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Research Article

Planar Graphs without Cycles of Length 3, 4, and 6 are (3, 3)-Colorable

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For non-negative integers d_1 and d_2, if V_1 and V_2 are two partitions of a graph G's vertex set $V(G)$, such that V_1 and V_2 induce two subgraphs of G, called $G[V_1]$ with maximum degree at most d_1 and $G[V_2]$ with maximum degree at most d_2, respectively, then the graph G is said to be improper (d_1, d_2)-colorable, as well as (d_1, d_2)-colorable. A class of planar graphs without C_3, C_4, and C_6 is denoted by $\mathscr{C}$. In 2019, Dross and Ochem proved that G is $(0, 6)$-colorable, for each graph G in $\mathscr{C}$. Given that $d_1 + d_2 \geq 6$, this inspires us to investigate whether G is (d_1, d_2)-colorable, for each graph G in $\mathscr{C}$. In this paper, we provide a partial solution by showing that G is (3, 3)-colorable, for each graph G in $\mathscr{C}$.

1. Introduction

The term improper (d_1, d_2)-colorable, as well as (d_1, d_2)-colorable, refers to a graph G, for non-negative integers d_1 and d_2, if V_1 and V_2 are two partitions of a graph G's vertex set $V(G)$, such that V_1 and V_2 induce two subgraphs of G, called $G[V_1]$ with maximum degree at most d_1 and $G[V_2]$ with maximum degree at most d_2, respectively. Planar graphs and their conditions suffice to be (d_1, d_2)-colorable are widely studied. The cycle of length n is denoted as C_n or n-cycle, for each $n \geq 3$. For each h, p, Montassier and Ochem [1] constructed planar graphs without C_3 that are not (h, p)-colorable. For any h, planar graphs without C_3, C_4, and C_5 that are not $(0, h)$-colorable were constructed by Borodin et al. [2]. As opposed, some sufficient conditions for planar graphs to be (d_1, d_2)-colorable for specific d_1 and d_2 are found. In 2022, Ma et al. [3] showed that every planar graph without C_4 and C_6 is (2, 9)-colorable. For any planar graph G with girth (the length of the shortest cycle in the graph) at least 5, Borodin and Kostochka [4] proved that G is (2, 6)-colorable. Recently, Zhang et al. [5] showed that planar graphs with girth at least 5 without adjacent 5-cycles are (1, 6)-colorable. Furthermore, Wang et al. [6] proved that planar graphs with girth at least 5 without adjacent 5-cycles are (3, 3)-colorable. This inspires us to investigate whether planar graphs without C_3, C_4, and C_6 have this property.

Let $\mathscr{C}$ be the class of planar graphs without C_3, C_4, and C_6. In [7], Choi et al. proved that G is (0, 45)-colorable, for each graph G in $\mathscr{C}$. Dross and Ochem [8] recently enhanced the findings by proving that G is (0, 6)-colorable, for each graph G in $\mathscr{C}$. Given that $d_1 + d_2 \geq 6$, this inspires us to investigate whether G is (d_1, d_2)-colorable, for each graph G in $\mathscr{C}$. It should be noted that preventing C_3, C_4, and C_6 from existing as subgraphs or induced subgraphs yields the same graph class. The following theorem provides a partial answer in this work.

Theorem 1. *Every graph in $\mathscr{C}$ is $(3,3)$-colorable.*

2. Notations and Helpful Properties

First, we present useful notions, observations, lemmas, and corollaries about a minimal planar graph G in $\mathscr{C}$ that is not (3, 3)-colorable.

We refer to the vertex set, edge set, and face set, respectively, as $V(G)$, $E(G)$, and $F(G)$. To indicate a face f's boundary, we use the notation $B(f)$. In addition, the number of edges on the boundary of a face f determines its degree. Let an l-vertex (face), an l^+-vertex (face), or an l^--vertex (face) be three different types of vertices (faces), each with a vertex (face) that has a degree of l, at least l, or at most l, respectively.

Lemma 2. *G is connected.*

Proof. Assume that G is not connected. Then each connected component of G is smaller than G. Therefore, a $(3,3)$-coloring is admissible for every connected component of G. A $(3,3)$-coloring of G results from the union of these $(3,3)$-colorings, which is contradictory. □

Lemma 3. *Every vertex in G is a 2^+-vertex.*

Proof. Assume that v is a 1-vertex in G. $G-v$ has a $(3,3)$-coloring c by employing the color set $\{1,2\}$ according to G's minimality. Since we have two colors, we can extend c to G by properly coloring v, a contrary to the choice of G. □

Observation 4. Every k-face is bounded by k-cycle for $k \le 9$.

As a result of the absence of some cycles in G, the following is easily observed.

Corollary 5. *G has neither 3-, 4-, nor 6-faces.*

Lemma 6 (See [[9], in Lemmas 2.1, 2.2, and 2.3]). *Let G be a minimal graph which is not (d_1,d_2)-colorable where $d_1 \le d_2$. Then, we have the following:*

(i) If v is a 3^--vertex in G, then v is adjacent to at least two $(d_1+2)^+$-vertices where one of which is a $(d_2+2)^+$-vertex.

(ii) If v is a $(d_1+d_2+1)^-$-vertex in G, then v is adjacent to at least one $(d_1+2)^+$-vertex.

Lemma 6 yields the following lemma.

Lemma 7. *Let G be a minimal graph which is not (3, 3)-colorable. Then, we have the following:*

(i) If v is a 3^--vertex in G, then v is adjacent to at least two 5^+-vertices.

(ii) If v is a 5-vertex in G, then v is adjacent to at most four 3^--vertices.

Lemma 8. *If v is a 2-vertex in G, then v is not incident to two 5-faces.*

Proof. Let v be a 2-vertex in G incident to two faces f_1 and f_2. Suppose to the contrary that f_1 and f_2 are 5-faces. By Observation 4, we may assume that a 5-cycle $B(f_1) = vv_1x_1x_2v_2$ and a 5-cycle $B(f_2) = vv_1u_1u_2v_2$. Then, $|V(B(f_1)) \cap V(B(f_2))| \ge 3$. If $x_1 = u_1$, then $d(v_1) = 2$, a contrary to Lemma 7 (i). If $x_1 = u_2$, then $v_1u_1u_2$ is a 3-cycle, a contradiction. By symmetry, $x_2 \notin \{u_1,u_2\}$. Thus, $|V(B(f_1)) \cap V(B(f_2))| = 3$. Hence, $B(f_1) \cup B(f_2)$ contains a 6-cycle. This contradiction completes the proof. □

3. Proof of Theorem 1

Proof of Theorem 1. Suppose the contrary to Theorem 1 and let G be a minimal counterexample. The following are our discharging processes.

Let $\mu(x)$ be an initial charge of x for all $x \in V(G) \cup F(G)$ where $\mu(x) = d(x) - 4$.

Using Euler's formula and Handshaking lemma

$$\begin{aligned}\sum_{x\in V(G)\cup F(G)} \mu(x) &= \sum_{x\in V(G)} (d(x)-4) + \sum_{x\in F(G)} (d(x)-4) \\ &= (2|E(G)| - 4|V(G)|) \\ &\quad + (2|E(G)| - 4|F(G)|) \\ &= 4|E(H)| - 4|V(H)| - 4|F(H)| \\ &= -8.\end{aligned} \tag{1}$$

Next, we redistribute the charge of vertices and faces to have $\mu^*(x)$ by transferring charge from one element to another later so that the total charge remains the same $(\sum_{x\in V(G)\cup F(G)} \mu^*(x) = -8)$. However, upon the completion of the discharging, the final charge satisfies $\mu^*(x) \ge 0$ for all $x \in V(G) \cup F(G)$ which is contradictory.

Discharging rules are as follows.

Let v be a vertex of a graph G.

(R1) For $d(v) = 2$, its incident 5-faces give charge $(1/2)$ to v.

(R2) For $d(v) = 2$, its incident 7^+-faces give charge 1 to v.

(R3) For $d(v) = 3$, its incident 5^+-faces give charge $(1/6)$ to v.

(R4) For $d(v) \le 3$, its adjacent 5^+-vertices give charge $(1/4)$ to v.

For each $x \in V(G) \cup F(G)$, the remaining part of the proof shows that $\mu^*(x) \ge 0$.

If x is a 4-vertex by (R1)-(R4), then it is obvious that $\mu^*(x) = \mu(x) = 0$.

(i) Consider a 2-vertex v.

Lemma 7 (i) states that two 5^+-vertices are adjacent to v. Moreover, v has at least one incident 7^+-face according to Lemma 8. So, using (R1), (R2), and (R4), we get the following result: $\mu^*(v) \ge \mu(v) + (1/2) + 1 + 2 \times (1/4) = 0$.

(ii) Consider a 3-vertex v.

Lemma 7 (i) states that at least two 5^+-vertices are adjacent to a vertex v. It should be noted that v has three incident 5^+-faces. So, using (R3) and (R4), we get the following result: $\mu^*(v) \geq \mu(v) + 3 \times (1/6) + 2 \times (1/4) = 0$.

(iii) Consider a 5-vertex v.

Lemma 7 (ii) states that at most four 3^--vertices can be adjacent to a vertex v. So, using (R4), we get the following result: $\mu^*(v) \geq \mu(v) - 4 \times (1/4) = 0$.

(iv) Consider a 6^+-vertex v.

Give v a k-vertex with $k \geq 6$. So, using (R4) and $k \geq 6$, we get the following result: $\mu^*(v) \geq \mu(v) - k \times (1/4) > 0$.

(v) Consider a 5-face f.

Lemma 7 (i) states that a face f has no more than two incident 2-vertices.

Case 1: There are two incident 2-vertices of a face f.
Lemma 7 (i) states that there are no 3-vertices incident to a face f. So, using (R1), we get the following result: $\mu^*(f) \geq \mu(f) - 2 \times (1/2) = 0$.
Case 2: There is one incident 2-vertex of a face f.
Lemma 7 (i) states that at most two 3-vertices are incident to a face f. So, using (R1) and (R3), we get the following result: $\mu^*(f) \geq \mu(f) - (1/2) - 2 \times (1/6) > 0$.
Case 3: There are no incident 2-vertices on a face f.
So, using (R3), we get the following result: $\mu^*(f) \geq \mu(f) - 5 \times (1/6) > 0$.

(vi) Consider a 7-face f.

Lemma 7 (i) states that a face f has at most three incident 2-vertices.

Case 1: Three 2-vertices are incident to a face f.
Lemma 7 (i) states that there are no 3-vertices incident to a face f. So, using (R2), we get the following result: $\mu^*(f) \geq \mu(f) - 3 \times 1 = 0$.
Case 2: At most two 2-vertices are incident to a face f.
So, using (R2) and (R3), we get the following result: $\mu^*(f) \geq \mu(f) - 2 \times 1 - 5 \times (1/6) > 0$.

(vii) Consider a k-face f where $k \geq 8$.

Consider f bounded by $v_0 v_1 \ldots v_{k-1}$ which each subscript is taken modulo k. This case depends on (R2) and (R3). To facilitate the calculation, we provide a new rule in which $\mu^*(f)$ is non-negative while its incident receive charges not less than by ones from (R2) and (R3). First, f sends charge 1/2 to each incident vertices (v_i for each $0 \leq i \leq k-1$). Considering that f is an 8^+-face, it implies that $\mu^*(f) \geq 0$. Now, we redistribute the charge from f as in the following new rule:

(R^*) A 5^+-vertex v_i gives (1/4) (received from f) to v_{i-1} or v_{i+1} if it is a 2-vertex.

Case 1: Consider v_i a 2-vertex.
Lemma 7 (i) states that a vertex v_i has two adjacent 5^+-vertices which are on the boundary of f, say v_{i-1} and v_{i+1}. So, using (R^*), we get the following result: two its adjacent 5^+-vertices send charge $2 \times (1/4)$ to v_i. Thus, v_i receives charge $(1/2) + 2 \times (1/4) = 1$ from f as in (R2).
Case 2: Consider v_i a 3-vertex.
So, using (R^*), v_i receives charge at least $(1/2) \geq (1/6)$ as in (R3).
Case 3: Consider v_i a 5^+-vertex.
So, using (R^*), we get the following result: a vertex v_i send charge (1/4) to v_{i-1} or v_{i+1} when it is a 2-vertex. Thus v_i receives charge at least $(1/2) - 2 \times (1/4) = 0$ from f as in (R2).
Hence, $\mu^*(f) \geq 0$.

One can observe that our proof is finished when $\mu^*(x) \geq 0$ for every $x \in V(G) \cup F(G)$. □

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this work.

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References

[1] M. Montassier and P. Ochem, "Near-colorings: non-colorable graphs and NP-completeness," *Electronic Journal of Combinatorics*, vol. 22, no. 1, p. 1.57, 2015.

[2] O. V. Borodin, A. O. Ivanova, M. Montassier, P. Ochem, and A. Raspaud, "Vertex decompositions of sparse graphs into an edgeless subgraph and a subgraph of maximum degree at most k," *Journal of Graph Theory*, vol. 65, no. 2, pp. 83–93, 2010.

[3] J. Ma, M. Huang, and X. Zhang, "Every planar graph without 4-cycles and 6-cycles is (2,9)-colorable," *Italian Journal of Pure and Applied Mathematics*, vol. 48, pp. 659–670, 2022.

[4] O. V. Borodin and A. V. Kostochka, "Defective 2-colorings of sparse graphs," *Journal of Combinatorial Theory- Series B*, vol. 104, pp. 72–80, 2014.

[5] G. Zhang, M. Chen, and W. Wang, "A sufficient condition for planar graphs with girth 5 to be (1,6)-colorable," *Applied Mathematics and Computation*, vol. 474, Article ID 128706, 2024.

[6] Y. Wang, D. Huang, and S. Finbow, "On the vertex partition of planar graphs into forests with bounded degree," *Applied Mathematics and Computation*, vol. 374, Article ID 125032, 2020.

[7] I. Choi, C.-H. Liu, and S.-I. Oum, "Characterization of cycle obstruction sets for improper coloring planar graphs," *SIAM Journal on Discrete Mathematics*, vol. 32, no. 2, pp. 1209–1228, 2018.

[8] F. Dross and P. Ochem, "Vertex partitions of (C_3, C_4, C_6)-free planar graphs," *Discrete Mathematics*, vol. 342, no. 11, pp. 3229–3236, 2019.

[9] I. Choi and A. Raspaud, "Planar graphs with girth at least 5 are (3,5)-colorable," *Discrete Mathematics*, vol. 338, no. 4, pp. 661–667, 2015.