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Research Article

BVP with a Load in the Form of a Fractional Integral

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A boundary value problem for a nonhomogeneous heat equation with a load in the form of a fractional Riemann–Liouville integral of an order $\beta \in (0, 1)$ is considered. By inverting the differential part, the problem is reduced to an integral equation with a kernel with a special function. The special function is presented as a generalized hypergeometric function. The limiting cases of the order β of the fractional derivative are studied: it is shown that the interval for changing the order of the fractional derivative can be expanded to integer values $\beta \in [0, 1]$. The results of the study remain unchanged. The kernel of the integral equation is estimated. Conditions for the solvability of the integral equation are obtained.

1. Introduction

The field of fractional calculus is rapidly developing, and it is currently being applied in all science fields [1–3]. Research in the following areas is relevant: study of direct problems with nonlocal conditions for partial differential equations of fractional order and study of special functions' properties related to the solutions of fractional differential equations and their application in solving various boundary value problems for a partial differential equation [4–12]. Detailed description on the application of fractional calculus in various fields of science and technology at this stage is given in the monograph [13]. Research on various direct and inverse boundary value problems for partial differential equations, as well as the study of the properties of special functions and operators of fractional integro-differentiation and their generalizations, is conducted in major scientific centers and world higher educational institutions. Boundary value problems with spatial fractional integrals are also interesting for the current study and are planned for further research. From the applied side, for example, works by authors in [14, 15] are relevant.

Also, loaded equations are an important section in the differential equation theory. If we consider a one-dimensional limited medium, at one of the edges of which there is a heat source, the power of which is proportional to the temperature value, then the process of heat propagation in this medium is described by the following loaded heat equation [16]:

$$u_t = u_{xx} + \lambda u(0, t). \tag{1}$$

If a section of the medium (for example, a rod), the temperature of which we are interested in, is located near one end of the rod and far from the other end, then the temperature is determined by the temperature regime of the nearest end (for example, the left end) and the initial condition (problem for a semiinfinite rod). The process of heat propagation will also be described by (1). The study of dynamic processes shows that the future course of many processes depends not only on the present but also is determined by the process prehistory. A mathematical model of these dynamic processes can be built using differential equations with memory of various kinds, which are also called equations with aftereffect or loaded differential equations. Loaded differential equations in a broad sense are usually called equations that contain any functional (functions) of the solution in the coefficients or on the right side. Note that the presence of a loaded term containing the values of the solution often leads to the phenomenon when the phase state of the process can affect the dynamics of the entire process at any point and at any moment. In addition, from the theoretical side, loaded equations represent a special class of equations with their own specific problems. The most general definition of a loaded equation was first given by A.M. Nakhushev. Using numerous examples, he showed the practical and theoretical importance of research on loaded equations [17]. In [18], the solvability of the nonlocal-in-time boundary-value problem for the nonlinear parabolic equation is proved.

At the intersection of the theory of fractional calculus and loaded heat equations, various interesting problems and studies can arise. From a mathematical point of view, nonclassical models of mathematical physics deserve attention, which are represented by equations that include the values of the desired function and its fractional derivatives or fractional integrals on some manifolds from the domain of boundary value problems. It is possible that the process of heat propagation in inhomogeneous media, where heat sources can vary depending on the location or time, will also be described by an equation of type 1, and only the loaded term will be presented in the form of a fractional integral or derivative. The aim of this work [19] is to clarify the nature of the fractional order load in the issues of solvability of the first boundary value problem for the heat equation. The resulting pseudo-Volterra integral equation has a nonempty spectrum for some values of the fractional derivative order. In [20], a loaded term has the form of Riemann–Liouville's fractional derivative with respect to the time variable, and the order of the derivative in the loaded term is less than the order of the differential part. The kernel of the obtained integral equation contains the Wright function. The conditions for the unique solvability of the integral equation are obtained. In [21–23], it is shown that the existence and uniqueness of solutions for fractionally loaded boundary value problems in certain functional classes depend on the order of a fractional derivative in the loaded term.

In [19–23], the loaded term in the equation was presented in the form of a fractional derivative, and the conditions for the solvability of the boundary value problems under consideration were obtained depending on the order of the derivative that is included in the loaded term, that is, the BVP could have a nonunique solution for some values of the fractional derivative's order. Now, we examine the solvability of a boundary value problem with a loaded equation and the solvability of an accompanying integral equation in the case where the loaded term was represented as a fractional integral. As a result, the obtained results are fundamentally different from the results of works by authors in [19–22], namely, it is shown that the BVP under the conditions of the proven theorem is uniquely solvable for any value of a fractional integral's order $\beta \in [0,1]$. In [23], sufficient conditions for the unique solvability of the boundary value problem with a fractional load were established. Moreover, an example is given showing that violation of these conditions can lead to nonuniqueness of the solution. Note that the solution to the problem was found in an explicit form.

The paper is organized as follows: in Section 2, we introduce some necessary definitions and mathematical preliminaries of fractional calculus, special functions, and boundary value problems which will be needed in the forthcoming sections. We present the statement of a boundary value problem for a nonhomogeneous heat equation with a load in the form of a fractional Riemann–Liouville integral in Section 3. In Section 4, the BVP is reduced to a Volterra integral equation of the second kind. The kernel of the equation contains a special function, namely, a generalized hypergeometric series. Checking the limit cases of the fractional integral's order in the loaded term of the BVP is done in Section 5. It is shown that the BVP at limit values of the fractional integral's order is reduced to an integral equation with a kernel that coincides with the limit value of the kernel of the integral equation obtained in Section 4. In Section 6, we estimate the integral equation's kernel and establish conditions under which it has a weak singularity. This implies the solvability conditions for the BVP which are provided in Section 7.

The equation of the boundary value problem includes a loaded term in the form of a fractional integral, and the kernel of the resulting integral equation contains a special function. The posed problem for the heat conduction equation with a fractional load, as well as the resulting integral equation, is new both in their formulation and in the methods for solving it. The results obtained in this work are also new and differ significantly from previous studies. Until now, such problems have not been fully considered and have not been systematically studied.

2. Preliminaries

Definition 1. Let $\varphi(x) \in L_1[a;b]$. Then, the integral

$$\left(I_{ax}^{\alpha}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\,dt, \quad x > a, \tag{2}$$

is called the fractional Riemann–Liouville integral of α order; here, $\alpha > 0$. For $\alpha = 0$, the integral $\left(I_{ax}^{0}\varphi\right)(x) = \varphi(x)$ [24].

Remark 1. It follows from the definition that for the existence of an integral $\left(I_{ax}^{\alpha}\varphi\right)(x)$ in the sense of Riemann–Liouville integral, it is sufficient that $\varphi(x)$ belongs to the class of summable functions: $\varphi(x) \in L_1[a;b]$.

We study boundary value problems for the loaded heat equation, when the loaded term is represented in the form of a fractional integral. The considered problem is reduced to an integral equation by inverting the integral part.

It is known [25] that in the domain $Q = \{(x,t) \mid x > 0, t > 0\}$, the solution to the boundary value problem of heat conduction

$$\begin{gathered} u_t = \mathrm{a}u_{xx} + \Phi(x,t), \\ u|_{t=0} = f(x), u|_{x=0} = g(x), \end{gathered} \tag{3}$$

is described by the following formula:

$$u(x,t) = \int_0^\infty f(\xi)G(x,\xi,t)d\xi + \int_0^t g(\tau)H(x,t-\tau)d\tau + \int_0^t \int_0^\infty \Phi(\xi,\tau)G(x,\xi,t-\tau)d\xi\, d\tau, \tag{4}$$

where

$$G(x,\xi,t) = \frac{1}{2\sqrt{\pi a t}}\left(\exp\left[-\frac{(x-\xi)^2}{4at}\right] - \exp\left[-\frac{(x+\xi)^2}{4at}\right]\right),$$
$$H(x,t) = \frac{x}{2\sqrt{\pi a t}^{(3/2)}}\exp\left(-\frac{x^2}{4at}\right). \tag{5}$$

The Green function $G(x,\xi,t-\tau)$ satisfies the following relation:

$$\int_0^\infty G(x,\xi,t)d\xi = \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right). \tag{6}$$

We also give definitions and some properties of special functions.

Error function and complementary error function have the following forms:

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}}\int_0^z \exp\left(-\zeta^2\right)d\zeta,$$
$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}}\int_z^\infty \exp\left(-\zeta^2\right)d\zeta = 1 - \operatorname{erf} z. \tag{7}$$

A generalized hypergeometric series is defined by the following formula [26]:

$${}_pF_q\left(\alpha_1,\alpha_2,\ldots,\alpha_p;\rho_1,\rho_2,\ldots,\rho_q;z\right) = \sum_{k=0}^\infty \frac{(\alpha_1)_k(\alpha_2)_k,\ldots,\left(\alpha_p\right)_k}{(\rho_1)_k(\rho_2)_k,\ldots,\left(\rho_q\right)_k}\frac{z^k}{k!}, \tag{8}$$

where

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}, \tag{9}$$

is a Pochhammer symbol.

If $p \le q$, the singular points of (8) are at $z = 0$ and $z = \infty$; $z = 0$ is a regular singularity and $z = \infty$ is an irregular singularity ([26] p. 137). Then, series (8) converges for all finite values z.

Generalized hypergeometric function (8) arises, for example, when calculating the integral by formula 3.478 (4) ([27] p. 356):

$$\int_0^u \xi^{\nu-1}(u-\xi)^{\mu-1}\exp\left(\beta\xi^n\right)d\xi$$
$$= B(\mu;\nu)u^{\mu+\nu-1}\,{}_nF_n\left(\frac{\nu}{n},\frac{\nu+1}{n},\ldots,\frac{\nu+n-1}{n};\frac{\mu+\nu}{n},\frac{\mu+\nu+1}{n},\ldots,\frac{\mu+\nu+n-1}{n};\beta u^n\right), \tag{10}$$

$$\operatorname{Re}\mu > 0, \operatorname{Re}\nu > 0, \quad n = 2,3,\ldots. \tag{11}$$

Here and everywhere else, $\Gamma(z)$ and $B(\mu;\nu)$ are Euler integrals.

Let us present some inequalities. Preliminarily, following the study by Luke [26], we introduce the following notation:

$$\theta = \frac{\alpha_p}{\rho_p},$$
$$\phi = \frac{\alpha_p + 1}{\rho_p + 1}, \quad (12)$$
$$\eta = \frac{\alpha_p + 2}{\rho_p + 2},$$

where, for function (19),

$$\phi = \prod_{j=1}^{p} \frac{\alpha_j + 1}{\rho_j + 1},$$
$$\alpha_p = (a_1, a_2, \ldots, a_p), \quad (13)$$
$$\rho_q = (b_1, b_2, \ldots, b_q).$$

Theorem 3. *Generalized hypergeometric function (8) ${}_pF_p(\alpha_p; \rho_p; -z)$ can be evaluated as follows:*

$$e^{-\theta z} < {}_pF_p(\alpha_p; \rho_p; -z) < 1 - \theta + \theta e^{-z},$$
$$e^{\theta z} < {}_pF_p(\alpha_p; \rho_p; z) < 1 - \theta + \theta e^{z},$$
$$1 - \theta\, z \left(1 - \frac{\phi}{2} + \frac{\phi}{2} e^{-z}\right) < {}_pF_q(\alpha_p; \rho_p; -z) < 1 - \theta\, z \exp\left(-\frac{\phi z}{2}\right), \quad (14)$$
$$1 + \theta\, z \exp\left(-\frac{\phi z}{2}\right) < {}_pF_p(\alpha_p; \rho_p; z) < 1 + \theta z\left(1 - \frac{\phi}{2} + \frac{\phi}{2} e^{z}\right),$$
$$z > 0, \rho_k \geq \alpha_j > 0, \quad j = 1, 2, \ldots, p.$$

3. Statement of the Problem

In a domain $Q = \{(x,t)\colon x > 0, t > 0\}$, we consider a BVP

$$u_t - u_{xx} + \lambda I_{0x}^{\beta} u(x,t)\Big|_{x=\gamma(t)} = f(x,t), \quad (15)$$

$$u(x,0) = 0, u(0,t) = 0, \quad (16)$$

where λ is a complex parameter.

$$I_{0x}^{\beta} u(x,t) = \frac{1}{\Gamma(\beta)} \int_0^x \frac{u(\xi,t)}{(x-\xi)^{1-\beta}} d\xi \quad (17)$$

is a fractional Riemann–Liouville integral (2) of an order β, $0 < \beta < 1$, $\gamma(t)$ is a continuous increasing function, $\gamma(0) = 0$, or $\gamma(t)$ is a positive const.

So, we assume that the solution $u(x,t)$ belongs to the following class:

$$u(x,t) \in L_1(x \geq 0). \quad (18)$$

The right side of the BVP equation vanishes at $t < 0$ and belongs to the following class:

$$f(x,t) \in L_{\infty}(A) \cap C(B), \quad (19)$$

where $A = \{(x,t) \mid x > 0, t \in [0,T]\}$, $B = \{(x,t) \mid x > 0, t \geq 0\}$, and T is a positive constant.

We also assume

$$f_1(x,t) = \int_0^t \int_0^{\infty} G(x,\xi,t-\tau) f(\xi,\tau) d\xi\, d\tau \in L_1(x \geq 0). \quad (20)$$

These classes are determined from the natural requirement for the existence and convergence of improper integrals arising in the study of the problem.

4. Reducing the BVP to an Integral Equation

Lemma 4. *Boundary value problems (15) and (16) are equivalently reduced to a Volterra integral equation of the second kind.*

Proof. By virtue of the condition (19), a solution of problems (15) and (16) can be represented by formula (3).

$$u(x,t) = -\lambda \int_0^t \int_0^{\infty} G(x,\xi,t-\tau)\mu(\tau) d\xi\, d\tau + f_1(x,t), \quad (21)$$

where

$$\mu(t) = I_{0x}^{\beta} u(x,t)\Big|_{x=\gamma(t)}, \quad 0 < \beta < 1, \quad (22)$$

$$f_1(x,t) = \int_0^t \int_0^{\infty} G(x,\xi,t-\tau) f(\xi,\tau) d\xi\, d\tau. \quad (23)$$

Function (23) exists and is bounded due to condition (19) and belongs to class (20) by assumption.

Taking into account relation (4) from (21), we obtain the following:

$$u(x,t) = -\lambda \int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu(\tau) d\tau + f_1(x,t). \quad (24)$$

By assumption, function (14) $f_1(x,t)$ and function $u(x,t)$ satisfy inclusions (20) and (18). Now, we apply a fractional integral operator of an order $\beta, 0<\beta<1$ on representation (24) with respect to variable x by formula (2) and put $a=0$. Then, we put $x=\gamma(t)$. On the left side, we get the function $\mu(t)$ according to (22).

To apply the operation of fractional integration of an order $\beta, 0<\beta<1$ to the right side of equality (24), we first calculate the following integral:

$$I_{0x}^{\beta}\left(\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\mu(\tau)d\tau\right)\right)(x). \tag{25}$$

So,

$$\begin{aligned} I_{0x}^{\beta}&\left(\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\mu(\tau)d\tau\right)\right)(x) \\ &= \frac{1}{\Gamma(\beta)}\int_0^x\left(\int_0^t \operatorname{erf}\left(\frac{\theta}{2\sqrt{t-\tau}}\mu(\tau)d\tau\right)\right)\frac{1}{(x-\theta)^{1-\beta}}d\theta \\ &= \frac{1}{\Gamma(\beta)}\int_0^t \mu(\tau)\left(\int_0^x \operatorname{erf}\left(\frac{\theta}{2\sqrt{t-\tau}}\right)(x-\theta)^{\beta-1}d\theta\right)d\tau \\ &= \frac{1}{\Gamma(\beta)}\int_0^t \mu(\tau)J(t,\tau,x;\beta)d\tau, \end{aligned} \tag{26}$$

or

$$\begin{aligned} I_{0x}^{\beta}&\left(\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\mu(\tau)d\tau\right)\right)(x) \\ &= \frac{1}{\Gamma(\beta)}\int_0^t \mu(\tau)J(t,\tau,x;\beta)d\tau. \end{aligned} \tag{27}$$

Since the study by authors in [28],

$$\begin{aligned} J(t,\tau,x;\beta) &= \int_0^x \operatorname{erf}\left(\frac{\theta}{2\sqrt{t-\tau}}\right)(x-\theta)^{\beta-1}d\theta \\ &= \frac{x^{\beta+1}}{\sqrt{\pi(t-\tau)}}B(2,\beta)\,{}_3F_3\left(1,\frac{3}{2},\frac{1}{2};\frac{\beta+2}{2},\frac{\beta+3}{2},\frac{3}{2};-\frac{x^2}{4(t-\tau)}\right) \\ &= \left(x^{\beta+1}\Gamma(\beta)/\sqrt{\pi(t-\tau)}\,\Gamma(\beta+2)\right){}_2F_2\left(1,\frac{1}{2};\frac{\beta+2}{2},\frac{\beta+3}{2};-\frac{x^2}{4(t-\tau)}\right), \end{aligned} \tag{28}$$

where ${}_pF_q$ is a generalized hypergeometric sequence (8) from definition (1.2) [26], and then integral (25) takes the following form:

$$I_{0x}^{\beta}\left(\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\mu(\tau)d\tau\right)\right)(x) = \int_0^t \frac{x^{\beta+1}}{\sqrt{\pi(t-\tau)}\Gamma(\beta+2)}\,{}_2F_2\left(1,\frac{1}{2};\frac{\beta+2}{2},\frac{\beta+3}{2};-\frac{x^2}{4(t-\tau)}\right)\mu(\tau)d\tau. \tag{29}$$

After applying the operation of fractional integration of an order $\beta, 0<\beta<1$ to equality (26), we get the following:

$$I_{0x}^{\beta}u(x,t)(x) = -\lambda I_{0x}^{\beta}\left(\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\mu(\tau)d\tau\right)\right)(x) + I_{0x}^{\beta}f_1(x,t)(x). \tag{30}$$

Taking into account notation (22) and equality (26), after substituting $x = \gamma(t)$ into the last equality, we obtain the following integral equation:

$$\mu(t) + \lambda \int_0^t K_\beta(t,\tau)\mu(\tau)d\tau = f_2(t), \quad (31)$$

where

$$K_\beta(t,\tau) = \left((\gamma(t))^{\beta+1} / \sqrt{\pi(t-\tau)}\,\Gamma(\beta+2) \right)_2 F_2\left(\frac{1}{2}, 1; \frac{\beta+2}{2}, \frac{\beta+3}{2}; -\frac{(\gamma(t))^2}{4(t-\tau)} \right) \quad (32)$$

and

$$f_2(t) = I_{0x}^{\beta} f_1(x,t)\Big|_{x=\gamma(t)}. \quad (33)$$

Here, $F_2(a_1, a_2; b_1, b_2; z)$ is a convergent generalized hypergeometric series (8) for all finite z.

BVP (15), (16) is reduced to integral (34). Lemma is proven. □

5. Study of Continuity in Order of a Fractional Integral in the Interval of the Order's Change

Then, we study the continuity in the order β of the fractional integral in the loaded term of the equation from BVP (15), (16).

Lemma 5. *For BVP (15), (16), there is continuity in the order β in the loaded term of equation (15).*

Proof. The lemma is proved by checking the limit cases of the fractional integral's order in the loaded term of equation (15).

(i) We consider the case $\beta = 0$. From (31) and (32), we get the BVP when $\beta = 0$:

$$\begin{gathered} u_t - u_{xx} + \lambda\mu(t) = f(x,t), \\ u|_{x=0} = 0; u|_{t=0} = 0, \end{gathered} \quad (34)$$

where $\mu(t) = I_{0x}^0 (u(x,t))|_{x=\gamma(t)} = u(\gamma(t);t)$.

The solution of the problem can be represented by (8):

$$u(x,t) = -\lambda \int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}} \right) \mu(\tau)d\tau + f_1(x,t). \quad (35)$$

By applying operation I_{0x}^0 and substituting $x = \gamma(t)$, we get:

$$\mu(t) + \lambda \int_0^t \operatorname{erf}\left(\frac{\gamma(\tau)}{2\sqrt{t-\tau}} \right) \mu(\tau)d\tau = f_2(t), \quad (36)$$

where

$$f_2(t) = f_1(\gamma(t), t). \quad (37)$$

Now, let us find the limit from kernel (32) for β tending to zero from the right:

$$\begin{aligned} \lim_{\beta \longrightarrow 0+0} K_\beta(t,\tau) &= \frac{\gamma(t)}{\sqrt{\pi(t-\tau)}\Gamma(2)}\, {}_2F_2\left(1, \frac{1}{2}; 1, \frac{3}{2}; -\frac{(\gamma(t))^2}{4(t-\tau)} \right) \\ &= \left(\frac{\gamma(t)}{\sqrt{\pi(t-\tau)}} \right) {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -\frac{(\gamma(t))^2}{4(t-\tau)} \right). \end{aligned} \quad (38)$$

It is known that [29]

$${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -z^2 \right) = \frac{\sqrt{\pi}}{2z} \operatorname{erf}(z). \quad (39)$$

Thus,

$$\lim_{\beta \longrightarrow 0+0} K_\beta(t,\tau) = \operatorname{erf}\left(\frac{\gamma(t)}{2\sqrt{t-\tau}} \right). \quad (40)$$

From here we conclude that (31) coincides with (36) at $\beta = 0$.

(ii) We consider the case $\beta = 1$. From (31) and (32), we get the BVP when $\beta = 1$:

$$\begin{gathered} u_t - u_{xx} + \lambda\mu(t) = f(x,t), \\ u|_{x=0} = 0; u|_{t=0} = 0, \end{gathered} \quad (41)$$

where

$$\mu(t) = I_{ox}^1 (u(x,t))\Big|_{x=\gamma(t)} = \int_0^{\gamma(t)} u(\theta,t)d\theta. \quad (42)$$

The solution of the problem can be represented by (5):

$$u(x,t) = -\lambda \int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu(\tau)d\tau + f_1(x,t). \tag{43}$$

Before applying operation I_{0x}^1 and substituting $x = \gamma(t)$ to representation (43), we calculate the following integral:

$$\begin{aligned} I_{0x}^1\left\{\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu(\tau)d\tau\right\} &= \int_0^x\left(\int_0^t \operatorname{erf}\left(\frac{\theta}{2\sqrt{t-\tau}}\right)\mu(\tau)d\tau\right)d\theta \\ &= \int_0^t \mu(\tau)\left(\int_0^x \operatorname{erf}\left(\frac{\theta}{2\sqrt{t-\tau}}\right)d\theta\right)d\tau \\ &= \int_0^t \frac{x^2}{2\sqrt{\pi(t-\tau)}}\, {}_2F_2\left(\frac{1}{2},1;\frac{3}{2},2;-\frac{x^2}{4(t-\tau)}\right)\mu(\tau)d\tau. \end{aligned} \tag{44}$$

Now applying operation I_{0x}^1 and substituting $x = \gamma(t)$ to representation (43), we get the following:

$$\mu(t) + \lambda \int_0^t \frac{\gamma^2(t)}{2\sqrt{\pi(t-\tau)}}\, {}_2F_2\left(\frac{1}{2},1;\frac{3}{2},2;-\frac{\gamma^2(t)}{4(t-\tau)}\right)\mu(\tau)d\tau = f_2(t). \tag{45}$$

The limit of kernel (19) for β tending to 1 from the left coincides with the kernel of the last integral equation.

Lemma is proven. □

Remark 6. Taking into account the formula from the study of authors in [28]

$${}_2F_2\left(\frac{1}{2},1;\frac{3}{2},2;-z\right) = \frac{1}{z}\left(\sqrt{\pi z}\, erfi(\sqrt{z}) + 1 - e^z\right), \tag{46}$$

where

$$erfi(x) = -i\, erf(ix) = \frac{2}{\sqrt{\pi}}\int_0^x e^{t^2}dt, \tag{47}$$

integral equation (45) can be rewritten as

$$\mu(t) + \lambda\int_0^t K_1(t,\tau)\mu(\tau)d\tau = f_2(t), \tag{48}$$

where

$$\begin{aligned} K_1(t,\tau) &= \gamma(t) erf\left(\frac{\gamma(t)}{2\sqrt{t-\tau}}\right) \\ &\quad - \frac{2\sqrt{t-\tau}}{\sqrt{\pi}}\left(1 - \exp\left(-\frac{\gamma^2(t)}{4(t-\tau)}\right)\right). \end{aligned} \tag{49}$$

If $\gamma(t) \sim t^{\omega}$ (near the point $t = 0$), $\omega \ge 0$, then the kernel $K_1(t,\tau)$ is bounded for all $t \in [0;T], 0 \le \tau \le t$.

6. Evaluating the Kernel of the Integral Equation

Kernel (32) of (31) has singularities when $\tau = t$ and $t = 0$.

To estimate the kernel, we consider a generalized hyperbolic function in the kernel as the form of the integral representation (6) [27].

$$\begin{aligned} &\int_0^u x^{\nu-1}(u-x)^{\mu-1}\exp(cx^n)dx \\ &\quad = B(\mu,\nu)u_n^{\mu+\nu-1}F_n\left(\frac{\nu}{n},\frac{\nu+1}{n},\ldots,\frac{\nu+n-1}{n};\frac{\mu+\nu}{n},\frac{\mu+\nu+1}{n},\ldots,\frac{\mu+\nu+n-1}{n};cu^n\right), \\ &\operatorname{Re}\mu > 0, \operatorname{Re}\nu > 0, \quad n = 2,3\ldots. \end{aligned} \tag{50}$$

Substituting variables' values $n = 2, \nu = 1, \mu = \beta + 1$ and $u = \gamma(t), c = -(1/4(t-\tau)), x = \xi$, we get:

$$K_\beta(t,\tau) = \frac{1}{\Gamma(\beta+1)\sqrt{\pi(t-\tau)}} \int_0^{\gamma(t)} (\gamma(t)-\xi)^\beta \exp\left(-\frac{\xi^2}{4(t-\tau)}\right) d\xi. \tag{51}$$

Then, we estimate the kernel $K_\beta(t,\tau)$ when $0 \le \beta \le 1$:

$$\left|K_\beta(t,\tau)\right| \le \frac{1}{\Gamma(\beta+1)\sqrt{\pi(t-\tau)}} \int_0^{\gamma(t)} (\gamma(t)-\xi)^\beta d\xi = \frac{1}{\Gamma(\beta+2)\sqrt{\pi(t-\tau)}} (\gamma(t))^{\beta+1}. \tag{52}$$

Then, we consider the connection between the kernel's features with the order of the fractional integral in the loaded term of the BVP's equation and with the behavior of the load for small values of the time variable.

Theorem 7. *Integral equation (31) is uniquely solvable in the class of functions $C([0;T])$ for the right-hand side $f_2(t) \in C([0;T])$ defined by formula (33), if $\gamma(t) \sim t^\omega$ (near the point $t = 0$), $\omega > 0$, and $0 \le \beta \le 1$.*

Proof. We introduce the following notation:

$$L_\beta(t,\tau) = \frac{(\gamma(t))^{\beta+1}}{\Gamma(\beta+2)} {}_2F_2\left(\frac{1}{2}, 1; \frac{\beta+2}{2}, \frac{\beta+3}{2}; -\frac{(\gamma(t))^2}{4(t-\tau)}\right), \tag{53}$$

that is,

$$K_\beta(t,\tau) = \frac{L_\beta(t,\tau)}{\sqrt{\pi(t-\tau)}}. \tag{54}$$

Let $\gamma(t) \sim t^\omega$ for $t \longrightarrow 0+0$ when $\omega \ge 0$. Then,

$$L_\beta(t,\tau) = \frac{t^\omega(\beta+1)}{\Gamma(\beta+2)} {}_2F_2\left(\frac{1}{2}, 1; \frac{\beta+2}{2}, \frac{\beta+3}{2}; -\frac{t^{2w}}{4(t-\tau)}\right). \tag{55}$$

This function has singularities on the line $t - \tau = 0$. Now, we investigate it on continuity

$$\lim_{\tau\to t-0} L_\beta(t,\tau) = \lim_{\tau\to t-0} \frac{t^\omega(\beta+1)}{\Gamma(\beta+2)} {}_2F_2\left(\frac{1}{2}, 1; \frac{\beta+2}{2}, \frac{\beta+3}{2}; -\frac{t^{2w}}{4(t-\tau)}\right). \tag{56}$$

Let us introduce the following variable:

$$z = \frac{t^{2w}}{4(t-\tau)} \Rightarrow \tau = t - \frac{t^{2w}}{4z}. \tag{57}$$

We have the following cases:

(a) If $2w - 1 > 0$ then $z \longrightarrow 0+0$ for $\tau \longrightarrow t-0$

(b) If $2w - 1 = 0$ then $z \longrightarrow (1/4)$ for $\tau \longrightarrow t-0$

(c) If $2w - 1 < 0$ then $z \longrightarrow +\infty$ for $\tau \longrightarrow t-0$

Now, we use the inequality from Theorem 3 for the following values of variables:

$$p = 2,\ \alpha_1 = \frac{1}{2}, \alpha_2 = 1, \rho_1 = \frac{\beta+2}{2}, \rho_2 = \frac{\beta+3}{2}, \quad z = \frac{t^{2w}}{4(t-\tau)}, \theta = \frac{\alpha_1\alpha_2}{\rho_1\rho_2} = \frac{2}{(\beta+2)(\beta+3)}. \tag{58}$$

We get

$$\exp\left(-\frac{2z}{(\beta+2)(\beta+3)}\right) < {}_2F_2\left(\frac{1}{2}, 1; \frac{\beta+2}{2}, \frac{\beta+3}{2}; -z\right) < 1 - \frac{2}{(\beta+2)(\beta+3)} + \frac{2}{(\beta+2)(\beta+3)} \exp(-z). \tag{59}$$

Then, the cases of values ω are revised.

(a) $2w-1>0$ for $\tau \longrightarrow t-0, z \longrightarrow 0+0$. From inequality (60), we obtain

$$ {}_2F_2\left(\frac{1}{2},1;\frac{\beta+2}{2},\frac{\beta+3)}{2};-\frac{t^{2w}}{4(t-\tau)}\right) \sim 1, \quad \text{for } \tau \longrightarrow t-0. \tag{60}$$

(b) $2w-1=0$. From inequality (60), we obtain

$$\exp\left(-\frac{1}{2(\beta+2)(\beta+3)}\right) < {}_2F_2\left(\frac{1}{2},1;\frac{\beta+2}{2},\frac{\beta+3}{2};-\frac{t^{2w}}{4(t-\tau)}\right) < 1-\frac{2}{(\beta+2)(\beta+3)}+\frac{2}{(\beta+2)(\beta+3)}\exp\left(-\frac{1}{4}\right), \tag{61}$$

or

$$\exp\left(-\frac{1}{2(\beta+2)(\beta+3)}\right) < {}_2F_2\left(\frac{1}{2},1;\frac{\beta+2}{2},\frac{\beta+3}{2};-\frac{t^{2w}}{4(t-\tau)}\right) < 1. \tag{62}$$

(c) $2w-1<0$. From inequality (60), we obtain

$$0 < {}_2F_2\left(\frac{1}{2},1;\frac{\beta+2}{2},\frac{\beta+3}{2};-\frac{t^{2w}}{4(t-\tau)}\right) < 1-\frac{2}{(\beta+2)(\beta+3)}. \tag{63}$$

Then, function (59) is bounded for any values $w\geq 0$ and $0\leq\beta\leq 1$. Moreover,

$$\lim_{\tau\longrightarrow t-0} L_\beta(t,\tau) = \lim_{\tau\longrightarrow t-0} \frac{t^w(\beta+1)}{\Gamma(\beta+2)}\,{}_2F_2\left(\frac{1}{2},1;\frac{\beta+2}{2},\frac{\beta+3}{2};-\frac{t^{2w}}{4(t-\tau)}\right) = 0. \tag{64}$$

As for kernel (32),

$$K_\beta(t,\tau) = \frac{L_\beta(t,\tau)}{\sqrt{\pi(t-\tau)}}. \tag{65}$$

Then, $\forall w\geq 0$ and $\beta\in[0;1]$ kernel (32) has a weak singularity in the domain $D=\{(t,\tau)\colon 0\leq t\leq T, 0\leq\tau\leq t\}$. This result agrees with the above inequality (24).

Integral equations with a weak singularity can be solved by the method of successive approximations [30]. It can be shown that for (31), successive iterated kernels are bounded starting from some numbers.

Since

$$\int_0^T\int_0^T K_\beta^2(t,\tau)d\tau\,dt \leq A, \text{const } A>0, \tag{66}$$

then according to [31] we have that the kernel $K_\beta(t,\tau)$ is continuous in the whole. If the right side of the integral equation is continuous on the segment [0, T], and the kernel is continuous in the whole, then any solution of the integral equation is continuous on the segment [0, T] [31]. □

Remark 8. Under the conditions of Theorem 7, kernel (32) of integral equation (31) has a weak singularity. Therefore, by applying the method of successive approximations, we can obtain a solution to integral equation (31) in the class of continuous functions. And the corresponding boundary value problem is well-posed in natural classes of functions, i.e., the loaded term of the problem's equation is a weak perturbation.

In [32], the authors consider a singular integral equation of the second kind of the Volterra type, but the method of successive approximations is not applicable to it. The boundary value problem is considered in a degenerating domain. In [33], the authors also consider the solvability of a nonhomogeneous boundary value problem for the Burgers equation in a degenerating domain, namely, in an infinite angular domain. The existence of nontrivial solutions of reduced homogeneous integral equations is shown.

Remark 9. It can be shown that

$$\lim_{t\longrightarrow 0}\int_0^t K_\beta(t,\tau)\,d\tau = 0, \quad 0<t<T. \tag{67}$$

Then, the norm of the integral operator from equation (31) is less than 1. Therefore, due to the principle of contraction mappings, there is a unique solution to equation (31) in the space of continuous functions.

7. On a Solution to BVP (15)-(16)

According to (43), we write the solution of problems (31)-(32) in the following form:

$$\begin{aligned} u(x,t) = {} & -\lambda\int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)\mu(\tau)d\tau \\ & + \int_0^t\int_0^\infty G(x,\xi,t-\tau)f(\xi,\tau)d\xi\, d\tau, \end{aligned} \tag{68}$$

where $f(x,t)$ belongs to class (19), and the solution of (34) $\mu(t)$ is a continuous and bounded function under the conditions of Theorem 7. Taking into account the nonnegativity of the functions $G(x,\xi,t-\tau)$ and $\operatorname{erf}(x/2\sqrt{t-\tau})$, taking into account an equality (it can be found after introducing the replacement $\xi = (x/2\sqrt{t-\tau})$ by integration, by parts, and by applying formula 3.461(5) from [27] (on page 351).

$$\begin{aligned} \int_0^t \operatorname{erf}\left(\frac{x}{2\sqrt{t-\tau}}\right)d\tau = {} & t\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) \\ & + \frac{x\sqrt{t}}{\sqrt{\pi}}\exp\left(-\frac{x^2}{4t}\right) - \frac{x^2}{2}\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right), \end{aligned} \tag{69}$$

directly from (68), we obtain the following estimate:

$$|u(x,t)| \le C(\lambda)x\sqrt{t}, \tag{70}$$

where $C(\lambda) = C_1|\lambda| + C_2$.

The derivatives of the solution $u(x,t)$ (68) satisfy the following inclusion:

$$u_t - u_{xx} = -\lambda\mu(t) + f(x,t) \in L_\infty(A)\cap C(B), \quad T = const > 0, \tag{71}$$

where $A = \{(x,t)\,|\,x>0, t\in[0,T]\}$, $B = \{(x,t)\,|\,x>0, t\ge 0\}$, and $T - const > 0$ (this fact follows from (18) and notation (13)).

So, function (68) satisfies (18) in the sense of relation (29). Obviously, solution (68) satisfies the initial and boundary conditions (8). Thus, function (68) according to (69) and (70) satisfies BVP (15)-(16) and belongs to the following class:

$$\begin{aligned} \mathfrak{U} = \Bigg\{ u \,\Bigg|\, & (x\sqrt{t})^{-1}u \in L_\infty(A)\cap C(B);\, u_t - u_{xx} \in L_\infty(A)\cap C(B); \\ & \left\{\frac{1}{\Gamma(\beta)}\int_0^x \frac{u(\xi,t)}{(x-\xi)^{1-\beta}}d\xi\right\}\Bigg|_{x=\gamma(t)} \in C([0;T]), \quad T = const>0, 0\le\beta\le 1 \Bigg\}, \end{aligned} \tag{72}$$

where $A = \{(x,t)\,|\,x>0, t\in[0,T]\}$, $B = \{(x,t)\,|\,x>0, t\ge 0\}$, and $T = const > 0$.

So, the following theorem is true.

Theorem 10. *Let conditions (19) and (20) be satisfied for the function $f(x,t)$, the function $\mu(t)\in C([0;T])$ is the solution of integral equation (31) with the right hand side $f_2(t)\in C([0;T])$ defined by formula (33). Then, BVP (15)-(16) has the only solution (69) in the class (72), if $\gamma(t)\sim t^\omega$ (near the point $t=0$), $\omega\ge 0$, and $0\le\beta\le 1$.*

8. Results and Discussion

Under the conditions of the theorem proved, there is an exact solution to BVP (15)-(16) in the class of sufficiently smooth function (72). Following the study by Beshtokov [34], to solve the problem using the method of energy inequalities, it is possible to obtain a priori estimates for the differential and difference equations. The obtained estimates ensure the uniqueness of the solution and the continuous dependence of the solution on the input data of the problem. The linearity of the problem under consideration with the obtained a priori estimates will ensure the convergence of the approximate solution to the exact solution at a certain velocity.

So, to model and substantiate theoretical conclusions, it is possible to obtain a priori estimates in differential and difference interpretations as in [34] from which the uniqueness and stability of the solution will follow from the initial data and the right side of the equation, as well as the convergence of the solution of the difference problem to the solution of the differential problem.

9. Conclusion

Note that in this work, theoretical results were obtained, the totality of which is important in the theory of loaded parabolic equations. Let us list these results. The posed boundary value problem is reduced to the Volterra integral equation of the second kind by inverting the differential part of the problem. The peculiarity of the resulting integral equation is that its kernel contains a special function. Therefore, it is difficult to directly solve the integral equation. Then, its kernel is evaluated. This process is accompanied by a description of functional classes of the solution and the right side of the equation.

Limiting cases of the fractional derivative's order for continuity with respect to the order of the fractional derivative are also studied. Based on the obtained results, the interval for changing the order of the fractional derivative in the loaded term of the equation was established, for which the existence and uniqueness theorem of the solution to the problem was proven. In addition, for limiting cases of the fractional derivative's order, it is possible to find an explicit solution to the boundary value problem. In some cases of the type of load containing the operator of fractional integro-differentiation, one can find a solution to the boundary value problem in explicit form, for example, in [35].

The results of this theoretical study have limited applicability to practical situations due to the underlying assumptions. Numerical studies, in contrast, can be more general and applicable to a wide range of conditions. The main weakness of this work is the following: the theoretical research is based on idealizations that do not fully reflect real conditions. However, we should not forget that theoretical research is still an important tool for understanding basic principles and patterns underlying the problem being studied. While numerical simulations can provide more accurate and detailed results, obtained theoretical results can contribute to the theory of fractional calculus and loaded differential equations. In this work, a boundary value problem that arose at the intersection of the theory of fractional calculus and the theory of loaded equations was investigated, and existence-uniqueness theorem for the solution in the class of continuous functions was established, but further extensions of this research are possible, such as proof of qualitative properties and boundaries, as well as the construction of explicit solutions in some special cases using other methods. Loaded partial differential equations involving Ψ-tempered operators can also be studied since there is a sufficient amount of research on the theory of Ψ-fractional calculus.

Data Availability

The theoretical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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