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Research Article

Almost Existentially Closed Models in Positive Logic

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This paper explores the concept of almost positively closed models in the framework of positive logic. To accomplish this, we initially define various forms of the positive amalgamation property, such as h-amalgamation and symmetric and asymmetric amalgamation properties. Subsequently, we introduce certain structures that enjoy these properties. Following this, we introduce the concepts of Δ-almost positively closed and Δ-weekly almost positively closed. The classes of these structures contain and exhibit properties that closely resemble those of positive existentially closed models. In order to investigate the relationship between positive almost closed and positive strong amalgamation properties, we first introduce the sets of positive algebraic formulas E_T and Alg_T and the properties of positive strong amalgamation. We then show that if a model A of a theory T is a $E_{T^+(A)}$-weekly almost positively closed, then A is a positive strong amalgamation basis of T, and if A is a positive strong amalgamation basis of T, then A is $Al_{T^+(A)}$-weekly almost positively closed.

1. Introduction

The notion of strong amalgamation base in the framework of the general model theory was defined by Bacsich in ([1, 2]). He proved that every strong amalgamation basis of a universal theory T with the amalgamation property is algebraically closed in the sense of Jonsson ([3]) and Robinson ([4]). In ([5]), Eklof has shown that the converse is true in general. He established necessary conditions for members of a special class K to be strong amalgamation basis, even when K is not an elementary class. These conditions are expressed in terms of a strong notion of algebraically closed structures, introduced in ([5]), and utilizing the concept of closure operators.

In the conventional models theory, the strong amalgamation property is a characteristic that a structure M can possess within its class of extensions. This essentially means that, for any extensions A and B of M, there exists a common extension C of A and B such that $A \cap B = M$.

The positive model theory is concerned essentially with the study of h-inductive theories which are built without the use of the negation. Considering positive formulas and homomorphisms instead of embeddings, positive logic generates new situations beyond the scope of logic with negation.

Consequently, when examining the property of amalgamation, homomorphisms are predominantly utilized instead of embeddings. Therefore, the application of the principle of strong amalgamation mentioned at the end of the first paragraph of this introduction naturally leads to introduce the concepts of "positive strong amalgamation" and "h-strong amalgamation."

In this paper, we will explore one of the specific aspects of positive logic which embodies the notions of algebraic closedness and strong amalgamation and undertake to study some interactions between these new notions inspired directly or indirectly from the works of Bacsich ([1]). In the first section, after summarising the necessary background of the positive model theory, we introduce the general form of symmetric and asymmetric amalgamations. We show that the model completeness of an h-inductive theory can be characterized by a form of symmetric amalgamation. The second section is devoted to the notions of almost positively closed models and a special class of positive formulas called (A, T)-closed formulas. Note that the terminology "closed formula" here has different meaning of the notion of formulas without free variables. We analyse the class of almost positively closed model and present a characterization through some properties of the class of the (A, T)-closed formulas. In the third section, we introduce the notions of positive strong amalgamation and h-strong amalgamation properties. We show that the class of almost positively closed has the positive strong amalgamation property. Furthermore, we give a syntactic characterization of positive strong amalgamation bases.

2. Positive Model Theory

2.1. Basic Definitions and Notations. In this subsection, we briefly introduce the basic terminology related to the positive logic. For more details, the reader is referred to ([6–8]).

Let L be a first order language that contains the symbol of equality and a constant $\perp$ denoting the antilogy. The quantifier-free positive formulas are obtained from atomic formulas using the connectives $\wedge$ and $\vee$. The positive formulas are built from quantifier-free positive formulas using the logical operators and quantifier $\wedge, \vee$ and $\exists$, respectively. Eventually, the positive formulas are of the form $\exists \overline{y}\ \phi(\overline{x}, \overline{y})$, where $\phi(\overline{x}, \overline{y})$ is quantifier-free formula. The variables $\overline{x}$ are said to be free. Also, a sentence is a formula without free variables.

A sentence is said to be h-inductive if it is a finite conjunction of sentences of the following form:

$$\forall \overline{x} (\varphi(\overline{x}) \longrightarrow \psi(\overline{x})), \tag{1}$$

where $\varphi(\overline{x})$ and $\psi(\overline{x})$ are positive formulas. The h-universal sentences are the negation of positive sentences.

Let A and B be two L-structures. A map h from A to B is a homomorphism if for every tuple $\overline{a} \in A$ and for every atomic formula ϕ; if $A \models \phi(\overline{a})$, then $B \models \phi(h(\overline{a}))$. So, we say that B is a continuation of A.

An embedding of A into B is a homomorphism $h: A \longrightarrow B$ such that for every $\overline{a} \in A$ and ϕ, an atomic formula, if $B \models \phi(h(\overline{a}))$, then $A \models \phi(\overline{a})$. The homomorphism $h: A \longrightarrow B$ is said to be an immersion whenever for every $\overline{a} \in A$ and ϕ a positive formula, if $B \models \phi(h(\overline{a}))$, then $A \models \phi(\overline{a})$.

A class of L-structures is said to be h-inductive if it is closed under the inductive limit of homomorphisms. For more details on the notion of h-inductive sequences and limits, the reader is invited to ([7]).

Parallel to the role of existentially closed structures in the framework of logic with negation, for every h-inductive theory T, there exists a class of models of T which represent the theory marvellously, and which enjoy the properties desired by every structures; namely, the h-inductive property of the class, the maximality of types (positive formulas satisfied by an element), amalgamation property, and others. These modules are called positively closed.

Definition 1. A model A of an h-inductive theory T is said to be positively closed (in short, pc) if every homomorphism from A to B, a model of T, is an immersion.

The following lemmas announce the main properties of pc models. They will be used without mention.

Lemma 2 (Lemma 14, see [7]). *A model A of an h-inductive theory is pc if and only if for every positive formula φ and $\overline{a} \in A$, if $A \not\models \varphi(\overline{a})$, then there exists a positive formula ψ such that $T \vdash \neg\exists \overline{x} (\varphi(\overline{x}) \wedge \psi(\overline{x}))$ and $A \models \psi(\overline{a})$.*

Lemma 3 (Theorem 2, see [7]). *Every model on an h-inductive theory T is continued in a pc model of T.*

For every positive formula φ, we denote by $Ctr_T(\varphi)$ the set of positive formulas ψ such that

$$T \vdash \neg\exists \overline{x} (\varphi(\overline{x}) \wedge \psi(\overline{x})). \tag{2}$$

Two h-inductive theories are said to be companion if every model of one of them can be continued into a model of the other or equivalently if the theories have the same pc models.

Every h-inductive theory T has a maximal companion denoted by $T_k(T)$ called the Kaiser's hull of T. $T_k(T)$ is the set of h-inductive sentences satisfied in each pc models of T. Likewise, T has a minimal companion denoted by $T_u(T)$, formed by its h-universal consequence sentences.

Remark 4. Let T_1 and T_2 two h-inductive theories. The following propositions are equivalent:

(i) T_1 and T_2 are companion.
(ii) $T_k(T_1) \equiv T_k(T_2)$.
(iii) $T_u(T_1) \equiv T_u(T_2)$.

Definition 5. Let T be an h-inductive theory.

(i) T is said to be model complete if every model of T is a pc model of T.
(ii) We say that T has a model companion whenever $T_k(T)$ is a model-complete theory.

Let A be a model of T. We shall use the following notations:

(i) $\mathrm{Diag}^+(A)$, the set of positive quantifier-free sentences satisfied by A in the language $L(A)$.
(ii) $\mathrm{Diag}(A)$, the set of atomic and negated atomic sentences satisfied by A in the language $L(A)$.
(ii) We denote by $T^+(A)$ the $L(A)$-theory $T \cup \{\mathrm{Diag}^+(A)\}$.
(iv) $T_i(A)$ (resp. $T_u(A)$) denote the set of h-inductive (resp. h-universal) $L(A)$-sentences satisfied by A.
(v) $T_i^\star(A)$ (resp. $T_u^\star(A)$) denote the set of h-inductive (resp. h-universal) L-sentences satisfied by A.
(vi) $T_k(A)$ (resp. $T_k^\star(A)$) denote the Kaiser's hull of $T_i(A)$ (resp. of $T_i^\star(A)$).
(vii) For every subset B of A, we denote by $T_i(A, B)$ (resp. $T_u(A, B)$) the set of h-inductive (resp. h-universal) $L(B)$-sentences satisfied by A.

Definition 6. Let A and B be two L-structures and h a homomorphism from A into B. h is said to be a strong immersion (in short s-immersion) if h is an immersion and B is a model of $T_i(A)$.

Remark 7. Let A and B two L-structures. We have the following properties:

(1) If A is immersed in B, then $T_u^\star(A) = T_u^\star(B)$ and $T_i^\star(B) \subseteq T_i^\star(A)$.

(2) A is immersed in B if and only if $T_i(B, A) \subseteq T_i(A)$.

(3) A is strongly immersed in B if and only if $T_i(B, A) = T_i(A)$.

(4) If A and B are two pc models of T, then every homomorphism from A into B is a strong immersion. Indeed, let $\varphi(\overline{a}, \overline{x})$ and $\psi(\overline{a}, \overline{x})$ be two positive formulas and let χ the h-inductive sentence $\forall\overline{x}(\varphi(\overline{a}, \overline{x}) \longrightarrow \psi(\overline{a}, \overline{x}))$. Suppose that $A \vdash \chi$ and $B \nvdash \chi$, then there is $\overline{b} \in B$ such that $B \models \varphi(\overline{a}, \overline{b})$ and $B \nvDash \psi(\overline{a}, \overline{b})$. Given that B is a pc model, there exists $\psi'(\overline{x}, \overline{y}) \in Ctr_T(\psi(\overline{x}, \overline{y}))$ such that $B \models \psi'(\overline{a}, \overline{b})$. Since A is immersed in B, then there is $\overline{a}' \in A$ such that $A \models \varphi(\overline{a}, \overline{a}') \wedge \psi'(\overline{a}, \overline{a}')$, which implies $A \models \varphi(\overline{a}, \overline{a}')$ and $A \nvDash \psi(\overline{a}, \overline{a}')$, a contradiction.

(5) The pc models of the $L(A)$-theory $T^+(A)$ are the pc models of T that are the continuation of A. Indeed, it is clear that every pc model of T in which A is continued is a model of $T^+(A)$ and then a pc model of $T^+(A)$. Conversely, let B be a pc model $T^+(A)$ and C a pc model of T in which B is continued by a homomorphism f. Then, C is a continuation of A, so C is a model of $T^+(A)$. Thereby, f is an immersion, which implies that B is a pc model of T.

Let A and B two be L-structures and f a mapping from A into B. We will use the following notations:

(i) $\text{Hom}(A, B)$: the set homomorphisms from A into B.

(ii) $\text{Emb}(A, B)$: the set embeddings from A into B.

(iii) $\text{Imm}(A, B)$: the set immersions from A into B.

(iv) $\text{Sm}(A, B)$: the set s-immersions from A into B.

Remark 8. Let A and B be two L-structures and f a mapping from A into B. Consider B as a $L(A)$-structure by interpreting the elements of A in B by f. We have the following:

(i) $f \in \text{Hom}(A, B)$ if and only if B is a model of $\text{Diag}^+(A)$.

(ii) $f \in \text{Emb}(A, B)$ if and only if B is a model of $\text{Diag}(A)$.

(iii) $f \in \text{Imm}(A, B)$ if and only if B is a model of $\text{Diag}^+(A) \cup T_u(A)$.

(iv) $f \in \text{Sm}(A, B)$ if and only if B is a model of $T_i(A)$.

2.2. Positive Amalgamation. To abbreviate the nominations of homomorphism, embedding, immersion, and strong immersion in the definition of the notions of amalgamation, we will use the first letter of each mapping defined above.

Definition 9. Let Γ be a class of L-structures and A a member of Γ. We say that A is $[h, e, i, s]$-amalgamation basis of Γ, if for every B, C members of Γ, if A is continued into B by f and embedded into C by g, there exist $D \in \Gamma$, $g' \in \text{Imm}(C, D)$, and $f' \in \text{Sm}(B, D)$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g'} \\ B & \xrightarrow[f']{} & D \end{array}. \tag{3}$$

By the same way, we define the notion of $[\alpha, \beta, \gamma, \delta]$-amalgamation property for every $(\alpha, \beta, \gamma, \delta) \in \{h, e, i, s\}^4$.

We say that A is an $[\alpha]$-amalgamation basis of Γ, if A is an $[\alpha, \alpha, \alpha, \alpha]$-amalgamation basis of Γ.

We say that A is $[\alpha, \beta]$-symmetric amalgamation basis of Γ whenever A is an $[\alpha, \beta, \beta, \alpha]$-amalgamation basis of Γ.

We say that A is $[\alpha, \gamma]$-asymmetric amalgamation basis of Γ, whenever A is an $[\alpha, \beta, \alpha, \beta]$-amalgamation basis of Γ.

The following remark lists some properties of diver forms of amalgamation with the notations and terminology given in the definition above.

Remark 10

(1) Every L-structure A is an $[i, h, s, h]$-amalgamation basis in the class of L-structures (lemma 4, [6]). Since every strong immersion is an immersion, it follows that every L-structure A is an $[s, h]$-asymmetric amalgamation basis in the class of L-structures.

(2) Every L-structure A is an $[s, i]$-asymmetric amalgamation basis in the class of L-structures (lemma 5, [6]).

(3) Every L-structure A is an $[e, s]$-asymmetric amalgamation basis in the class of L-structures (lemma 4, [9]).

(4) Every L-structure A is an $[i, h]$-asymmetric amalgamation basis in the class of L-structures (lemma 8, [7]).

(5) Every pc model of an h-inductive theory T is an $[h]$-amalgamation basis in the class of model of T.

Lemma 11. *Every L-structure is $[s, x]$-asymmetric amalgamation basis in the class of L-structure, where x is a homomorphism, an embedding, or an immersion.*

Proof. The proof of the lemma directly follows from the Remark 10. More precisely, the cases where x is a homomorphism is addressed in bullet 1 of the Remark 10, while the case where x is an embedding is covered in bullet 3. The case where x is an immersion is addressed in bullet 2. ☐

Lemma 12. *A model of T is pc if and only if it has the $[h, i]$-symmetric amalgamation property in the class of models of T.*

Proof. Let A be an $[h, i]$-symmetric amalgamation basis of T. Assume that $A \nvDash \varphi(\overline{a})$, where $\overline{a} \in A$ and φ a positive formula. Given that A is an $[h, i]$-symmetric amalgamation basis, we claim that $T \cup \text{Diag}^+(A) \cup \{\varphi(\overline{a})\}$ is inconsistent.

Otherwise, we can find two continuations of A in which one of them satisfies $\varphi(\overline{a})$ and the other does not satisfy $\varphi(\overline{a})$, which contradicts the assumption that A has the $[h,i]$-symmetric amalgamation property. □

Proposition 13. *An h-inductive theory T has a model companion if and only if $T_k(T)$ has the $[h,i]$- symmetric amalgamation property.*

Proof. Suppose that T has a model companion, then every model of $T_k(T)$ is a pc model. Since the pc models have the $[h]$-amalgamation property and the homomorphisms between the pc models are immersion, it follows from the fifth bullet of the Remark 10 that $T_k(T)$ has the $[h,i]$-symmetric amalgamation property.

The opposite direction follows easily from Lemma 12. □

3. Almost Positively Closed Structures

In this section, we introduce the notions of almost and Δ-almost positively closed models, and we give a syntactic characterization and a characterization via the closed formulas which turns out to be an essential tool in the study of the notion of Δ-almost positively closedness.

Definition 14. Let T be an h-inductive theory and A a model of T. Let Δ be a subset of L-quantifier-free positive formulas such that for every $\varphi(\overline{x}) \in \Delta$, the set $T \cup \{\text{Diag}^+(A) \cup \exists\overline{x}\varphi(\overline{x})\}$ is consistent. The model A is said to be

(i) Almost positively closed (apc in short), if for every model $B \models T$, $f \in \text{Hom}(A,B)$, and $\varphi(\overline{x},\overline{y})$ a quantifier-free positive formula, if $B \models \exists\overline{y}\varphi(\overline{a},\overline{y})$ and $\overline{a} \in A$, then there is $\overline{a}' \in A$ such that $B \models \varphi(\overline{a},\overline{a}')$.

(ii) Δ-almost positively closed (Δ-apc in short), if for every model $B \models T$, $f \in \text{Hom}(A,B)$, and $\varphi(\overline{x},\overline{y}) \in \Delta$, if $B \models \exists\overline{y}\varphi(\overline{a},\overline{y})$ and $\overline{a} \in A$, then there is $\overline{a}' \in A$ such that $B \models \varphi(\overline{a},\overline{a}')$.

(iii) Weakly almost positively closed (wpc in short), if for every pc model $B \models T$, $f \in \text{Hom}(A,B)$, and $\varphi(\overline{x},\overline{y})$ a quantifier-free positive formula if $B \models \exists\overline{y}\varphi(\overline{a},\overline{y})$ and $\overline{a} \in A$, then there is $\overline{a}' \in A$ such that $B \models \varphi(\overline{a},\overline{a}')$.

(iv) Δ-weakly almost positively closed (Δ-wpc in short), if for every pc model $B \models T$, $f \in \text{Hom}(A,B)$, and $\varphi(\overline{x},\overline{y}) \in \Delta$, if $B \models \exists\overline{x}\varphi(\overline{a},\overline{x})$, then there is $\overline{a}' \in A$ such that $B \models \varphi(\overline{a},\overline{a}')$.

Theorem 15. *Let A be a model of an h-inductive L-theory T. Let Δ be a set of $L(A)$-quantifier free positive formula that satisfies the condition of Definition 14. The model A is Δ-apc of T if and only if for every $\varphi(\overline{a},\overline{x}) \in \Delta$, there exist $n \in \mathbb{N}$ and a quantifier-free positive formula $\psi(\overline{a},\overline{a}',\overline{a}_1,\ldots,\overline{a}_n) \in \text{Diag}^+(A)$ such that*

$$T \vdash \forall\overline{x},\overline{y},\overline{y}_1,\ldots,\overline{y}_n\left(\left(\psi(\overline{x},\overline{y},\overline{y}_1,\ldots,\overline{y}_n) \wedge \exists\overline{z}\varphi(\overline{x},\overline{z})\right) \longrightarrow \vee_{i=1}^n \varphi(\overline{x},\overline{y_i})\right). \tag{4}$$

Proof. Assume that A is an Δ-apc model of T and let $\varphi(\overline{a},\overline{x}) \in \Delta$ such that $T^* = T \cup \text{Diag}^+(A) \cup \{\exists\overline{x}\varphi(\overline{a},\overline{x})\}$ is consistent. Given that A is Δ-apc, then $T^* \cup \{\neg\varphi(\overline{a},\overline{a}') | \overline{a}' \in A\}$ is inconsistent. Thus, there are $\overline{a}',\overline{a}_1,\overline{a}_2,\ldots,\overline{a}_n \in A$ and $\psi(\overline{a},\overline{a}',\overline{a}_1,\overline{a}_2,\ldots,\overline{a}_n) \in \text{Diag}^+(A)$ such that

$$T \cup \{\psi(\overline{a},\overline{a}',\overline{a}_1,\overline{a}_2,\ldots,\overline{a}_n), \neg(\vee_{i=1}^n \varphi(\overline{a},\overline{a}_i)), \exists\overline{x}\varphi(\overline{a},\overline{x})\}, \tag{5}$$

is inconsistent, which implies

$$T \vdash \forall\overline{x},\overline{y},\overline{y}_1,\ldots,\overline{y}_n\left(\left(\psi(\overline{x},\overline{y},\overline{y}_1,\ldots,\overline{y}_n) \wedge \exists\overline{z}\varphi(\overline{x},\overline{z})\right) \longrightarrow \vee_{i=1}^n \varphi(\overline{x},\overline{y_i})\right). \tag{6}$$

For the other direction, let A be a model of T that satisfies the hypothesis of the theorem. Let $\varphi(\overline{a},\overline{x}) \in \Delta$ and $f \in \text{Hom}(A,B)$, where B is a model of T and $B \models \exists\overline{x}\varphi(\overline{a},\overline{x})$. Given that $A \models \psi(\overline{a},\overline{a}',\overline{a}_1,\ldots,\overline{a}_n)$, then $B \models \psi(\overline{a},\overline{a}',\overline{a}_1,\ldots,\overline{a}_n) \wedge \exists\overline{x}\varphi(\overline{a},\overline{x})$. By the hypothesis of the theorem, we obtain $B \models \vee_{i=1}^n \varphi(\overline{a},\overline{a}_i)$. So, A is an Δ-apc model of T. □

Corollary 16. *Let A be a model of T and Δ a set of quantifier-free positive $L(A)$-formulas. If A is immersed in an Δ'-apc model B of T and $\Delta \subseteq \Delta'$, then A is an Δ-apc model of T.*

Proof. Let $\varphi(\overline{a},\overline{x}) \in \Delta$. Given that $\varphi(\overline{a},\overline{x}) \in \Delta'$ and B is an Δ'-apc model of T, by Theorem 15, there is $\psi(\overline{a},\overline{b},\overline{b}_1,\ldots,\overline{b}_n) \in \text{Diag}^+(B)$ such that

$$T \vdash \forall\overline{x},\overline{y},\overline{y}_1,\ldots,\overline{y}_n\left(\left(\psi(\overline{x},\overline{y},\overline{y}_1,\ldots,\overline{y}_n) \wedge \exists\overline{z}\varphi(\overline{x},\overline{z})\right) \longrightarrow \vee_{i=1}^n \varphi(\overline{x},\overline{y_i})\right). \tag{7}$$

On the other hand, since A is immersed in B, then there are $\overline{a}', \overline{a}_1, \ldots, \overline{a}_n \in A$ such that $\psi(\overline{a}, \overline{a}', \overline{a}_1, \ldots, \overline{a}_n) \in \mathrm{Diag}^+(A)$. By Theorem 15, A is an Δ-apc of T. □

Lemma 17. *Let $(A_i, f_{ij})_{i \le j \in I}$ be an inductive sequence of models of an h-inductive theory T. Suppose that for every $i \in I$, the model A_i is Δ_i-apc where Δ_i is a set of quantifier-free positive $L(A_i)$-formulas such that $\forall i \le j \in I, \Delta_i \subseteq \Delta_j$. Then, the inductive limit A of the sequence $(A_i, f_{ij})_{i \le j \in I}$ is $\cup_{i \in I} \Delta_i$-apc of T.*

Proof. Let B be a model of T and $f \in \mathrm{Hom}(A, B)$. Let $\overline{a} \in A$ and $\varphi(\overline{a}, \overline{y}) \in \cup_{i \in I} \Delta_i$ such that $B \models \varphi(\overline{a}, \overline{b})$. Let $i \in I$ such that $\overline{a} \in A_i$ and $\varphi(\overline{a}, \overline{y}) \in \Delta_i$. Given that $f^\circ f_i \in \mathrm{Hom}(A_i, B)$ where f_i is the canonical homomorphism defined from A_i in A, then there is $\overline{a}' \in A_i$ such that $B \models \varphi(\overline{a}, \overline{a}')$. So, A is $\cup_{i \in I} \Delta_i$-apc. □

Remark 18. Let T be an h-inductive L-theory and Δ a set of quantifier-free positive L-formulas. We have the following properties:

(1) If A is apc, then A is wpc of T.

(2) Every pc model of T is an apc (resp. Δ-apc) model of T.

(3) The classes of apc and wpc (resp. Δ-apc and Δ-wpc) models of T are h-inductive.

(4) If A is an apc model of T and B a model of T, then $\mathrm{Emb}(A, B) = \mathrm{Imm}(A, B)$.

(5) Let $\Delta \subseteq D$ be two sets of free quantifier positive formulas. If A is D-apc (resp. D-wpc) then A is Δ-apc (resp. Δ-wpc).

(6) Every apc model of T has the property of $[e, h]$-asymmetric amalgamation (property 4 of Remark 18, and the property 4 of Remark 10).

Example 1

(1) Let $L = \{f\}$ be a functional language. Let T be the h-inductive theory.

$$\{\forall x, y (f(x) = f(y) \longrightarrow x = y)\}. \tag{8}$$

The theory T has a model companion axiomatized by $T_k(T) = T \cup \{\forall x y (x = y)\}$. The class of apc model of T is elementary and axiomatized by the h-inductive theory.

$$T \cup \{\exists x, f(x) = x\} \cup \{\forall x \exists y (f(y) = x)\}. \tag{9}$$

(2) Let L and T be the functional language and the theory defined in the bullet above. Let T'' the h-inductive theory $T \cup \{\neg \exists x (f(x) = x)\}$. The class of apc model of T'' is axiomatized by the h-inductive theory.

$$T \cup \{\forall x \exists y (f(y) = x)\} \cup \{\exists x f^p(x) = x \mid \ p \text{ a prime number}\}. \tag{10}$$

(3) Let T_f the theory of fields. Since the negation of equality $x = y$ is defined by the positive formula $\exists z (x - y) \cdot z = 1$ and every homomorphism is an embedding then the classes of apc fields, pc fields, and existentially closed fields are equals.

Definition 19. Let T be an h-inductive L-theory and A a model of T.

(i) A positive formula $\varphi(x)$ is said to be T-algebraic if $\varphi(x) \not\equiv \perp$ modulo T (i.e., $\varphi(x)$ has a realisation in some model of T) and there exists a positive formula $\psi(y_1, \ldots, y_n)$ such that $\varphi(x) \wedge \psi\ (y_1, \ldots, y_n) \not\equiv \perp$ modulo $T^+(A)$, and

$$T \vdash \forall x, \overline{y} \Big((\varphi(x) \wedge \psi(\overline{y})) \longrightarrow \bigvee_i x = y_i \Big). \tag{11}$$

We denote by Al_T the set of T-algebraic quantifier-free positive L-formulas.

(ii) For every positive formula $\varphi(\overline{x})$, we denote by $E(\varphi, T)$ the set of positive formulas $\psi(\overline{y})$ such that $\varphi(\overline{x}) \wedge \psi(\overline{y}) \not\equiv \perp$ modulo T and satisfy the following property:

$$T \vdash \forall \overline{x}\, \overline{y} \Big((\varphi(\overline{x}) \wedge \psi(\overline{y})) \longrightarrow \bigvee_{i,j} x_i = y_j \Big). \tag{12}$$

(iii) A positive formula $\varphi(\overline{x}, \overline{y})$ is said to be (A, T)-closed if $\varphi \not\equiv \perp$ modulo T, and for every pc model continuation B of A, if $B \models \varphi(\overline{a}, \overline{b})$ for some $\overline{a} \in A$, then $\overline{b} \in A$.

Remark 20

(1) A quantifier-free positive formula is T-algebraic if and only if its algebraic is in the sense of Robinson ([4]).

(2) Given that the class of pc models of $T^+(A)$ coincides with the class of pc models of T that are continuation of A (bullet 5 of Remark 7), then a formula is (A, T)-closed if and only if it is $(A, T^+(A))$-closed.

(3) Let A be a model of T. Denote by C_A the set of quantifier-free formulas that are (A, T)-closed. Then, A is C_A-wpc.

(4) If every formula in $Al_{T^+(A)}$ is (A, T)-closed, by the bullet 2 above, the model A is $Al_{T^+(A)}$-wpc, and since $Al_T \subset Al_{T^+(A)}$, A is also Al_T-wpc.

We denote by E_T the set of quantifier-free positive formulas $\varphi(\overline{x})$ such that $E(\varphi, T) \neq \varnothing$.

Lemma 21. *Let A be an h-amalgamation basis of T. If A is $E_{T^+(A)}$-wpc (resp. $E_{T^+(A)}$-apc), then every formula in $Al_{T^+(A)}$ is (A, T)-closed.*

Proof. Let A be a $E_{T^+(A)}$-wpc and an h-amalgamation basis of T. Assume the existence of a formula $\varphi(\overline{a}, y) \in Al_{T^+(A)}$ such that $\varphi(\overline{a}, y)$ is not (A, T)-closed. So, there exist a pc models B of $T^+(A)$ and $b \in B - A$ such that $B \models \varphi(\overline{a}, b)$. Let $\psi(\overline{a}, \overline{x}) \in E(\varphi, T^+(A)))$, and let C be a pc model of $T^+(A)$ and $\overline{c} \in C$ such that $C \models \psi(\overline{a}, \overline{c})$. Given that $\psi(\overline{a}, \overline{x}) \in E_{T^+(A)}$ and A is an $Al_{T^+(A)}$-wpc model of T, then there is $\overline{a}'$ in A such that $C \models \psi(\overline{a}, \overline{a}')$. Let D be a model of T that amalgamate commutatively the diagram $C \longleftarrow A \longrightarrow B$. Thus, B is immersed in D and so $B \models \varphi(\overline{a}, b) \wedge \psi(\overline{a}, \overline{a}')$, which implies $\vee_i b = a'_i$, a contradiction.

The proof of the case where A is Al_T-apc is an immediate. ☐

4. Strong Amalgamation

In this section, we introduce the notions of positive strong amalgamation and h-strong amalgamation. We investigate their properties and interactions with the notion of almost positively closedness.

4.1. Positive Strong Amalgamation

Definition 22. Let T be an h-inductive theory. A model A of T is said to be a positive strong amalgamation basis (in short PSA) (resp. h-strong amalgamation basis (in short h-SA)) of T, if for every pc models (resp. models) B and C of T, if A is continued into B and C by two homomorphisms f and g, respectively, then there exist D a model of T and f', g', two homomorphisms, such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'}, \\ C & \xrightarrow[g']{} & D \end{array} \qquad (13)$$

and satisfies the following property (P):

$\forall (b, c) \in B \times C$, if $g'(c) = f'(b)$, then there is $a \in A$ such that $c = g(a)$ and $b = f(a)$.

Remark 23. Note that in the definition of PSA basis, we can reformulate the property (P) as follows:

$\forall (b, c) \in B \times C$, if $g'(c) = f'(b)$, then there exist $a, a' \in A$ such that $c = g(a)$ and $b = f(a')$.

Indeed, let $a, a' \in A$ such that $c = g(a)$ and $b = f(a')$, then

$$f'(b) = f'^{\circ} f(a') = g'^{\circ} g(a') = g'(c) = g'^{\circ} g(a). \qquad (14)$$

Given that g' is an immersion, we have

$$\begin{aligned} g(a') &= g(a) = c, \\ f(a') &= b. \end{aligned} \qquad (15)$$

Example 2

(1) Every h-inductive theory for which the unique pc model is the trivial model $A_e = \{a\}$ has the positive strong amalgamation property. As examples of these theories, we have the theory of groups and the theory of partially ordered sets.

(2) Let $L = \{p, q\}$ where p and q are two unary relation symbols. Let T be the h-inductive theory $\{\forall x, y((p(x) \wedge q(y)) \longrightarrow x = y)\}$. The trivial structure $E = \{e\}$ such that $E \models p(e) \wedge q(e)$ is the unique pc model of T; thus, T has the positive strong amalgamation property. However, the structure $A = \{a\}$ where $A \not\models (p(a) \vee q(a))$ has no h-strong amalgamation property. Indeed, let $B = \{a, b\}, C = \{a, c\}$, $B \models p(b)$, and $C \models q(c)$, then the diagram $B \longleftarrow A \longrightarrow C$ cannot be h-strongly amalgamate.

Lemma 24. *Let $A, B,$ and C be three L-structures. Let $i \in Imm(A, B)$ and $h \in Hom(A, C)$. Then, there exist D a L-structure, h' a homomorphism, and s an s-immersion such that the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'}, \\ C & \xrightarrow[s]{} & D \end{array} \qquad (16)$$

and satisfies the following property:

$\forall (b, c) \in B \times C$, and if $h'(b) = s(c)$, then there exists $a \in A$ such that $c = h(a)$ and $b = i(a)$.

Proof. The proof consists in the verification that the following set is $L(B \cup C)$-consistent.

$$T' = T_i(C) \cup \mathrm{Diag}^+(B) \cup \mathrm{Diag}^+(C) \cup \{b \neq c \mid b \in B - A, c \in C - h(A)\}, \qquad (17)$$

where every elements of A is interpreted by the same symbols of constant in B and C.

Assume that T' is $L(B \cup C)$-inconsistent. Then, there exist $\varphi(h(\overline{a}), \overline{c}) \in \mathrm{Diag}^{+}(C)$, $\psi(\overline{a}, \overline{b}) \in \mathrm{Diag}^{+}(B)$ where $\overline{c} \in C - h(A)$ and $\overline{b} \in B - A$ such that

$$T_i(C) \vdash \forall \overline{y}\left((\varphi(h(\overline{a}), \overline{c}) \wedge \psi(\overline{a}, \overline{y})) \longrightarrow \bigvee_{i,j} y_i = c_j \right). \quad (18)$$

Given that $B \models \psi(\overline{a}, \overline{b})$ and A is immersed in B, so there is $\overline{a}' \in A$ such that $A \models \psi(\overline{a}, \overline{a}')$. Consequently, $C \models \varphi(h(\overline{a}), \overline{c}) \wedge \psi(h(\overline{a}), h(\overline{a}'))$. Thereby, $C \models \vee_{i,j} h(\overline{a}')_i = c_j$, which is a contradiction.

Let D be a model of T', then the following digram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ C & \xrightarrow[\ s\ ]{} & D \end{array}, \quad (19)$$

where h' is a homomorphism and s a strong immersion. Let $b \in B$ and $c \in C$ such that $s(c) = h'(b)$, so there is a, a' in A such that $c = h(a)$ and $b = i(a')$. By the commutativity of the diagram above, we have

$$s^{\circ} h(a) = h'^{\circ} i(a') = s^{\circ} h(a'). \quad (20)$$

Given that s is an immersion, we obtain

$$\begin{aligned} b &= i(a'), \\ c &= h(a) = h(a'). \end{aligned} \quad (21)$$

□

Corollary 25. *Every pc model A of T is a h-strong amalgamation basis of T.*

Proof. Immediate from Lemma 24. □

Proposition 26. *Let A and B be two models of an h-inductive theory T. If A is immersed in B, a h-SA basis of T, then A is a h-SA basis of T.*

Proof. Let A_1 and A_2 be two models of T, $f \in \mathrm{Hom}(A, A_1)$, and $g \in \mathrm{Hom}(A, A_2)$. By applying Lemma 24 to the diagrams $A_1 \leftarrow A \longrightarrow B$ and $A_1 \leftarrow A \longrightarrow B$, we get the commutative diagrams (1) and (2), where f', g' are homomorphisms and i_1, i_2 are strong immersions.

Now, given that B has the h-strong amalgamation property, we get the commutative diagram (3):

$$\begin{array}{ccccc} A_1 & \xrightarrow{\ i_1\ } & B_1 & & \\ {\scriptstyle f}\uparrow & (1) & \uparrow{\scriptstyle f'} & \searrow{\scriptstyle f''} & \\ A & \xrightarrow{\ i\ } & B & (3) & C, \\ {\scriptstyle g}\downarrow & (2) & \downarrow{\scriptstyle g'} & \nearrow{\scriptstyle g''} & \\ A_2 & \xrightarrow[\ i_2\ ]{} & B_2 & & \end{array} \quad (22)$$

where f'', g'' are homomorphisms.

We claim that C makes the diagram $A_1 \leftarrow A \longrightarrow A_2$ strongly amalgamate. Indeed, let $a_1 \in A_1$ and $a_2 \in A_2$ such that $f''^{\circ} i_1(a_1) = g''^{\circ} i_2(a_2)$. By the h-strong amalgamation property of the diagram (3), there is $b \in B$ such that $f'(b) = i_1(a_1)$ and $g'(b) = i_2(a_2)$. Considering the properties of the diagrams (1) and (2), we get two elements a and a' from A such that

$$\begin{cases} f(a) = a_1, & i(a) = b, \\ g(a') = a_2, & i(a') = b. \end{cases} \quad (23)$$

Given that i is an immersion, then $a = a'$ and $f(a) = a_1, g(a) = a_2$. So, A is a h-SA basis of T. □

Lemma 27. *An h-amalgamation basis A of T is a PSA basis if and only if for every pc model of $T^{+}(A)$ and for every $\varphi(\overline{a}, \overline{x}) \in E_{T^{+}(A)}$, we have $B \nvDash \varphi(\overline{a}, b_1 \cdots b_n)$ for every $b_1, \ldots, b_n \in B - A$.*

Proof. Let A be a PSA basis of T. Suppose that there are $\varphi(\overline{a}, \overline{x}) \in E_{T^{+}(A)}$ and B a pc model of $T^{+}(A)$ such that $B \models \varphi(\overline{a}, b_1 \cdots b_n)$ where $b_1, \ldots, b_n \in B - A$. Let $\psi(\overline{a}, \overline{y}) \in E(\varphi, T^{+}(A))$ and C a pc model of $T^{+}(A)$ such that $C \models \psi(\overline{a}, \overline{c})$.

Given that A is a PSA basis of T, we obtain the following commutative diagram.

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i} \\ C & \xrightarrow[\ i'\ ]{} & D \end{array}, \quad (24)$$

where i and i' are immersion, and D a model of T that satisfies the following property:

$$\forall (b, c) \in B \times C; i(b) = i'(c) \Rightarrow \exists a \in A, (f(a) = b \wedge g(a) = c). \quad (25)$$

Now, since $D \models \varphi(\overline{a}, i(\overline{b})) \wedge \psi(\overline{a}, i'(\overline{c}))$, then $D \models \vee_{i,j} i(b_i) = i'(c_j)$, which implies the existence of an element $a' \in A$ such that $i(b_i) = i'(c_j) = i(a')$, a contradiction.

For the other direction, let B and C be two pc models of T, $f \in \mathrm{Hom}(A, B)$, and $g \in \mathrm{Hom}(A, C)$ such that the following diagram is not h-strongly amalgamable.

$$C \xleftarrow{\ g\ } A \xrightarrow{\ f\ } B. \quad (26)$$

Thus, there exist $\varphi(f(\overline{a}), b_1 \cdots b_n) \in \mathrm{Diag}^{+}(B)$, $\psi(g(\overline{a}), c_1 \cdots c_m) \in \mathrm{Diag}^{+}(C)$ where $b_1, \ldots, b_n \in B - A$ and $c_1, \ldots, c_m \in C - A$ such that

$$T^{+}(A) \vdash \forall \overline{y}\left((\varphi(\overline{a}, \overline{x}) \wedge \psi(\overline{a}, \overline{y})) \longrightarrow \bigvee_{i,j} x_i = y_j \right). \quad (27)$$

Then, $\varphi(\overline{a}, \overline{x}) \in E_{T^{+}(A)}$ and $B \models \varphi(\overline{a}, b_1 \cdots b_n)$. □

Theorem 28. *Let A be an h-amalgamation basis of T, then we have the following properties:*

(1) *If A is a $E_{T^{+}(A)}$-wpc model of T, then A is a PSA basis of T.*

(2) If A is a PSA basis of T, then A is $Al_{T^+(A)}$-wpc.

Proof

(1) Let A be an h-amalgamation basis and a $E_{T^+(A)}$-wpc model of T. Let B and C be two pc models of T, $f \in \mathrm{Hom}(A, B)$, and $g \in \mathrm{Hom}(A, C)$. Let D a model of T such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g\downarrow & & \downarrow i_1 \\ C & \xrightarrow[i_2]{} & D \end{array}, \quad (28)$$

where i_1 and i_2 are immersions.

We claim that the set $T \cup \mathrm{Diag}^+(B) \cup \mathrm{Diag}^+(C) \cup \{b \neq c \mid b \in B - A, c \in C - A\}$ is $L(B \cup C)$-consistent (note that the element of A are interpreted by the same symbols of constants in B and C). Assume that the set above is inconsistent. Then there are $\overline{a} \in A, \overline{b} \in B - A, \overline{c} \in C - A$, $\varphi(\overline{a}, \overline{b}) \in \mathrm{Diag}^+(B)$ and $\psi(\overline{a}, \overline{c}) \in \mathrm{Diag}^+(C)$ such that

$$T^+(A) \cup \left\{\varphi(\overline{a}, \overline{b}), \psi(\overline{a}, \overline{c}), \bigwedge_{i,j} b_i \neq c_j\right\}, \quad (29)$$

is $L(B \cup C)$-inconsistent, thereby

$$T^+(A) \vdash \forall \overline{y}, \overline{z}\left((\varphi(\overline{a}, \overline{y}) \wedge \psi(\overline{a}, \overline{z})) \longrightarrow \bigvee_{i,j} y_i = z_j\right). \quad (30)$$

Now, since $C \models \psi(\overline{a}, \overline{c})$, $\psi \in E_{T^+(A)}$, and A is an $E_{T^+(A)}$-wpc model, then there is $\overline{a}' \in A$ such that $C \models \psi(\overline{a}, \overline{a}')$. Thereby, $D \models \psi(\overline{a}, \overline{a}')$, so $B \models \psi(\overline{a}, \overline{a}') \wedge \varphi(\overline{a}, \overline{b})$, which implies $B \models \vee_{i,j} b_i = a'_j$, a contradiction. Thus, A is a PSA basis of T.

(2) Suppose that A is PSA of T. Since $Al_{T^+(A)} \subseteq E_{T^+(A)}$, by Lemma 27, every formula in $Al_{T^+(A)}$ is (A, T)-closed, which implies that A is a $Al_{T^+(A)}$-wpc model of T by Remark 20 bullet (4). □

Data Availability

No data were used to support the findings of this study.

Disclosure

An earlier version of this article was previously published as preprint on https://arxiv.org/ [10].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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