x^\gamma)^{-\gamma}})^2}, \]

(6)

The various shapes (decreasing, right-skewed, left-skewed, or unimodal) of the densities of the MAPTBXII distribution are shown in Figure 1.

The various shapes (decreasing or upside down bathtub) of the hazard function are displayed in Figure 2.

The PDF of the MAPTBXII distribution in linear form is given as follows:

\[ g(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \eta_{ijm} \xi^{i+j}(1 + x^\gamma)^{-\gamma(m+1)-1}, \]

(7)

where \( \eta_{ijm} = (\alpha(-1)^m(i+1)^{m+1} \log(\alpha)^{m+1}) j^i / (\alpha - 1) \) \( (\alpha + 1)^{m+1} j^i, x < 0, \xi > 0, \gamma > 0, \) and \( \alpha > 0. \)

The asymptotic nature of the CDF, PDF, and hazard function is as follows:

\[ G_{\text{MAPT}}(x) \rightarrow 0, \quad x \rightarrow -\infty \]

\[ g_{\text{MAPT}}(x) \rightarrow 0, \quad x \rightarrow -\infty \]

\[ h_{\text{MAPT}}(x) \rightarrow 0, \quad x \rightarrow -\infty \]

The quantile function of MAPTBXII distribution is given as follows:

\[ x_p = \left\{ \frac{1 - \log(p(\alpha^2 - 1) + 1/\alpha^2 + 1))}{\log(\alpha)} \right\}^{-1/\gamma} \]

(8)

where \( p \in (0, 1) \) and \( Q(p) = x_p \) is the quantile function.
3. Statistical Properties

3.1. Moments. Moments are a crucial component of statistical theory, and they allow for the examination of many essential characteristics of any distribution.

Mathematically, the $r^{th}$ moments of $X$ are given by

$$
\mu_r = E(X^r) = \int_0^\infty x^r f(x) \, dx.
$$

We substitute equation (7) into equation (9), in order to obtain

$$
\mu_r = \xi y \sum_{i=0}^\infty \sum_{m=0}^\infty \eta_{ijm} \int_0^\infty x^{r+\xi-1} (1+x^\xi)^{-y(m+1)-1} \, dx.
$$

Letting $v = x^\xi$, which implies $x = v^{1/\xi}$ and $dx = (1/\xi)v^{1-1/\xi}dv$, we obtain the following expression:

$$
\mu_r = y \sum_{i=0}^\infty \sum_{m=0}^\infty \eta_{ijm} \int_0^\infty v^{r/\xi} (1+v)^{-y(m+1)-1} \, dv.
$$

Utilising the identity used in [29],

$$
\mathcal{B}(c, d) = \int_0^\infty v^{c-1} (1+v)^{-c+d} \, dv, c > 0, d > 0,
$$

then gives

$$
\mu_r = \sum_{i=0}^\infty \sum_{m=0}^\infty \sum_{j=0}^{\infty} \eta_{ijm} \mathcal{B}\left(\frac{r}{\xi} + 1, y(m+1) - \frac{r}{\xi}\right).
$$

It follows that the $r^{th}$ moment of the MAPTBXII distribution can be expressed as follows:

$$
\mu_r = \sum_{i=0}^\infty \sum_{m=0}^\infty \sum_{j=0}^{\infty} \eta_{ijm} \mathcal{B}\left(\frac{r}{\xi} + 1, y(m+1) - \frac{r}{\xi}\right).
$$

where $\mathcal{B}(\cdot, \cdot)$ is given as the beta function and $r = 1, 2, \ldots$

Metrics such as variance ($\sigma^2$), coefficient of variation (CVAR), skewness (CSK), and kurtosis (CKUR) can be derived by utilizing moment-based computations. $\mu$, $\sigma^2$, CVAR, CSK, and CKUR, respectively, are obtained using the following expression:

$$
\sigma^2 = \mu_2^\prime - (\mu)^2,
$$

$$
\text{CVAR} = \frac{\sqrt{\mu_2^\prime - (\mu)^2}}{\mu},
$$

$$
\text{CSK} = \frac{\mu_3^\prime - 3\mu^\prime \mu_2^\prime + 2\mu^3}{\sigma^2},
$$

$$
\text{CKUR} = \frac{\mu_4^\prime - 4\mu^\prime \mu_3^\prime + 6\mu^2 \mu_2^\prime - 3\mu^4}{\sigma^4}.
$$

Table 1 displays the $\sigma^2$, CVAR, CSK, and CKUR of the MAPTBXII distribution, calculated using noncentral moments for specific parameter values. The results suggest that the MAPTBXII distribution exhibits a varying degree of skewness, ranging from highly skewed to moderately skewed. It is noteworthy that the skewness of the distribution may differ depending on the chosen parameter values. Some combinations lead to a positively skewed distribution, while others result in a negatively skewed distribution. This variability underscores the distribution's

![Figure 1: Densities of the MAPTBXII distribution.](image1)

![Figure 2: The hazard function plots of the MAPTBXII.](image2)
flexibility in capturing different skewness patterns based on the
specified parameters. The MAPTBXII distribution is
leptokurtic (CKUR > 3) and thus has heavier tails and a
sharper peak.

3.2. Incomplete Moments. The incomplete moments can be
used to calculate the Lorenz curve, the Bonferroni curve, the
mean deviation, and the median deviation, among others.

Mathematically, the incomplete moment is given as follows:

\[ m_r(z) = E(X^r | X \leq z) = \int_0^z x^r f(x) \, dx. \]  (16)

We substitute equation (7) into equation (16), in order to obtain

\[ m_r(z) = \xi y \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \eta_{ijm} \int_0^z x^{r+i-1} (1 + x^\xi)^{-y(m+1)-1} \, dx. \]  (17)

Letting \( v = x^\xi \), which implies \( x = v^{1/\xi} \) and
\( dx = (1/\xi) v^{1-1/\xi} \, dv \), we obtain the following expression:

\[ m_r(z) = \gamma \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \eta_{ijm} \int_0^z v^{r+i-1} (1 + v)^{-y(m+1)-1} \, dv. \]  (18)

Utilising the identity used in [29],

\[ \mathcal{B}(c, d) = \int_0^\infty v^{c-1} (1 + v)^{-(c+d)} \, dv, \quad c > 0, \quad d > 0, \]  (19)

helps to obtain

\[ m_r(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \eta_{ijm} \mathcal{B}(r + 1/\xi, y(m+1) - r/\xi). \]  (20)

It follows that the \( r^{\text{th}} \) incomplete moment of the
MAPTBXII distribution is given by the following expression:

\[ m_r(z) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \eta_{ijm} \mathcal{B}(z^{1/\xi} + 1, y(m+1) - r/\xi). \]  (21)

where \( \mathcal{B}(\cdot, \cdot) \) is given as the incomplete beta function and
\( r = 1, 2, 3, \ldots \).

3.3. Moment Generating Function. The moment generating
function (MGF) is used to determine the distribution’s moments, if any. Mathematically,

\[ M(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r E(X^r)}{r!} = \sum_{r=0}^{\infty} \frac{t^r \gamma_r}{r!}. \]  (22)

When we substitute equation (14) into equation (22), we
obtain the MGF of the MAPTBXII distribution. Hence, the
MGF of the MAPTBXII distribution is given by the fol-

\[ M(t) = \sum_{r=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \eta_{ijm} \mathcal{B}(r + 1, y(m+1) - r/\xi). \]  (23)

4. Estimation of Parameters

The maximum likelihood estimation (MLE) method is used
as an estimator. The log-likelihood function is therefore
given as

\[ \ell(x, c, \xi, \gamma) = \sum_{i=1}^{n} \left[ -\left(1 + x_i^\xi\right)^\gamma \log(\alpha) + n \log(\gamma) \right] + \left(\xi - 1\right) \sum_{i=1}^{n} \log x_i \]
\[ - (y + 1) \sum_{i=1}^{n} \left(1 + x_i^\xi\right)^{-y} - n \log(\alpha - 1) - 2 \sum_{i=1}^{n} \log \left(1 + \alpha - \alpha^{1-x_i^{\xi}\gamma}\right). \]  (24)

By maximising equation (24), we obtain the parameter
estimates.

5. Modified Alpha Power Transformed Burr XII
Regression Model

Suppose \( X \) follows the MAPTBXII distribution and \( Y = \log(pX) \). The PDF of \( Y \) can be derived by substituting
\( c = 1/\sigma \) and \( \beta = \exp(\mu) \).

\[ f(y) = \frac{\alpha^{2-(1+\exp((y-\mu)/\sigma)^-\gamma)} ((\log(\alpha)y)/\sigma)\exp((y-\mu)/\sigma)(1 + \exp((y-\mu)/\sigma))^{-\gamma-1}}{(\alpha - 1)(1 + \alpha - \alpha^{1-(1+\exp((y-\mu)/\sigma)^-\gamma)})^2}, \]  (25)
Table 1: First five moments varying parameter values.

<table>
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<tr>
<th>$\alpha$</th>
<th>$\xi$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\nu$</th>
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</tbody>
</table>

where $y \in R$, $\sigma > 0$ is the scale parameter, $\gamma > 0$ is the shape parameter, $\alpha > 0$, and $\mu \in R$ is the location parameter.

Equation (25) is referred to as the log-MAPTB XII (LMAPTBXII) distribution. If $X \sim$ MAPTBXII($\alpha, \xi, \gamma$), then $Y = \log(\beta X) \sim$ LMAPTBXII($\alpha, \xi, \gamma, \mu$).

Figure 3 shows various shapes of the densities of the LMAPTBXII distribution.

The survival function of the LMAPTBXII distribution can be expressed as follows:

$$S(y) = 1 - \frac{\alpha\left(1 + \exp\left(\frac{y - \mu}{\sigma}\right)\right)^{-1}}{(\alpha - 1)(1 + \alpha - \alpha\left(1 + \exp\left(\frac{y - \mu}{\sigma}\right)\right))^{-1}}.$$

The proposed location-scale regression model is defined with the dependent variable $y_j$ and predictor variables $y_j = (1, z_{j1}, \ldots, z_{jp})$, where $1$ is the intercept and expressed as follows:

$$y_j = Z_j' \beta + \sigma W_j,$$

where $j = 1, 2, 3, \ldots, n$, $\beta = (\beta_1, \beta_2, \beta_3, \ldots, \beta_p)$ are the regression parameters, and $W_j$ denotes the random error.

The log-likelihood function of the LMAPTBXII model is given by the following expression:

$$\ell = \sum_{j=1}^{n} \left(2 - \left(1 + \exp\left(\frac{y_j - \mu_j}{\sigma}\right)\right)^{-1}\right)\log(\alpha) + n\log^2(\alpha) + n\log\left(\frac{1}{\alpha}\right) + \sum_{j=1}^{n} \left(\frac{y_j - \mu_j}{\sigma}\right) - (y + 1) \sum_{j=1}^{n} \left(1 + \exp\left(\frac{y_j - \mu_j}{\sigma}\right)\right) - n\log(\alpha - 1) - 2 \sum_{j=1}^{n} \log\left(1 + \alpha - \alpha\left(1 + \exp\left(\frac{y_j - \mu_j}{\sigma}\right)\right)\right).$$

7. Applications

In this section, we provide the applications of the MAPTBXII distribution using two uncensored datasets.

The precipitation (in inches) in the Minneapolis dataset was used in [30, 31]. The second dataset represents runoff amounts at Jug Bridge, Maryland, and was used by Makubate et al. [32]. The performance of the MAPTBXII distribution is compared with the Topp–Leone Burr-XII distribution (TLBXII) [33], Weibull Burr XII distribution (WBXII) [29], inverse Weibull distribution (INW) [34], exponentiated exponential Burr XII distribution (EEBXII) [35], Cauchy [36], and inverse Gompertz distribution (IGD) [37].

Analytical measures such as Kolmogorov–Smirnov (K-S) test, Anderson–Darling test (AD), and Cramér–von Mises test (CVM), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICc), and Bayesian Information Criterion (BIC) are considered in evaluating the
goodness of fit of the proposed distribution and the other fitted models.

The maximum likelihood estimates and other goodness-of-fit statistics for the two datasets are presented in Tables 3 and 4.

The MLEs of the MAPTBXII model for datasets 1 and 2 are both unique and represent real maxima, as demonstrated by the profile log-likelihood plots shown in Figures 4 and 5.

The fitted densities are presented in Figures 6 and 7, whereas the fitted CDFs are shown in Figures 8 and 9. It can be observed from these results that the MAPTBXII provides a better fit to the two datasets than the other competing lifetime distributions.

7.1. Application of Modified Alpha Power Transformed Burr XII Regression Model. The LMAPTBXII model was employed in the analysis of a dataset related to stock market liquidity. Dataset can be retrieved from https://instruction.bus.wisc.edu/jfrees/jfreesbooks/RegressionModeling/BookWebDec2010/data.html (accessed on 8 January 2023). The competing models are the log-harmonic mixture Burr XII (LHMBXII) distribution [31] and the log-Gumbel Burr XII (LGBXII) distribution [38]. The response variable $y_j$ is the total number of shares that were traded on an exchange during a specific period (volume), while the covariate is the number of shares outstanding as of December 31, 1984, expressed in millions of shares (shares) ($z_j$). The fitted model is given by the following expression:

$$y_j = \beta_0 + \beta_1 z_{j1}. \quad (29)$$

Table 2: Simulation results.

<table>
<thead>
<tr>
<th>n</th>
<th>Parameter</th>
<th>I: $\alpha = 6.85$, $\xi = 0.10$, $\gamma = 0.50$</th>
<th>II: $\alpha = 10.65$, $\xi = 6.10$, $\gamma = 10.50$</th>
<th>III: $\alpha = 1.65$, $\xi = 1.90$, $\gamma = 6.50$</th>
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<td>$\gamma$</td>
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Table 3: MLEs and goodness-of-fit statistics for data 1.

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<th>Distribution</th>
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<th>Estimates (standard error)</th>
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<th>BIC</th>
<th>AICc</th>
<th>CVM</th>
<th>AD</th>
<th>KS</th>
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<tbody>
<tr>
<td>MAFTBXII</td>
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<td>3.5493 (3.42083)</td>
<td>−38.67247</td>
<td>83.34494</td>
<td>87.54853</td>
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<td>0.1289 (0.9996)</td>
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<td>84.6453</td>
<td>88.8489</td>
<td>85.5683</td>
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<td>0.3116 (0.9286)</td>
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<td>$k$</td>
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<td>−1.2170 (0.6186)</td>
<td>−39.0394</td>
<td>84.0788</td>
<td>88.2824</td>
<td>85.0018</td>
<td>0.0198 (0.9561)</td>
<td>0.1909 (0.9320)</td>
<td>0.0901 (0.9190)</td>
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<td></td>
<td>$k$</td>
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<tr>
<td>EEBXII</td>
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<td>6.5384 (5.1732)</td>
<td>−38.5193</td>
<td>85.0386</td>
<td>90.6434</td>
<td>86.6386</td>
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<td>0.1703 (0.9966)</td>
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<td></td>
<td>$\xi$</td>
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<td>$\gamma$</td>
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<td>Cauchy</td>
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<td>94.485</td>
<td>97.6509</td>
<td>95.2923</td>
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<td>0.7734 (0.4994)</td>
<td>0.1464 (0.5407)</td>
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<td>1.42381 (0.16552)</td>
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<td>IGD</td>
<td>$\alpha$</td>
<td>0.8155 (0.2265)</td>
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<td>96.3582</td>
<td>94.0002</td>
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<td>$\beta$</td>
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Table 4: MLEs and goodness-of-fit statistics for data 2.

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<th>BIC</th>
<th>AICc</th>
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<th>AD</th>
<th>KS</th>
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<td>-14.82152</td>
<td>35.6431</td>
<td>39.3000</td>
<td>36.7859</td>
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<td>0.0988 (0.9999)</td>
<td>0.0658 (0.9999)</td>
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<td></td>
<td>ξ</td>
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<tr>
<td></td>
<td>γ</td>
<td>1.2310 (1.0149)</td>
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<tr>
<td>TLBXII</td>
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<td>3.0443 (7.3218)</td>
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<td>36.6325</td>
<td>0.0136 (0.9999)</td>
<td>0.1049 (0.9891)</td>
<td>0.0681 (0.9998)</td>
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<td></td>
<td>k</td>
<td>1.6508 (1.6972)</td>
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<tr>
<td>WBXII</td>
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<td>39.8956</td>
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<tr>
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<td>λ</td>
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<td>35.6544</td>
<td>39.3100</td>
<td>36.7963</td>
<td>0.1929 (0.9789)</td>
<td>0.1009 (0.9620)</td>
<td>0.0701 (0.9567)</td>
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<td></td>
<td>c</td>
<td>4.4347 (1.5007)</td>
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<td>k</td>
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<td>39.5073</td>
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<td>0.0720 (0.9999)</td>
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<td>γ</td>
<td>10.3041 (14.3761)</td>
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<tr>
<td>Cauchy</td>
<td>α</td>
<td>0.2466 (0.0655)</td>
<td>43.6537</td>
<td>46.0914</td>
<td>44.1991</td>
<td>0.1077 (0.5518)</td>
<td>0.1077 (0.5518)</td>
<td>0.1484 (0.6411)</td>
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<td>c</td>
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<tr>
<td>IGD</td>
<td>α</td>
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<td>42.4802</td>
<td>44.9179</td>
<td>43.0256</td>
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<td>0.1726 (0.4457)</td>
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<td>β</td>
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Figure 4: Profile log-likelihood of the MAPTBXII model for dataset 1.

Figure 5: Profile log-likelihood of the MAPTBXII model for dataset 2.

Figure 6: Fitted densities for data 1.
Figure 7: Fitted densities for data 2.

Figure 8: Fitted CDFs for data 1.

Figure 9: Fitted CDFs for data 2.
Table 5: Parameter estimates and selection criteria.

<table>
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<tr>
<th>Parameter Estimates</th>
<th>p values</th>
<th>Selection Criteria</th>
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<td>0.0574 (0.0070)</td>
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<td>0.8959 (0.3582)</td>
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<td>$k$</td>
<td>0.2272 (0.0853)</td>
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<td>$\sigma$</td>
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<td>BIC</td>
<td>812.4162</td>
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</table>

Table 5 presents the maximum likelihood estimates, standard errors (in parentheses), and $p$ values for the fitted model. The model selection criteria clearly indicate that the LMAPTBXII model is the most suitable. Leveraging the parameter estimates derived from the LMAPTBXII model, we formulate the following equation:

$$\hat{y}_j = 2.3975 + 0.0574z_j^1.$$  (30)

We can deduce that the effect shares had on the total number of shares that were traded on an exchange during a specific period was positively significant.

To assess the appropriateness of the LMAPTBXII, LGBXII, and LHMBXII models, Cox–Snell residuals were generated. Upon examination of the probability-probability (P-P) plot illustrated in Figure 10, it is evident that the residuals of the LMAPTBXII model exhibit closer alignment to the diagonal line in comparison to those of the LGBXII and LHMBXII models. This observation signifies that the LMAPTBXII model provides a better fit to the data.
Table 6 summarises the diagnostics results (p values in parentheses), which affirm the conclusions made using the P-P plots.

### Data Availability

The data used to support the findings of this study are included within the article.

### Conflicts of Interest

The authors declare that there are no conflicts of interest with regard to this article.

### References


