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Research Article

Rings in Which Every Element Is a Sum of a Nilpotent and Three 7-Potents

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In this article, we define and discuss strongly $S_{3,7}$ nil-clean rings: every element in a ring is the sum of a nilpotent and three 7-potents that commute with each other. We use the properties of nilpotent and 7-potent to conduct in-depth research and a large number of calculations and obtain a nilpotent formula for the constant a. Furthermore, we prove that a ring R is a strongly $S_{3,7}$ nil-clean ring if and only if $R = R_1 \oplus R_2 \oplus R_3 \oplus R_4 \oplus R_5 \oplus R_6$, where R_1, R_2, R_3, R_4, R_5, and R_6 are strongly $S_{3,7}$ nil-clean rings with $2 \in \text{Nil}(R_1)$, $3 \in \text{Nil}(R_2)$, $5 \in \text{Nil}(R_3)$, $7 \in \text{Nil}(R_4)$, $13 \in \text{Nil}(R_5)$, and $19 \in \text{Nil}(R_6)$. The equivalent conditions of strongly $S_{3,7}$ nil-clean rings in some cases are discussed.

1. Introduction

In Nicholson's paper [1], the concept of a clean ring was first mentioned. In [2], Nicholson proposed the concept of strongly clean rings: every element in a ring is the sum of an idempotent and a unit that commute. The classical and interesting generalization is to use a nilpotent instead of a unit to obtain strongly nil-clean elements. Diesl described and characterized strongly nil-clean rings in [3]. Every element of a ring R is said to be strongly nil-clean if it is the sum of an idempotent and a nilpotent that commute with one another. The authors in [4, 5] determined the structure of strongly nil-clean rings. The deformation and generalization of idempotents in nil-clean rings can be extended to many new classes of rings. In [6], Chen and Sheibani identified the rings for which every element is the sum of a nilpotent and a tripotent that commute with one another. In [7], Ying et al. determined the rings for which every element is the sum of a nilpotent and two tripotents that commute with one another. In [8], Cui and Xia characterized the rings for which every element is the sum of a nilpotent and three tripotents that commute. Diesl unified the symbols in [9], summarized and extended the previous results, and determined the structure of strongly $S_{n,3}$ nil-clean rings ($n \leq 7$) and the structure of strongly $S_{2,m}$ nil-clean rings. So far, scholars at home and abroad have revealed many interesting results in this field. These results extend the theory of algebraic structures, especially the theory of rings. It provides a new perspective for us to understand the internal structure and properties of rings, which is very important to solve the problem of the classification and properties of rings.

Therefore, this paper will continue this kind of research and use 7-potent to replace 3-potent in the [8] to promote and determine strongly $S_{3,7}$ nil-clean rings: every element in a ring is a nilpotent and three 7-potents that commute. At the same time, we prove that a ring R is a strongly $S_{3,7}$ nil-clean ring if and only if $R = R_1 \oplus R_2 \oplus R_3 \oplus R_4 \oplus R_5 \oplus R_6$, where R_1, R_2, R_3, R_4, R_5, and R_6 are strongly $S_{3,7}$ nil-clean rings with $2 \in \text{Nil}(R_1)$, $3 \in \text{Nil}(R_2)$, $5 \in \text{Nil}(R_3)$, $7 \in \text{Nil}(R_4)$, $13 \in \text{Nil}(R_5)$, and $19 \in \text{Nil}(R_6)$. The equivalent conditions of strongly $S_{3,7}$ nil-clean rings in some cases are discussed.

2. Structure of Strongly $S_{3,7}$-Nil-Clean Rings

Definition 1. An element p in a ring is 3-potent if it satisfies $p^3 = p$.

Definition 2. An element p in a ring is 7-potent if it satisfies $p^7 = p$.

Definition 3. A ring R is a strongly $S_{3,7}$ nil-clean ring if every element in R is the sum of a nilpotent and three 7-potents that commute with one another.

In order to facilitate writing, the following provisions are given: let $a \in R$ and write $a = b + e + f + g$, where $b \in \text{Nil}(R)$, $e^7 = e$, $f^7 = f$, $g^7 = g \in R$ and b, e, f, g commute with one another.

Lemma 4. *The following equations are true:*

$$e^2 + f^2 + g^2 = a^2 - 2(ef + eg + fg) + bt_1; \tag{1}$$

$$e^3 + f^3 + g^3 = a^3 - 3\left(e^2 f + e^2 g + ef^2 + eg^2 + f^2 g + fg^2\right) - 6efg + bt_2; \tag{2}$$

$$\begin{aligned} e^4 + f^4 + g^4 = {} & a^4 - 4\left(e^3 f + e^3 g + ef^3 + eg^3 + f^3 g + fg^3\right) \\ & - 6\left(e^2 f^2 + e^2 g^2 + f^2 g^2\right) - 12\left(e^2 fg + ef^2 g + efg^2\right) + bt_3; \end{aligned} \tag{3}$$

$$e^6 f + e^6 g + ef^6 + eg^6 + f^6 g + fg^6 = a\left(e^6 + f^6 + g^6\right) - a + bt_4; \tag{4}$$

$$\begin{aligned} & e^5 f^2 + e^5 g^2 + e^2 f^5 + e^2 g^5 + f^5 g^2 + f^2 g^5 \\ & = a^2\left(e^5 + f^5 + g^5\right) - 2a\left(e^6 + f^6 + g^6\right) - 2efg\left(e^4 + f^4 + g^4\right) + a + bt_5; \end{aligned} \tag{5}$$

$$\begin{aligned} & e^4 f^3 + e^4 g^3 + e^3 f^4 + e^3 g^4 + f^4 g^3 + f^3 g^4 \\ & = 3a\left(e^6 + f^6 + g^6\right) - 3a^2\left(e^5 + f^5 + g^5\right) + a^3\left(e^4 + f^4 + g^4\right) \\ & \quad + 3efg\left(e^4 + f^4 + g^4\right) - 3aefg\left(e^3 + f^3 + g^3\right) - a + bt_6; \end{aligned} \tag{6}$$

$$efg\left(e^2 + f^2 + g^2\right) = a^2 efg - 2e^2 f^2 g^2\left(e^5 + f^5 + g^5\right) + bt_7; \tag{7}$$

$$efg\left(e^3 + f^3 + g^3\right) = a^3 efg - 3ae^2 f^2 g^2\left(e^5 + f^5 + g^5\right) + 3e^2 f^2 g^2 + bt_8; \tag{8}$$

$$\begin{aligned} 7efg\left(e^4 + f^4 + g^4\right) = {} & 16a^2 e^6 f^6 g^6\left(e^5 + f^5 + g^5\right) + 8a^2 e^4 f^4 g^4\left(e^5 + f^5 + g^5\right) \\ & + 4a^2 e^2 f^2 g^2\left(e^5 + f^5 + g^5\right) - a^4 efg - 8ae^2 f^2 g^2 - 2a^4 e^3 f^3 g^3 \\ & - 16ae^4 f^4 g^4 - 4a^4 e^5 f^5 g^5 - 32ae^6 f^6 g^6 + bt_9. \end{aligned} \tag{9}$$

Proof. Since

$$e^2 + f^2 + g^2 = (e + f + g)^2 - 2(ef + eg + fg) = a^2 - 2(ef + eg + fg) + bt_1, \tag{10}$$

so the equation (1) is established. Similarly, the equations (2) and (3) are established. As

$$\begin{aligned} & e^6 f + e^6 g + e f^6 + e g^6 + f^6 g + f g^6 \\ & \quad = \left(e^6 + f^6 + g^6\right)(e + f + g) - \left(e^7 + f^7 + g^7\right) \\ & \quad = a\left(e^6 + f^6 + g^6\right) - a + b t_4. \end{aligned} \tag{11}$$

Therefore, the equation (4) is established. By the equations (1) and (4), we get.

$$\begin{aligned} & e^5 f^2 + e^5 g^2 + e^2 f^5 + e^2 g^5 + f^5 g^2 + f^2 g^5 \\ & \quad = \left(e^5 + f^5 + g^5\right)\left(e^2 + f^2 + g^2\right) - \left(e^7 + f^7 + g^7\right) \\ & \quad = \left(e^5 + f^5 + g^5\right)\left[(e + f + g)^2 - 2(ef + eg + fg)\right] - (a - b) \\ & \quad = (a - b)^2\left(e^5 + f^5 + g^5\right) - 2\left(e^6 f + e^6 g + e f^6 + e g^6 + f^6 g \right. \\ & \qquad \left. + f g^6\right) - 2efg\left(e^4 + f^4 + g^4\right) - a + b \\ & \quad = a^2\left(e^5 + f^5 + g^5\right) - 2a\left(e^6 + f^6 + g^6\right) - 2efg\left(e^4 + f^4 + g^4\right) \\ & \qquad + a + b t_5. \end{aligned} \tag{12}$$

So, the equation (5) is established. By the equations (1)–(5), we obtain

$$\begin{aligned} & e^4 f^3 + e^4 g^3 + e^3 f^4 + e^3 g^4 + f^4 g^3 + f^3 g^4 \\ & \quad = \left(e^4 + f^4 + g^4\right)\left(e^3 + f^3 + g^3\right) - \left(e^7 + f^7 + g^7\right) \\ & \quad = \left(e^4 + f^4 + g^4\right)\left[(e + f + g)^3 - 3\left(e^2 f + e^2 g + e f^2 + e g^2 + f^2 g \right.\right. \\ & \qquad \left.\left. + f g^2\right) - 6efg\right] - (a - b) \\ & \quad = (a - b)^3\left(e^4 + f^4 + g^4\right) - 3\left(e^6 f + e^6 g + e f^6 + e g^6 + f^6 g + f g^6\right) \\ & \qquad - 3\left(e^5 f^2 + e^5 g^2 + e^2 f^5 + e^2 g^5 + f^5 g^2 + f^2 g^5\right) - 3\left(e^4 f^2 g + e^4 f g^2 \right. \\ & \qquad \left. + e^2 f^4 g + e^2 f g^4 + e f^4 g^2 + e f^2 g^4\right) - 6efg\left(e^4 + f^4 + g^4\right) - a + b \\ & \quad = a^3\left(e^4 + f^4 + g^4\right) - 3a\left(e^6 + f^6 + g^6\right) + 3a - 3a^2\left(e^5 + f^5 + g^5\right) \\ & \qquad + 6a\left(e^6 + f^6 + g^6\right) + 6efg\left(e^4 + f^4 + g^4\right) - 3a - 3aefg\left(e^3 + f^3 \right. \\ & \qquad \left. + g^3\right) + 3efg\left(e^4 + f^4 + g^4\right) - 6efg\left(e^4 + f^4 + g^4\right) - a + b t_6 \\ & \quad = 3a\left(e^6 + f^6 + g^6\right) - 3a^2\left(e^5 + f^5 + g^5\right) + a^3\left(e^4 + f^4 + g^4\right) \\ & \qquad + 3efg\left(e^4 + f^4 + g^4\right) - 3aefg\left(e^3 + f^3 + g^3\right) - a + b t_6. \end{aligned} \tag{13}$$

Thus, the equation (6) is established. By the equation (1), we get

$$\begin{aligned} &efg\left(e^2+f^2+g^2\right) \\ &\quad = a^2efg-2efg(ef+eg+fg)+efgbt_1 \\ &\quad = a^2efg-2e^2f^2g^2\left(e^5+f^5+g^5\right)+bt_7. \end{aligned} \tag{14}$$

So, the equation (7) is established. By the equation (2), we obtain

$$\begin{aligned} &efg\left(e^3+f^3+g^3\right) \\ &\quad = a^3efg-3efg\left(e^2f+e^2g+ef^2+eg^2+f^2g+fg^2\right) \\ &\qquad -6e^2f^2g^2+efgbt_2 \\ &\quad = a^3efg-3e^2f^2g^2\left(e^5f+e^5g+ef^5+eg^5+f^5g+fg^5\right) \\ &\qquad -6e^2f^2g^2+efgbt_2 \\ &\quad = a^3efg-3e^2f^2g^2\left[\left(e^5+f^5+g^5\right)(e+f+g)-\left(e^6+f^6\right.\right. \\ &\qquad \left.\left.+g^6\right)\right]-6e^2f^2g^2+efgbt_2 \\ &\quad = a^3efg-3ae^2f^2g^2\left(e^5+f^5+g^5\right)+3e^2f^2g^2+bt_8. \end{aligned} \tag{15}$$

$$\begin{aligned} &efg\left(e^4+f^4+g^4\right) \\ &\quad = a^4efg-4efg\left(e^3f+e^3g+ef^3+eg^3+f^3g+fg^3\right)-6efg \\ &\qquad \cdot\left(e^2f^2+e^2g^2+f^2g^2\right)-12efg\left(e^2fg+ef^2g+efg^2\right)+efgbt_3 \\ &\quad = a^4efg-4e^2f^2g^2\left(e^5f^2+e^5g^2+e^2f^5+e^2g^5+f^5g^2+f^2g^5\right) \\ &\qquad -6e^3f^3g^3\left(e^4+f^4+g^4\right)-12e^2f^2g^2(e+f+g)+efgbt_3 \\ &\quad = a^4efg-4a^2e^2f^2g^2\left(e^5+f^5+g^5\right)+8ae^2f^2g^2\left(e^6+f^6+g^6\right) \\ &\qquad +8e^3f^3g^3\left(e^4+f^4+g^4\right)-4ae^2f^2g^2-4e^2f^2g^2bt_5-6e^3f^3g^3 \\ &\qquad \cdot\left(e^4+f^4+g^4\right)-12e^2f^2g^2(e+f+g)+efgbt_3 \\ &\quad = a^4efg-4a^2e^2f^2g^2\left(e^5+f^5+g^5\right)+2e^3f^3g^3\left(e^4+f^4+g^4\right) \\ &\qquad +8ae^2f^2g^2+bt. \end{aligned} \tag{16}$$

Therefore, $efg(e^4+f^4+g^4)=a^4efg-4a^2e^2f^2g^2(e^5+f^5+g^5)+2e^3f^3g^3(e^4+f^4+g^4)+8ae^2f^2g^2+bt$. According to this equation, we can easily obtain

$$\begin{aligned} 2e^3f^3g^3\left(e^4+f^4+g^4\right) &= 2a^4e^3f^3g^3-8a^2e^4f^4g^4\left(e^5+f^5+g^5\right) \\ &\quad +4e^5f^5g^5\left(e^4+f^4+g^4\right)+16ae^4f^4g^4+2e^2f^2g^2bt, \\ 4e^5f^5g^5\left(e^4+f^4+g^4\right) &= 4a^4e^5f^5g^5-16a^2e^6f^6g^6\left(e^5+f^5+g^5\right) \\ &\quad +8efg\left(e^4+f^4+g^4\right)+32ae^6f^6g^6+4e^4f^4g^4bt. \end{aligned} \tag{17}$$

Hence, the equation (8) is established. By the equations (3) and (5), we get.

The above equations are simplified to give

$$
\begin{aligned}
7efg\left(e^4+f^4+g^4\right) = {} & 16a^2e^6f^6g^6\left(e^5+f^5+g^5\right)+8a^2e^4f^4g^4\left(e^5+f^5+g^5\right)\\
&+4a^2e^2f^2g^2\left(e^5+f^5+g^5\right)-a^4efg-8ae^2f^2g^2\\
&-2a^4e^3f^3g^3-16ae^4f^4g^4-4a^4e^5f^5g^5-32ae^6f^6g^6+bt_9.
\end{aligned}
\tag{18}
$$

Hence, the equation (9) is established. □

Lemma 5. *The following formulas are nilpotent:*

$$
\begin{aligned}
&a^5-6\left(e^5+f^5+g^5\right)-30e^2f^2g^2\left(e^5+f^5+g^5\right)\\
&\quad+15a\left(e^4+f^4+g^4\right)-10a^2\left(e^3+f^3+g^3\right);
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&a^6-6a\left(e^5+f^5+g^5\right)+15a^2\left(e^4+f^4+g^4\right)-10a^3\left(e^3+f^3+g^3\right)\\
&\quad+30efg\left(e^3+f^3+g^3\right)-30aefg\left(e^2+f^2+g^2\right)-90ae^2f^2g^2;
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&a^7+20a-70a\left(e^6+f^6+g^6\right)+84a^2\left(e^5+f^5+g^5\right)-35a^3\left(e^4+f^4\right.\\
&\quad\left.+g^4\right)-70efg\left(e^4+f^4+g^4\right)+140aefg\left(e^3+f^3+g^3\right)-70a^2efg\left(e^2\right.\\
&\quad\left.+f^2+g^2\right)-70ae^2f^2g^2.
\end{aligned}
\tag{21}
$$

Proof. As

$$
\begin{aligned}
(a-b)^5 &= (e+f+g)^5\\
&= e^5+f^5+g^5+5\left(e^4f+e^4g+ef^4+eg^4+f^4g+fg^4\right)+10\\
&\quad\cdot\left(e^3f^2+e^3g^2+e^2f^3+e^2g^3+f^3g^2+f^2g^3\right)+20\left(e^3fg+ef^3g\right.\\
&\quad\left.+efg^3\right)+30\left(e^2f^2g+e^2fg^2+ef^2g^2\right)\\
&= e^5+f^5+g^5+5\left[\left(e^4+f^4+g^4\right)(e+f+g)-\left(e^5+f^5+g^5\right)\right]\\
&\quad+10\left[\left(e^3+f^3+g^3\right)\left(e^2+f^2+g^2\right)-\left(e^5+f^5+g^5\right)\right]+20efg\\
&\quad\cdot\left(e^2+f^2+g^2\right)+30e^2f^2g^2\left(e^5+f^5+g^5\right)\\
&= -14\left(e^5+f^5+g^5\right)+5a\left(e^4+f^4+g^4\right)+10a^2\left(e^3+f^3+g^3\right)\\
&\quad-20a\left(e^4+f^4+g^4\right)+20\left(e^5+f^5+g^5\right)-20efg\left(e^2+f^2+g^2\right)\\
&\quad+20efg\left(e^2+f^2+g^2\right)+30e^2f^2g^2\left(e^5+f^5+g^5\right)+bt_{10}\\
&= 6\left(e^5+f^5+g^5\right)+30e^2f^2g^2\left(e^5+f^5+g^5\right)-15a\left(e^4+f^4+g^4\right)\\
&\quad+10a^2\left(e^3+f^3+g^3\right)+bt_{10}.
\end{aligned}
\tag{22}
$$

Therefore, $a^5 - 6(e^5 + f^5 + g^5) - 30e^2f^2g^2(e^5 + f^5 + g^5) + 15a(e^4 + f^4 + g^4) - 10a^2(e^3 + f^3 + g^3)$ is nilpotent. Since

$$\begin{aligned}
(a-b)^6 &= (e+f+g)^6 \\
&= e^6 + f^6 + g^6 + 6\left(e^5f + e^5g + ef^5 + eg^5 + f^5g + fg^5\right) + 15\left(e^4f^2\right. \\
&\quad \left. + e^4g^2 + e^2f^4 + e^2g^4 + f^4g^2 + f^2g^4\right) + 20\left(e^3f^3 + e^3g^3 + f^3g^3\right) \\
&\quad + 30\left(e^4fg + ef^4g + efg^4\right) + 60\left(e^3f^2g + e^3fg^2 + e^2f^3g + e^2fg^3\right. \\
&\quad \left. + ef^3g^2 + ef^2g^3\right) + 90e^2f^2g^2 \\
&= e^6 + f^6 + g^6 + 6\left[\left(e^5 + f^5 + g^5\right)(e+f+g) - \left(e^6 + f^6 + g^6\right)\right] \\
&\quad + 15\left[\left(e^4 + f^4 + g^4\right)\left(e^2 + f^2 + g^2\right) - \left(e^6 + f^6 + g^6\right)\right] + 10\left[\left(e^3 + f^3\right.\right. \\
&\quad \left.\left. + g^3\right)\left(e^3 + f^3 + g^3\right) - \left(e^6 + f^6 + g^6\right)\right] + 30efg\left(e^3 + f^3 + g^3\right) \\
&\quad + 60efg\left[\left(e^2 + f^2 + g^2\right)(e+f+g) - \left(e^3 + f^3 + g^3\right)\right] + 90e^2f^2g^2 \\
&= -30\left(e^6 + f^6 + g^6\right) + 6(a-b)\left(e^5 + f^5 + g^5\right) + 15\left(e^4 + f^4 + g^4\right) \\
&\qquad \cdot \left[(e+f+g)^2 - 2(ef + eg + fg)\right] + 10\left(e^3 + f^3 + g^3\right)\left[(e+f+g)^3\right. \\
&\quad \left. - 3\left(e^2f + e^2g + ef^2 + eg^2 + f^2g + fg^2\right) - 6efg\right] + 30efg\left(e^3\right. \\
&\quad \left. + f^3 + g^3\right) + 60(a-b)efg\left(e^2 + f^2 + g^2\right) - 60efg\left(e^3 + f^3 + g^3\right) \\
&\quad + 90e^2f^2g^2 \\
&= -30\left(e^6 + f^6 + g^6\right) + 6a\left(e^5 + f^5 + g^5\right) + 15a^2\left(e^4 + f^4 + g^4\right) \\
&\quad - 30a\left(e^5 + f^5 + g^5\right) + 30\left(e^6 + f^6 + g^6\right) - 30efg\left(e^3 + f^3 + g^3\right) \\
&\quad + 10a^3\left(e^3 + f^3 + g^3\right) - 30a\left(e^5 + f^5 + g^5\right) + 30\left(e^6 + f^6 + g^6\right) \\
&\quad - 30a^2\left(e^4 + f^4 + g^4\right) + 30\left(e^6 + f^6 + g^6\right) + 60a\left(e^5 + f^5 + g^5\right) \\
&\quad - 60\left(e^6 + f^6 + g^6\right) + 60efg\left(e^3 + f^3 + g^3\right) - 30aefg\left(e^2 + f^2\right. \\
&\quad \left. + g^2\right) + 30efg\left(e^3 + f^3 + g^3\right) - 60efg\left(e^3 + f^3 + g^3\right) + 30efg \\
&\qquad \cdot \left(e^3 + f^3 + g^3\right) + 60aefg\left(e^2 + f^2 + g^2\right) - 60efg\left(e^3 + f^3 + g^3\right) \\
&\quad + 90e^2f^2g^2 + bt_{11} \\
&= 6a\left(e^5 + f^5 + g^5\right) - 15a^2\left(e^4 + f^4 + g^4\right) + 10a^3\left(e^3 + f^3 + g^3\right) \\
&\quad - 30efg\left(e^3 + f^3 + g^3\right) + 30aefg\left(e^2 + f^2 + g^2\right) + 90e^2f^2g^2 + bt_{11}.
\end{aligned} \tag{23}$$

So, $a^6 - 6a(e^5 + f^5 + g^5) + 15a^2(e^4 + f^4 + g^4) - 10a^3(e^3 + f^3 + g^3) + 30efg(e^3 + f^3 + g^3) - 30aefg(e^2 + f^2 + g^2) - 90ae^2f^2g^2$ is nilpotent. Since

$$
\begin{aligned}
(a-b)^7 &= (e+f+g)^7 \\
&= e^7 + f^7 + g^7 + 7\left(e^6 f + e^6 g + ef^6 + eg^6 + f^6 g + fg^6\right) + 21\left(e^5 f^2\right. \\
&\quad \left. + e^5 g^2 + e^2 f^5 + e^2 g^5 + f^5 g^2 + f^2 g^5\right) + 35\left(e^4 f^3 + e^4 g^3 + e^3 f^4\right. \\
&\quad \left. + e^3 g^4 + f^4 g^3 + f^3 g^4\right) + 42\left(e^5 fg + ef^5 g + efg^5\right) + 105\left(e^4 f^2 g\right. \\
&\quad \left. + e^4 fg^2 + e^2 f^4 g + e^2 fg^4 + ef^4 g^2 + ef^2 g^4\right) + 140\left(e^3 f^3 g + e^3 fg^3\right. \\
&\quad \left. + ef^3 g^3\right) + 210\left(e^3 f^2 g^2 + e^2 f^3 g^2 + e^2 f^2 g^3\right) \\
&= e + f + g + 7a\left(e^6 + f^6 + g^6\right) - 7a + 21a^2\left(e^5 + f^5 + g^5\right) - 42a\left(e^6\right. \\
&\quad \left. + f^6 + g^6\right) - 42efg\left(e^4 + f^4 + g^4\right) + 21a + 105a\left(e^6 + f^6 + g^6\right) \\
&\quad - 105a^2\left(e^5 + f^5 + g^5\right) + 35a^3\left(e^4 + f^4 + g^4\right) + 105efg\left(e^4 + f^4\right. \\
&\quad \left. + g^4\right) - 105aefg\left(e^3 + f^3 + g^3\right) - 35a + 42efg\left(e^4 + f^4 + g^4\right) \\
&\quad + 105aefg\left(e^3 + f^3 + g^3\right) - 105efg\left(e^4 + f^4 + g^4\right) + 70a^2 efg\left(e^2\right. \\
&\quad \left. + f^2 + g^2\right) - 140aefg\left(e^3 + f^3 + g^3\right) + 70efg\left(e^4 + f^4 + g^4\right) \\
&\quad - 140ae^2 f^2 g^2 + 210ae^2 f^2 g^2 + bt_{12} \\
&= -20a + 70a\left(e^6 + f^6 + g^6\right) - 84a^2\left(e^5 + f^5 + g^5\right) + 35a^3\left(e^4\right. \\
&\quad \left. + f^4 + g^4\right) + 70efg\left(e^4 + f^4 + g^4\right) - 140aefg\left(e^3 + f^3 + g^3\right) \\
&\quad + 70a^2 efg\left(e^2 + f^2 + g^2\right) + 70ae^2 f^2 g^2 + bt_{12}.
\end{aligned}
\tag{24}
$$

Hence, $a^7 + 20a - 70a(e^6 + f^6 + g^6) + 84a^2(e^5 + f^5 + g^5) - 35a^3(e^4 + f^4 + g^4) - 70efg(e^4 + f^4 + g^4) + 140aefg(e^3 + f^3 + g^3) - 70a^2efg(e^2 + f^2 + g^2) - 70ae^2f^2g^2$ is nilpotent. □

Lemma 6. *$f(a)efg$ is nilpotent, where $f(a) = 630a - 630$.*

Proof. Substituting (8) and (9) into (19) multiplied by $7efg$ gives

$$
\begin{aligned}
&- 78a^5 efg - 330a^2 e^2 f^2 g^2 - 30a^5 e^3 f^3 g^3 - 240a^2 e^4 f^4 g^4 - 60a^5 e^5 f^5 g^5 \\
&- 480a^2 e^6 f^6 g^6 - 42efg\left(e^5 + f^5 + g^5\right) + 270a^3 e^2 f^2 g^2\left(e^5 + f^5 + g^5\right) \\
&- 210e^3 f^3 g^3\left(e^5 + f^5 + g^5\right) + 120a^3 e^4 f^4 g^4\left(e^5 + f^5 + g^5\right) \\
&+ 240a^3 e^6 f^6 g^6\left(e^5 + f^5 + g^5\right),
\end{aligned}
\tag{25}
$$

is nilpotent. Substituting (7), (8), and (9) into (20) multiplied by $7efg$ gives

$$\begin{aligned}&-78a^6efg-330a^3e^2f^2g^2-\left(30a^6+630a-630\right)e^3f^3g^3-240a^3e^4f^4g^4\\&-60a^6e^5f^5g^5-480a^3e^6f^6g^6-42aefg\left(e^5+f^5+g^5\right)+270a^4e^2f^2g^2\left(e^5\right.\\&\left.+f^5+g^5\right)-210ae^3f^3g^3\left(e^5+f^5+g^5\right)+120a^4e^4f^4g^4\left(e^5+f^5+g^5\right)\\&+240a^4e^6f^6g^6\left(e^5+f^5+g^5\right),\end{aligned}\tag{26}$$

is nilpotent. Multiplying (25) by a in combination with (26) gives

$$(630a-630)e^3f^3g^3,\tag{27}$$

is nilpotent. That is, $(630a-630)efg$ is nilpotent. □

Lemma 7. *$h(a)(ef+eg+fg)$ is nilpotent, where $h(a)=f(a)A$, A is the following determinant*

$$\begin{vmatrix}20a-20a^7 & 140a^5 & -280a^3 & 140a & 0 & 0\\ 0 & 20a-20a^7 & 140a^5 & -280a^3 & 140a & 0\\ 0 & 0 & 20a-20a^7 & 140a^5 & -280a^3 & 140a\\ 140a & 0 & 0 & 20a-20a^7 & 140a^5 & -280a^3\\ -280a^3 & 140a & 0 & 0 & 20a-20a^7 & 140a^5\\ 140a^5 & -280a^3 & 140a & 0 & 0 & 20a-20a^7\end{vmatrix}.\tag{28}$$

Proof. As

$$\begin{aligned}&(e+f+g)^4\\&\quad=e^4+f^4+g^4+4\left(e^3f+e^3g+ef^3+eg^3+f^3g+fg^3\right)+6\left(e^2f^2\right.\\&\qquad\left.+e^2g^2+f^2g^2\right)+12\left(e^2fg+ef^2g+efg^2\right)\\&\quad=e^4+f^4+g^4+4(ef+eg+fg)\left(e^2+f^2+g^2\right)-4\left(e^2fg+ef^2g\right.\\&\qquad\left.+efg^2\right)+6\left(e^2f^2+e^2g^2+f^2g^2\right)+12\left(e^2fg+ef^2g+efg^2\right)\\&\quad=e^4+f^4+g^4+4(ef+eg+fg)\left[(e+f+g)^2-2(ef+eg+fg)\right]\\&\qquad+6\left(e^2f^2+e^2g^2+f^2g^2\right)+8efg(e+f+g)\\&\quad=e^4+f^4+g^4+4(e+f+g)^2(ef+eg+fg)-8\left(e^2f^2+e^2g^2+f^2g^2\right)\\&\qquad-16efg(e+f+g)+6\left(e^2f^2+e^2g^2+f^2g^2\right)+8efg(e+f+g)\\&\quad=e^4+f^4+g^4+4(e+f+g)^2(ef+eg+fg)-2\left(e^2f^2+e^2g^2+f^2g^2\right)\\&\qquad-8efg(e+f+g).\end{aligned}\tag{29}$$

So, $e^4 + f^4 + g^4 = a^4 - 4a^2(ef + eg + fg) + 2(e^2f^2 + e^2g^2 + f^2g^2) + 8aefg + bt_{13}$. Now, multiplying both sides of this formula by $f(a)$ and by Lemma 6, we obtain

$$f(a)\left(e^4 + f^4 + g^4\right) - f(a)\left[a^4 - 4a^2(ef + eg + fg) + 2\left(e^2f^2 + e^2g^2 + f^2g^2\right)\right], \tag{30}$$

is nilpotent. Since

$$\begin{aligned}
&(e + f + g)^6 \\
&= e^6 + f^6 + g^6 + 6\left(e^5f + e^5g + ef^5 + eg^5 + f^5g + fg^5\right) + 15\left(e^4f^2\right. \\
&\quad \left. + e^4g^2 + e^2f^4 + e^2g^4 + f^4g^2 + f^2g^4\right) + 20\left(e^3f^3 + e^3g^3 + f^3g^3\right) \\
&\quad + 30\left(e^4fg + ef^4g + efg^4\right) + 60\left(e^3f^2g + e^3fg^2 + e^2f^3g + e^2fg^3\right. \\
&\quad \left. + ef^3g^2 + ef^2g^3\right) + 90e^2f^2g^2 \\
&= e^6 + f^6 + g^6 + 6\left[\left(e^5 + f^5 + g^5\right)(e + f + g) - \left(e^6 + f^6 + g^6\right)\right] \\
&\quad + 15\left(e^2f^2 + e^2g^2 + f^2g^2\right)\left(e^2 + f^2 + g^2\right) - 45e^2f^2g^2 + 20\left(e^3f^3\right. \\
&\quad \left. + e^3g^3 + f^3g^3\right) + 30efg\left(e^3 + f^3 + g^3\right) + 60efg\left(e^2f + e^2g + ef^2\right. \\
&\quad \left. + eg^2 + f^2g + fg^2\right) + 90e^2f^2g^2 \\
&= -5\left(e^6 + f^6 + g^6\right) + 6(e + f + g)\left(e^5 + f^5 + g^5\right) + 15\left(e^2f^2 + e^2g^2\right. \\
&\quad \left. + f^2g^2\right)\left[(e + f + g)^2 - 2(ef + eg + fg)\right] + 20\left(e^3f^3 + e^3g^3 + f^3g^3\right) \\
&\quad + 30efg\left(e^3 + f^3 + g^3\right) + 60efg\left(e^2f + e^2g + ef^2 + eg^2 + f^2g + fg^2\right) \\
&\quad + 45e^2f^2g^2 \\
&= -5\left(e^6 + f^6 + g^6\right) + 6(e + f + g)\left(e^5 + f^5 + g^5\right) + 15(e + f + g)^2 \\
&\quad \cdot \left(e^2f^2 + e^2g^2 + f^2g^2\right) - 30\left(e^3f^3 + e^3g^3 + f^3g^3\right) - 30efg\left(e^2f + e^2g\right. \\
&\quad \left. + ef^2 + eg^2 + f^2g + fg^2\right) + 20\left(e^3f^3 + e^3g^3 + f^3g^3\right) \\
&\quad + 30efg\left(e^3 + f^3 + g^3\right) + 60efg\left(e^2f + e^2g + ef^2 + eg^2 + f^2g + fg^2\right) + 45e^2f^2g^2 \\
&= -5\left(e^6 + f^6 + g^6\right) + 6a\left(e^5 + f^5 + g^5\right) + 15a^2\left(e^2f^2 + e^2g^2 + f^2g^2\right) \\
&\quad - 10\left(e^3f^3 + e^3g^3 + f^3g^3\right) + 30efg\left(e^3 + f^3 + g^3\right) + 30efg\left(e^2f + e^2g\right. \\
&\quad \left. + ef^2 + eg^2 + f^2g + fg^2\right) + 45e^2f^2g^2 + bt_{14}.
\end{aligned} \tag{31}$$

So, $a^6 + 5(e^6 + f^6 + g^6) - 6a(e^5 + f^5 + g^5) - 15a^2(e^2f^2 + e^2g^2 + f^2g^2) + 10(e^3f^3 + e^3g^3 + f^3g^3) - 30efg(e^3 + f^3 + g^3) - 30efg(e^2f + e^2g + ef^2 + eg^2 + f^2g + fg^2) - 45e^2f^2g^2$ is nilpotent. Now, multiplying this formula by $f(a)$ and by Lemma 6, we obtain

$$\begin{aligned}
&f(a)\left[a^6 + 5\left(e^6 + f^6 + g^6\right) - 6a\left(e^5 + f^5 + g^5\right) - 15a^2\left(e^2f^2 + e^2g^2\right.\right. \\
&\quad \left.\left. + f^2g^2\right) + 10\left(e^3f^3 + e^3g^3 + f^3g^3\right)\right],
\end{aligned} \tag{32}$$

is nilpotent. Substituting (30) into (21) multiplied by $f(a)$ gives

$$f(a)\left[a^7 + 20a - 70a\left(e^6 + f^6 + g^6\right) + 84a^2\left(e^5 + f^5 + g^5\right) - 35a^3\left[a^4 - 4a^2(ef + eg + fg) + 2\left(e^2f^2 + e^2g^2 + f^2g^2\right)\right]\right], \tag{33}$$

is nilpotent. Multiplying (32) by $14a$ in combination with (33) gives

$$f(a)\left[-20a^7 + 20a + 140a^5(ef + eg + fg) - 280a^3\left(e^2f^2 + e^2g^2 + f^2g^2\right) + 140a\left(e^3f^3 + e^3g^3 + f^3g^3\right)\right], \tag{34}$$

is nilpotent. As

$$\begin{aligned}
f(a)(ef + eg + fg)^2 &= f(a)\left(e^2f^2 + e^2g^2 + f^2g^2\right) + f(a)efg\alpha_1,\\
f(a)(ef + eg + fg)^3 &= f(a)\left(e^3f^3 + e^3g^3 + f^3g^3\right) + f(a)efg\alpha_2,\\
f(a)(ef + eg + fg)^4 &= f(a)\left(e^4f^4 + e^4g^4 + f^4g^4\right) + f(a)efg\alpha_3,\\
f(a)(ef + eg + fg)^5 &= f(a)\left(e^5f^5 + e^5g^5 + f^5g^5\right) + f(a)efg\alpha_4,\\
f(a)(ef + eg + fg)^6 &= f(a)\left(e^6f^6 + e^6g^6 + f^6g^6\right) + f(a)efg\alpha_5,\\
f(a)(ef + eg + fg)^7 &= f(a)(ef + eg + fg) + f(a)efg\alpha_6.
\end{aligned} \tag{35}$$

Hence, multiplying (34) by $(ef + eg + fg)$, $(ef + eg + fg)^2$, $(ef + eg + fg)^3$, $(ef + eg + fg)^4$, $(ef + eg + fg)^5$, $(ef + eg + fg)^6$, respectively. By Lemma 6, we obtain

$$\begin{aligned}
&f(a)\left[\left(-20a^7 + 20a\right)(ef + eg + fg) + 140a^5\left(e^2f^2 + e^2g^2 + f^2g^2\right) - 280a^3\left(e^3f^3 + e^3g^3 + f^3g^3\right) + 140a\left(e^4f^4 + e^4g^4 + f^4g^4\right)\right],\\
&f(a)\left[\left(-20a^7 + 20a\right)\left(e^2f^2 + e^2g^2 + f^2g^2\right) + 140a^5\left(e^3f^3 + e^3g^3 + f^3g^3\right) - 280a^3\left(e^4f^4 + e^4g^4 + f^4g^4\right) + 140a\left(e^5f^5 + e^5g^5 + f^5g^5\right)\right],\\
&f(a)\left[\left(-20a^7 + 20a\right)\left(e^3f^3 + e^3g^3 + f^3g^3\right) + 140a^5\left(e^4f^4 + e^4g^4 + f^4g^4\right) - 280a^3\left(e^5f^5 + e^5g^5 + f^5g^5\right) + 140a\left(e^6f^6 + e^6g^6 + f^6g^6\right)\right],\\
&f(a)\left[\left(-20a^7 + 20a\right)\left(e^4f^4 + e^4g^4 + f^4g^4\right) + 140a^5\left(e^5f^5 + e^5g^5 + f^5g^5\right) - 280a^3\left(e^6f^6 + e^6g^6 + f^6g^6\right) + 140a(ef + eg + fg)\right],\\
&f(a)\left[\left(-20a^7 + 20a\right)\left(e^5f^5 + e^5g^5 + f^5g^5\right) + 140a^5\left(e^6f^6 + e^6g^6 + f^6g^6\right) - 280a^3(ef + eg + fg) + 140a\left(e^2f^2 + e^2g^2 + f^2g^2\right)\right],\\
&f(a)\left[\left(-20a^7 + 20a\right)\left(e^6f^6 + e^6g^6 + f^6g^6\right) + 140a^5(ef + eg + fg) - 280a^3\left(e^2f^2 + e^2g^2 + f^2g^2\right) + 140a\left(e^3f^3 + e^3g^3 + f^3g^3\right)\right],
\end{aligned} \tag{36}$$

are nilpotent, where coefficient determinant of exactly $h(a)$. So, $h(a)(ef + eg + fg)$ is nilpotent. □

Lemma 8. *$f(a)h(a)(20a - 20a^7)$ is nilpotent.*

Proof. Multiplying $h(a)$ by (34) and by Lemma 7, we obtain $f(a)h(a)(20a - 20a^7)$ is nilpotent. □

Lemma 9. *A ring R is a strongly $S_{3,7}$ nil-clean ring if and only if $R = R_1 \oplus R_2 \oplus R_3 \oplus R_4 \oplus R_5 \oplus R_6$, where R_1, R_2, R_3, R_4, R_5, and R_6 are strongly $S_{3,7}$ nil-clean rings with $2 \in Nil(R_1)$, $3 \in Nil(R_2)$, $5 \in Nil(R_3)$, $7 \in Nil(R_4)$, $13 \in Nil(R_5)$, and $19 \in Nil(R_6)$.*

Proof. The necessity is obvious. To prove sufficiency below, we first prove that $2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19$ is nilpotent. By Lemma 8, let $a = 3$, we deduce that $2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 103 \cdot 673$ is nilpotent. Let $a = 4$, we deduce that $2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 23 \cdot 31 \cdot 73 \cdot 1879$ is nilpotent. So, $2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19$ is nilpotent. Therefore, there exists an integer $n \geq 1$ such that $2^nR \cap 3^nR \cap 5^nR \cap 7^nR \cap 13^nR \cap 19^nR = 0$. By the Chinese Remainder Theorem, $R = R_1 \oplus R_2 \oplus R_3 \oplus R_4 \oplus R_5 \oplus R_6$, where $R_1 \cong R/2^nR$, $R_2 \cong R/3^nR$, $R_3 \cong R/5^nR$, $R_4 \cong R/7^nR$, $R_5 \cong R/13^nR$, $R_6 \cong R/19^nR$, and R_1, R_2, R_3, R_4, R_5, R_6 are strongly $S_{3,7}$ nil-clean rings. □

Lemma 10. *Let R be a strongly $S_{3,7}$ nil-clean ring with $2 \in Nil(R)$. The following is established:*

(1) $a^4 - a$ is nilpotent for all $a \in R$.

(2) $J(R)$ is nil and $R/J(R)$ is a subdirect product of rings isomorphic to $\mathbb{Z}_2$ or $\mathbb{F}_4$.

Proof. (1) $\Longleftrightarrow$ (2) (see [10], Theorem 3.5).

As $a \in R$ and $a = b + e + f + g$, where $b \in \mathrm{Nil}(R)$, $e^7 = e, f^7 = f, g^7 = g$ and b, e, f, g commute with one another. If x is a 7-potent of R, then $(x^4 - x)^2 = 2(x^2 - x^5)$, so $x^4 - x \in \mathrm{Nil}(R)$. As there exists polynomials $\alpha(x_1, x_2, x_3, x_4), \beta(x_1, x_2, x_3, x_4) \in \mathbb{Z}(x_1, x_2, x_3, x_4)$ such that $a^4 - a = (e^4 - e) + (f^4 - f) + (g^4 - g) + 2 \cdot \alpha\ (b, e, f, g) + b \cdot \beta(b, e, f, g)$. So, $a^4 - a$ is nilpotent. □

Lemma 11. *Let R be a ring with $3 \in Nil(R)$. The following is equivalent:*

(1) R is a strongly $S_{3,7}$ nil-clean ring.

(2) $a^3 - a$ is nilpotent for all $a \in R$.

(3) Every element of R is the sum of a nilpotent and a tripotent that commute.

(4) $J(R)$ is nil and $R/J(R)$ is a subdirect product of $\mathbb{Z}_3$'s.

Proof. (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) (see [11], Proposition 2.8).

(3) $\Longrightarrow$ (1) The implication is clear.

(1) $\Longrightarrow$ (2) As $a \in R$ and $a = b + e + f + g$, where $b \in \mathrm{Nil}(R)$, $e^7 = e, f^7 = f, g^7 = g$ and b, e, f, g commute with one another. If x is a 7-potent of R, then $(x^3 - x)^3 = 3(x^5 - x)$, so $x^3 - x \in \mathrm{Nil}(R)$. As there exist polynomials $\alpha(x_1, x_2, x_3, x_4), \beta(x_1, x_2, x_3, x_4) \in \mathbb{Z}(x_1, x_2, x_3, x_4)$ such that $a^3 - a = (e^3 - e) + (f^3 - f) + (g^3 - g) + 3 \cdot \alpha(b, e, f, g) + b \cdot \beta(b, e, f, g)$. So, $a^3 - a$ is nilpotent. □

Lemma 12. *Let R be a strongly $S_{3,7}$ nil-clean ring with $5 \in Nil(R)$, then $a^{25} - a$ is nilpotent for all $a \in R$.*

Proof. As $a \in R$ and $a = b + e + f + g$, where $b \in \mathrm{Nil}(R)$, $e^7 = e, f^7 = f, g^7 = g$ and b, e, f, g commute with one another. As there exist polynomials $\alpha(x_1, x_2, x_3, x_4), \beta(x_1, x_2, x_3, x_4) \in \mathbb{Z}(x_1, x_2, x_3, x_4)$ such that $a^{25} - a = 5 \cdot \alpha(b, e, f, g) + b \cdot \beta(b, e, f, g)$. So, $a^{25} - a$ is nilpotent. □

Lemma 13. *Let R be a ring with $7 \in Nil(R)$. The following is equivalent:*

(1) R is a strongly $S_{3,7}$ nil-clean ring.

(2) $a^7 - a$ is nilpotent for all $a \in R$.

(3) Every element of R is the sum of a nilpotent and three tripotents that commute.

(4) $J(R)$ is nil and $R/J(R)$ is a subdirect product of $\mathbb{Z}_7$'s.

Proof. (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) (see [8], Lemma 9).

(3) $\Longrightarrow$ (1) The implication is clear.

(1) $\Longrightarrow$ (2) As $a \in R$ and $a = b + e + f + g$, where $b \in \mathrm{Nil}(R)$, $e^7 = e, f^7 = f, g^7 = g$ and b, e, f, g commute with one another. As there exist polynomials $\alpha(x_1, x_2, x_3, x_4), \beta(x_1, x_2, x_3, x_4) \in \mathbb{Z}(x_1, x_2, x_3, x_4)$ such that $a^7 - a = 7 \cdot \alpha(b, e, f, g) + b \cdot \beta(b, e, f, g)$. So, $a^7 - a$ is nilpotent. □

Lemma 14 (see [7], Lemma 3.5). *Let $a \in R$. If $a^2 - a$ is nilpotent, then there exists a polynomial $\theta(t) \in \mathbb{Z}(t)$ such that $\theta(a)^2 = \theta(a)$ and $a - \theta(a)$ is nilpotent.*

Lemma 15 (see [11], Lemma 2.6). *If $2 \in U(R)$ and $a^3 - a$ is nilpotent, then there exists a polynomial $\theta(t) \in \mathbb{Z}(t)$ such that $\theta(a)^3 = \theta(a)$ and $a - \theta(a)$ is nilpotent.*

Lemma 16. *If R is a subdirect product of $\mathbb{Z}_{13}$'s and $x \in R$, then there exist polynomials $\alpha(t), \beta(t), \gamma(t), \eta(t), \lambda(t), \mu(t) \in \mathbb{Z}[t]$ such that $x = \alpha(x) + \beta(x) + \gamma(x) + \eta(x) + \lambda(x) + \mu(x)$ and $\alpha(x), \beta(x), \gamma(x), \eta(x), \lambda(x), \mu(x)$ are tripotents.*

Proof. Let R be a subdirect product of R_α with $\alpha \in \Lambda$, where $R_\alpha = \mathbb{Z}_{13}$. Then R is a subring of ΠR_α. We write $x = (x_\alpha) \in R$, let Λ be a disjoint union of $\Lambda_0, \Lambda_1, \ldots, \Lambda_{12}$ such that $x_\alpha = i \Longleftrightarrow \alpha \in \Lambda_i, i = 0, 1, \ldots, 12$. Let $x = (0_{\Lambda_0}, 1_{\Lambda_1}, 2_{\Lambda_2}, 3_{\Lambda_3}, 4_{\Lambda_4}, 5_{\Lambda_5}, 6_{\Lambda_6}, 7_{\Lambda_7}, 8_{\Lambda_8}, 9_{\Lambda_9}, 10_{\Lambda_{10}}, 11_{\Lambda_{11}}, 12_{\Lambda_{12}})$ and set

$$\begin{aligned}
e_1 &= \left(0_{\Lambda_0}, 1_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}, 0_{\Lambda_5}, 0_{\Lambda_6}, 0_{\Lambda_7}, 0_{\Lambda_8}, 0_{\Lambda_9}, 0_{\Lambda_{10}}, 0_{\Lambda_{11}}, 0_{\Lambda_{12}}\right),\\
e_2 &= \left(0_{\Lambda_0}, 0_{\Lambda_1}, 1_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}, 0_{\Lambda_5}, 0_{\Lambda_6}, 0_{\Lambda_7}, 0_{\Lambda_8}, 0_{\Lambda_9}, 0_{\Lambda_{10}}, 0_{\Lambda_{11}}, 0_{\Lambda_{12}}\right),\\
e_3 &= \left(0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 1_{\Lambda_3}, 0_{\Lambda_4}, 0_{\Lambda_5}, 0_{\Lambda_6}, 0_{\Lambda_7}, 0_{\Lambda_8}, 0_{\Lambda_9}, 0_{\Lambda_{10}}, 0_{\Lambda_{11}}, 0_{\Lambda_{12}}\right),\\
e_4 &= \left(0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 1_{\Lambda_4}, 0_{\Lambda_5}, 0_{\Lambda_6}, 0_{\Lambda_7}, 0_{\Lambda_8}, 0_{\Lambda_9}, 0_{\Lambda_{10}}, 0_{\Lambda_{11}}, 0_{\Lambda_{12}}\right),\\
e_5 &= \left(0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}, 1_{\Lambda_5}, 0_{\Lambda_6}, 0_{\Lambda_7}, 0_{\Lambda_8}, 0_{\Lambda_9}, 0_{\Lambda_{10}}, 0_{\Lambda_{11}}, 0_{\Lambda_{12}}\right),\\
e_6 &= \left(0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}, 0_{\Lambda_5}, 1_{\Lambda_6}, 0_{\Lambda_7}, 0_{\Lambda_8}, 0_{\Lambda_9}, 0_{\Lambda_{10}}, 0_{\Lambda_{11}}, 0_{\Lambda_{12}}\right),\\
e_7 &= \left(0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}, 0_{\Lambda_5}, 0_{\Lambda_6}, 1_{\Lambda_7}, 0_{\Lambda_8}, 0_{\Lambda_9}, 0_{\Lambda_{10}}, 0_{\Lambda_{11}}, 0_{\Lambda_{12}}\right),\\
e_8 &= \left(0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}, 0_{\Lambda_5}, 0_{\Lambda_6}, 0_{\Lambda_7}, 1_{\Lambda_8}, 0_{\Lambda_9}, 0_{\Lambda_{10}}, 0_{\Lambda_{11}}, 0_{\Lambda_{12}}\right),\\
e_9 &= \left(0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}, 0_{\Lambda_5}, 0_{\Lambda_6}, 0_{\Lambda_7}, 0_{\Lambda_8}, 1_{\Lambda_9}, 0_{\Lambda_{10}}, 0_{\Lambda_{11}}, 0_{\Lambda_{12}}\right),\\
e_{10} &= \left(0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}, 0_{\Lambda_5}, 0_{\Lambda_6}, 0_{\Lambda_7}, 0_{\Lambda_8}, 0_{\Lambda_9}, 1_{\Lambda_{10}}, 0_{\Lambda_{11}}, 0_{\Lambda_{12}}\right),\\
e_{11} &= \left(0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}, 0_{\Lambda_5}, 0_{\Lambda_6}, 0_{\Lambda_7}, 0_{\Lambda_8}, 0_{\Lambda_9}, 0_{\Lambda_{10}}, 1_{\Lambda_{11}}, 0_{\Lambda_{12}}\right),\\
e_{12} &= \left(0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}, 0_{\Lambda_5}, 0_{\Lambda_6}, 0_{\Lambda_7}, 0_{\Lambda_8}, 0_{\Lambda_9}, 0_{\Lambda_{10}}, 0_{\Lambda_{11}}, 1_{\Lambda_{12}}\right).
\end{aligned} \tag{37}$$

One can show that there exist polynomials $\theta_i(t) \in \mathbb{Z}[t]$ such that $e_i = \theta_i(x)$ for $i = 1, 2, \ldots, 12$. Indeed,

$$\begin{aligned}
e_1 &= x^{12} - y^{12}, y = x - x^{12},\\
e_2 &= y^{12} - z^{12}, z = y - y^{12},\\
e_3 &= z^{12} - w^{12}, w = z - z^{12},\\
&\cdots\\
e_{10} &= u^{12} - v^{12}, v = u - u^{12},\\
e_{11} &= 2v^{12} - v,\\
e_{12} &= v - v^{12}.
\end{aligned} \tag{38}$$

So, $e_i \in R$ for $i = 1,2,\ldots,12$. Let $e = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + 12e_7 + 12e_8 + 12e_9 + 12e_{10} + 12e_{11} + 12e_{12}$, $f = e_2 + e_3 + e_4 + e_5 + e_6 + 12e_7 + 12e_8 + 12e_9 + 12e_{10} + 12e_{11}$, $g = e_3 + e_4 + e_5 + e_6 + 12e_7 + 12e_8 + 12e_9 + 12e_{10}$, $h = e_4 + e_5 + e_6 + 12e_7 + 12e_8 + 12e_9$, $i = e_5 + e_6 + 12e_7 + 12e_8$, $j = e_6 + 12e_7$, then e, f, g, h, i, j are tripotents and $x = e + f + g + h + i + j$. With

$$\begin{aligned}
\alpha(t) &= \theta_1(t) + \theta_2(t) + \theta_3(t) + \theta_4(t) + \theta_5(t) + \theta_6(t) + 12\theta_7(t)\\
&\quad + 12\theta_8(t) + 12\theta_9(t) + 12\theta_{10}(t) + 12\theta_{11}(t) + 12\theta_{12}(t),\\
\beta(t) &= \theta_2(t) + \theta_3(t) + \theta_4(t) + \theta_5(t) + \theta_6(t) + 12\theta_7(t) + 12\theta_8(t)\\
&\quad + 12\theta_9(t) + 12\theta_{10}(t) + 12\theta_{11}(t),\\
\gamma(t) &= \theta_3(t) + \theta_4(t) + \theta_5(t) + \theta_6(t) + 12\theta_7(t) + 12\theta_8(t) + 12\theta_9(t)\\
&\quad + 12\theta_{10}(t),\\
\eta(t) &= \theta_4(t) + \theta_5(t) + \theta_6(t) + 12\theta_7(t) + 12\theta_8(t) + 12\theta_9(t),\\
\lambda(t) &= \theta_5(t) + \theta_6(t) + 12\theta_7(t) + 12\theta_8(t),\\
\mu(t) &= \theta_6(t) + 12\theta_7(t).
\end{aligned} \tag{39}$$

Thus, we have $e = \alpha(x), f = \beta(x), g = \gamma(x), h = \eta(x), i = \lambda(x), j = \mu(x)$. □

Lemma 17. *Let R be a ring with $13 \in Nil(R)$. The following is equivalent:*

(1) R is a strongly $S_{3,7}$ nil-clean ring.

(2) $a^{13}-a$ is nilpotent for all $a \in R$.

(3) $J(R)$ is nil and $R/J(R)$ is a subdirect product of $\mathbb{Z}_{13}$'s.

(4) Every element of R is the sum of a nilpotent and six tripotents that commute.

Proof. (2) $\Longleftrightarrow$ (3) (see [10], Lemma 3.2).

(3) $\Longrightarrow$ (4) By Lemma 16 and $a \in R$, then there exist polynomials $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\eta(t)$, $\lambda(t)$, $\mu(t) \in \mathbb{Z}[t]$ such that $\overline{a} = \alpha(\overline{a}) + \beta(\overline{a}) + \gamma(\overline{a}) + \eta(\overline{a}) + \lambda(\overline{a}) + \mu(\overline{a})$, where $\alpha(\overline{a}), \beta(\overline{a}), \gamma(\overline{a}), \eta(\overline{a}), \lambda(\overline{a}), \mu(\overline{a}) \in R/J(R)$ and they are tripotents. As $\alpha(\overline{a}) = \overline{\alpha(a)}, \beta(\overline{a}) = \overline{\beta(a)}, \gamma(\overline{a}) = \overline{\gamma(a)}, \eta(\overline{a}) = \overline{\eta(a)}, \lambda(\overline{a}) = \overline{\lambda(a)}, \mu(\overline{a}) = \overline{\mu(a)}$, so $\alpha(a)^3-\alpha(a), \beta(a)^3-\beta(a), \gamma(a)^3-\gamma(a), \eta(a)^3-\eta(a), \lambda(a)^3-\lambda(a), \mu(a)^3-\mu(a) \in J(R)$ are nilpotent. Since $2\cdot 7 = 13+1$ and $13 \in Nil(R)$, so $2 \in U(R)$. By Lemma 15, there exist tripotents $e, f, g, h, i, j \in \mathbb{Z}[a]$ such that $\alpha(a)-e, \beta(a)-f, \gamma(a)-g, \eta(a)-h, \lambda(a)-i, \mu(a)-j$ are nilpotent. Then, $b = a-e-f-g-h-i-j = (a-\alpha(a)-\beta(a)-\gamma(a)-\eta(a)-\lambda(a)-\mu(a)) + (\alpha(a)-e) + (\beta(a)-f) + (\gamma(a)-g) + (\eta(a)-h) + (\lambda(a)-i) + (\mu(a)-j)$ are nilpotent, where e, f, g, h, i, j commute with one another and $a = b+e+f+g+h+i+j$.

(4) $\Longrightarrow$ (1) The implication is clear.

(1) $\Longrightarrow$ (2) As $a \in R$ and $a = b+e+f+g$, where $b \in Nil(R)$, $e^7 = e, f^7 = f, g^7 = g$ and b, e, f, g commute with one another. As there exist polynomials $\alpha(x_1, x_2, x_3, x_4), \beta(x_1, x_2, x_3, x_4) \in \mathbb{Z}(x_1, x_2, x_3, x_4)$ such that $a^{13}-a = 13\cdot\alpha(b, e, f, g) + b\cdot\beta(b, e, f, g)$. So, $a^{13}-a$ is nilpotent. □

Lemma 18. *If R is a subdirect product of $\mathbb{Z}_{19}$'s and $x \in R$, then there exist polynomials $\alpha(t), \beta(t), \gamma(t), \eta(t), \lambda(t), \mu(t), \rho(t), \varphi(t), \upsilon(t) \in \mathbb{Z}[t]$ such that $x = \alpha(x) + \beta(x) + \gamma(x) + \eta(x) + \lambda(x) + \mu(x) + \rho(x) + \varphi(x) + \upsilon(x)$ and $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\eta(x)$, $\lambda(x)$, $\mu(x)$, $\rho(x)$, $\varphi(x)$, $\upsilon(x)$ are tripotents.*

Proof. The proof process is the same as Lemma 16. □

Lemma 19. *Let R be a ring with $19 \in Nil(R)$. The following is equivalent:*

(1) R is a strongly $S_{3,7}$ nil-clean ring.

(2) $a^{19}-a$ is nilpotent for all $a \in R$.

(3) $J(R)$ is nil and $R/J(R)$ is a subdirect product of $\mathbb{Z}_{19}$'s.

(4) Every element of R is the sum of a nilpotent and nine tripotents that commute.

Proof. (2) $\Longleftrightarrow$ (3) (see [10], Lemma 3.2).

(3) $\Longrightarrow$ (4) By Lemma 18 and $a \in R$, then there exist polynomials $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\eta(t)$, $\lambda(t)$, $\mu(t)$, $\rho(t)$, $\varphi(t)$, $\upsilon(t) \in \mathbb{Z}[t]$ such that $\overline{a} = \alpha(\overline{a}) + \beta(\overline{a}) + \gamma(\overline{a}) + \eta(\overline{a}) + \lambda(\overline{a}) + \mu(\overline{a}) + \rho(\overline{a}) + \varphi(\overline{a}) + \upsilon(\overline{a})$, where $\alpha(\overline{a})$, $\beta(\overline{a})$, $\gamma(\overline{a})$, $\eta(\overline{a})$, $\lambda(\overline{a})$, $\mu(\overline{a})$, $\rho(\overline{a})$, $\varphi(\overline{a})$, $\upsilon(\overline{a}) \in R/J(R)$ and they are tripotents. As $\alpha(\overline{a}) = \overline{\alpha(a)}, \beta(\overline{a}) = \overline{\beta(a)}, \gamma(\overline{a}) = \overline{\gamma(a)}, \eta(\overline{a}) = \overline{\eta(a)}, \lambda(\overline{a}) = \overline{\lambda(a)}, \mu(\overline{a}) = \overline{\mu(a)}, \rho(\overline{a}) = \overline{\rho(a)}, \varphi(\overline{a}) = \overline{\varphi(a)}, \upsilon(\overline{a}) = \overline{\upsilon(a)}$, so $\alpha(a)^3-\alpha(a), \beta(a)^3-\beta(a), \gamma(a)^3-\gamma(a), \eta(a)^3-\eta(a), \lambda(a)^3-\lambda(a), \mu(a)^3-\mu(a), \rho(a)^3-\rho(a), \varphi(a)^3-\varphi(a), \upsilon(a)^3-\upsilon(a) \in J(R)$ are nilpotent. Since $2\cdot 7 = 13+1$ and $13 \in Nil(R)$, so $2 \in U(R)$. By Lemma 15, there exist tripotents $e, f, g, h, i, j, k, l, m \in \mathbb{Z}[a]$ such that $\alpha(a)-e, \beta(a)-f, \gamma(a)-g, \eta(a)-h, \lambda(a)-i, \mu(a)-j, \rho(a)-k, \varphi(a)-l, \upsilon(a)-m$ are nilpotent. Then, $b = a-e-f-g-h-i-j-k-l-m = (a-\alpha(a)-\beta(a)-\gamma(a)-\eta(a)-\lambda(a)-\mu(a)-\rho(a)-\varphi(a)-\upsilon(a)) + (\alpha(a)-e) + (\beta(a)-f) + (\gamma(a)-g) + (\eta(a)-h) + (\lambda(a)-i) + (\mu(a)-j) + (\rho(a)-k) + (\varphi(a)-l) + (\upsilon(a)-m)$ are nilpotent, where $e, f, g, h, i, j, k, l, m$ commute with one another and $a = b+e+f+g+h+i+j+k+l+m$.

(4) $\Longrightarrow$ (1) The implication is clear.

(1) $\Longrightarrow$ (2) As $a \in R$ and $a = b+e+f+g$, where $b \in Nil(R)$, $e^7 = e, f^7 = f, g^7 = g$ and b, e, f, g commute with one another. As there exist polynomials $\alpha(x_1, x_2, x_3, x_4), \beta(x_1, x_2, x_3, x_4) \in \mathbb{Z}(x_1, x_2, x_3, x_4)$ such that $a^{19}-a = 19\cdot\alpha(b, e, f, g) + b\cdot\beta(b, e, f, g)$. So, $a^{19}-a$ is nilpotent.

We have completed the scheduled promotion. The following are some other properties of rings. □

Theorem 20. *The following are equivalent for a ring R:*

(1) $a-a^5$ is nilpotent for all $a \in R$.

(2) $a^4 \in R$ is strongly nil-clean for all $a \in R$.

Proof. (1) $\Longrightarrow$ (2) As $a \in R$ and $a-a^5 \in \mathrm{Nil}(R)$. Thus, $a^4-a^8 \in \mathrm{Nil}(R)$. By Lemma 14, there exists an idempotent $e \in R$ such that $a^4-e \in \mathrm{Nil}(R)$ and $a^4e = ae^4$. So $a^4 \in R$ is strongly nil-clean.

(2) $\Longrightarrow$ (1) Since $a \in R$ and $a^4 \in R$ is strongly nil-clean, then there exists an idempotent element such that $a^4-a^8 \in \mathrm{Nil}(R)$. So $a^3(a-a^5) \in \mathrm{Nil}(R)$. As $(a-a^5)^4 = a^3(a-a^5)(1-a^4)^3 \in \mathrm{Nil}(R)$. Therefore, $a-a^5 \in \mathrm{Nil}(R)$. □

Theorem 21. *The following are equivalent for a ring R:*

(1) $a-a^7$ is nilpotent for all $a \in R$.

(2) $a^6 \in R$ is strongly nil-clean for all $a \in R$.

Proof. (1) $\Longrightarrow$ (2) As $a \in R$ and $a-a^7 \in \mathrm{Nil}(R)$. Thus, $a^6-a^{12} \in \mathrm{Nil}(R)$. By Lemma 14, there exists an idempotent $e \in R$ such that $a^6-e \in \mathrm{Nil}(R)$ and $a^6e = ae^6$. So $a^6 \in R$ is strongly nil-clean.

(2) $\Longrightarrow$ (1) Since $a \in R$ and $a^6 \in R$ is strongly nil-clean, then there exists an idempotent element such that $a^6-a^{12} \in \mathrm{Nil}(R)$. So $a^5(a-a^7) \in \mathrm{Nil}(R)$. As $(a-a^7)^6 = a^5(a-a^7)(1-a^6)^5 \in \mathrm{Nil}(R)$. Therefore, $a-a^7 \in \mathrm{Nil}(R)$. □

Theorem 22. *The following are equivalent for a ring R:*

(1) $a-a^{13}$ is nilpotent for all $a \in R$.

(2) $a^{12} \in R$ is strongly nil-clean for all $a \in R$.

Proof. (1) $\Longrightarrow$ (2) As $a \in R$ and $a - a^{13} \in \mathrm{Nil}(R)$. Thus $a^{12} - a^{24} \in \mathrm{Nil}(R)$. By Lemma 14, there exists an idempotent $e \in R$ such that $a^{12} - e \in \mathrm{Nil}(R)$ and $a^{12}e = ae^{12}$. So $a^{12} \in R$ is strongly nil-clean.

(2) $\Longrightarrow$ (1) Since $a \in R$ and $a^{12} \in R$ is strongly nil-clean, then there exists an idempotent element such that $a^{12} - a^{24} \in \mathrm{Nil}(R)$. So $a^{11}(a - a^{13}) \in \mathrm{Nil}(R)$. As $(a - a^{13})^{12} = a^{11}(a - a^{13})(1 - a^{12})^{11} \in \mathrm{Nil}(R)$. Therefore, $a - a^{13} \in \mathrm{Nil}(R)$. □

3. Conclusion

In this paper, we mainly study strongly $S_{3,7}$ nil-clean rings and obtain the following important conclusions: a ring R is a strongly $S_{3,7}$ nil-clean ring if and only if $R = R_1 \oplus R_2 \oplus R_3 \oplus R_4 \oplus R_5 \oplus R_6$, where R_1, R_2, R_3, R_4, R_5, and R_6 are strongly $S_{3,7}$ nil-clean rings with $2 \in \mathrm{Nil}(R_1)$, $3 \in \mathrm{Nil}(R_2)$, $5 \in \mathrm{Nil}(R_3)$, $7 \in \mathrm{Nil}(R_4)$, $13 \in \mathrm{Nil}(R_5)$, and $19 \in \mathrm{Nil}(R_6)$.

Due to difficulties encountered in the calculations and time constraints, we regret that we have not been able to generalize this ring to more general cases. For example, (1) determine the rings for which every element is the sum of a nilpotent and n 7-potents that commute with one another. (2) Determine the rings for which every element is the sum of a nilpotent and three n-potents that commute with one another.

The above ideas provide new perspectives for the study of related rings and deserve further exploration.

Data Availability

The authors solemnly declare that the data and literature of our papers named "RINGS IN WHICH EVERY ELEMENT IS A SUM OF A NILPOTENT AND THREE 7-POTENTS" are from published literature. We are responsible for the authenticity of the data. This article is completely open to readers without confidentiality. The author agrees with the information in the paper and agrees that the author of the communication will make a decision on behalf of all the authors without controversy.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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