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Research Article

Analytical Approach to Polarization Mode Dispersion in Linearly Spun Fiber with Birefringence

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The behavior of Polarization Mode Dispersion (PMD) in spun optical fiber is a topic of great interest in optical networking. Earlier work in this area has focused more on approximate or numerical solutions. In this paper we present analytical results for PMD in spun fibers with triangular spin profile function. It is found that in some parameter ranges the analytical results differ from the approximations.

1. Introduction

The Polarization Mode Dispersion (PMD) is a well-known phenomenon in optical fibers and its role in the propagation of light pulse in various kinds of optical fibers has been a subject of intensive investigation [1–6] in the past. Its physical origin lies in the birefringence property of an optical fiber so that the orthogonal modes of the light electromagnetic wave acquire different propagation speeds resulting in a phase difference between them. The optical fiber at granular level is nonhomogeneous and also has other defects accumulated during the manufacturing process. Due to these issues, the birefringence in a physical fiber becomes random as pointed out by Foschini and Poole in [7]. In addition, Menyuk and Wai [8] have also considered the nonlinear effects arising from higher order susceptibility that also becomes important under certain physical conditions.

Sometime ago, Wang et al. [1] derived expressions for the Differential Group Delay (DGD) of a randomly birefringent fiber in the Fixed Modulus Model (FMM) in which the DGD has both modulus and the phase. The FMM assumes that the modulus of the birefringence vector is a random variable. They presented analytical results with the following assumptions: (i) the spin function is periodic (a sine function) and (ii) the periodicity length (p) of the fiber is much smaller than the fiber correlation length (L_F) or $p \ll L_F$. Later they also generalized the FMM and presented the Random Modulus Model (RMM), which includes the randomness in the direction of the birefringence vector. But then the RMM equations could only be solved numerically.

The present work is a contribution to the analytical calculations within FMM and so is only valid for a short fiber distance. This limitation arises because beyond that distance the birefringence randomness [7] becomes dominant. In the present work the full implications of the FMM have been explored under the following conditions: (i) The $p \ll L_F$ approximation has been relaxed, (ii) a nonzero twist has been included, and (iii) the periodic spin rate has been replaced with a constant spin rate. We give the analytical solutions of the exact FMM equations under these conditions and also present some numerical results based on them showing the effect of different physical conditions. The analytical methods are those applicable to the coupled mode theory calculations adapted to the optical fibers [9].

2. Theoretical Analysis

2.1. The Model with Periodic Spin Function. The starting point is the well-known vector equation describing the change in the Jones local electric field vector $\vec{A}(\omega, z)$ with the angular frequency ω and distance z along a twisted fiber. Consider

$$\begin{bmatrix} \dfrac{dA_1(z)}{dz} \\ \dfrac{dA_2(z)}{dz} \end{bmatrix} = \frac{i}{2}(\Delta\beta) \begin{bmatrix} 0 & e^{2i\Theta(z)} \\ e^{-2i\Theta(z)} & 0 \end{bmatrix} \begin{bmatrix} A_1(z) \\ A_2(z) \end{bmatrix}. \quad (1)$$

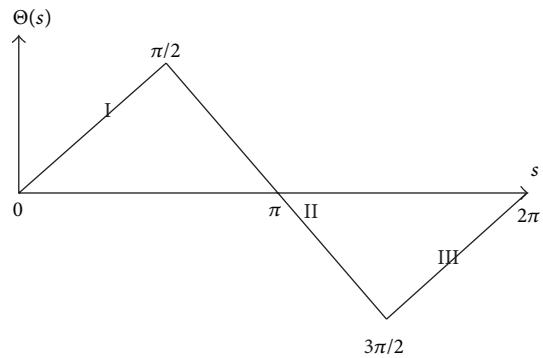

Figure 1: The 3-segment approximation to the periodic sine function.

Here $\Delta\beta(\omega)$ is the birefringence and

$$\Theta(z) = \frac{\alpha_0}{\eta}\sin(\eta z) \tag{2}$$

is the periodic spin profile function with spin magnitude α_0 and angular frequency of spatial modulation η.

The boundary conditions are

$$A_1(0) = 1, \quad \frac{dA_1(0)}{dz} = 0, \tag{3a}$$

$$A_2(0) = 0, \quad \frac{dA_2(0)}{dz} = i\left(\frac{\Delta\beta}{2}\right). \tag{3b}$$

Let $s = \eta z$ be a dimensionless variable. We use $(d/dz) = \eta(d/ds)$ to rewrite (1). Consider

$$\begin{bmatrix} A_{1s}(s) \\ A_{2s}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2ic\sin s} \\ e^{-2ic\sin s} & 0 \end{bmatrix} \begin{bmatrix} A_1(s) \\ A_2(s) \end{bmatrix}. \tag{4}$$

The subscripts denote differentiation ($A_{1s} = dA_{1s}/ds$, $A_{2s} = dA_{2s}/ds$). Also, $a = (\Delta\beta/2\eta)$ and $c = (\alpha_0/\eta)$ are dimensionless constants.

We express all parameters in terms of the lengths given as beat length ($L_B = 2\pi/\Delta\beta$), spin period ($\Lambda = 2\pi/\eta$), and coupling length ($l_0 = 2\pi/\alpha_0$). Then we can write $a = \Lambda/2L_B$, $c = L_B/l_0$.

The new boundary conditions are

$$A_1(0) = 1, \quad A_{1s}(0) = 0, \tag{5a}$$

$$A_2(0) = 0, \quad A_{2s}(0) = ia. \tag{5b}$$

These equations ((1) or equivalently (4)) do not have analytical solutions.

In the perturbative approximation (see Appendix B), an analytical result has been derived earlier [1]. In the present work we derive analytic solutions by replacing the sine function by linear segments and compare them to the perturbative solutions for the same segments.

2.2. Linear Segment Approximation to the Periodic Spin Function: Analytical Solutions for the Jones Amplitude Equations

The Model. The periods of the straight line segments shown in Figure 1 approximate the periodic sine function. Here a single period with 3-segment approximation is shown in Figure 1.

The field amplitudes for a given segment satisfy the following equations:

$$\begin{bmatrix} A_{1s}^{(m)}(s) \\ A_{2s}^{(m)}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2i\theta_m(s)} \\ e^{-2i\theta_m(s)} & 0 \end{bmatrix} \begin{bmatrix} A_1^{(m)}(s) \\ A_2^{(m)}(s) \end{bmatrix}. \tag{6}$$

The superscript and subscript m both indicate the segments for which the coupled equations hold. The limits of segments are given below.

We require that the endpoints of $\theta_m(s)$ should be the same as that of the sine-function spin profile $\Theta(s)|_{\text{spin}} = c\sin s$ for all segments. Define $\tilde{c} = (2c/\pi)$ so that the endpoint conditions for segments hold.

For $n = 1$, *Segment I* ($0 \le s \le \pi/2$),

$$\theta_1(s) = \tilde{c}s,$$
$$\Theta(s=0)|_{\text{spin}} = 0 = \theta_1(s=0), \tag{7}$$
$$\Theta\left(s = \frac{\pi}{2}\right)\Big|_{\text{spin}} = c = \theta_1\left(s = \frac{\pi}{2}\right).$$

For $n = 2$, *Segment II* ($\pi/2 \le s \le 3\pi/2$),

$$\begin{aligned} \theta_2(s) &= -\tilde{c}s + 2c, \\ \Theta\left(s = \frac{\pi}{2}\right)\Big|_{\text{spin}} &= c = \theta_2\left(s = \frac{\pi}{2}\right), \\ \Theta\left(s = \frac{3\pi}{2}\right)\Big|_{\text{spin}} &= -c = \theta_2\left(s = \frac{3\pi}{2}\right). \end{aligned} \tag{8}$$

For $n = 3$, *Segment III* ($3\pi/2 \le s \le 2\pi$),

$$\begin{aligned} \theta_3(s) &= \tilde{c}s - 4c, \\ \Theta\left(s = \frac{3\pi}{2}\right)\Big|_{\text{spin}} &= -c = \theta_3\left(s = \frac{3\pi}{2}\right), \\ \Theta(s = 2\pi)|_{\text{spin}} &= 0 = \theta_3(s = 2\pi). \end{aligned} \tag{9}$$

The General m-Segment Solutions. The solutions for the mth segment have the following general form:

$$\begin{bmatrix} e^{-i\theta_m(s)} A_1^{(m)}(s) \\ iae^{i\theta_m(s)} A_2^{(m)}(s) \end{bmatrix} = \begin{bmatrix} a_1^{(m)} + ib_1^{(m)} & a_2^{(m)} + ib_2^{(m)} \\ \left\{-\theta_{m/s} b_1^{(m)} + q_m a_2^{(m)} + i\left(\theta_{m/s} a_1^{(m)} + q_m b_2^{(m)}\right)\right\} & \left\{-\left(\theta_{m/s} b_2^{(m)} + q_m a_1^{(m)}\right) + i\left(\theta_{m/s} a_2^{(m)} - q_m b_1^{(m)}\right)\right\} \end{bmatrix} \begin{bmatrix} \cos q_m s \\ \sin q_m s \end{bmatrix} \tag{10}$$

with

$$\begin{aligned} q_m^2 &= a^2 + \theta_m^2(s), \\ \theta_{m/s} &= \frac{d\theta_m(s)}{ds}. \end{aligned} \tag{11}$$

The exact solutions for the coupled equations in one segment are related to those in the previous adjacent segment by the following chain-relations among the coefficients.

Define $u = (q_{m-1}/q_m)$, $v = (\theta_{m/s} - \theta_{m-1/s})/q_m$, and then the chain-relations are given by

$$\begin{bmatrix} a_1^{(m)} \\ a_2^{(m)} \\ b_1^{(m)} \\ b_2^{(m)} \end{bmatrix} = \left\{ \begin{bmatrix} t_1 & t_3 & 0 & 0 \\ t_2 & t_4 & 0 & 0 \\ 0 & 0 & t_1 & t_3 \\ 0 & 0 & t_2 & t_4 \end{bmatrix} + u \begin{bmatrix} t_4 & -t_2 & 0 & 0 \\ -t_3 & t_1 & 0 & 0 \\ 0 & 0 & t_4 & -t_2 \\ 0 & 0 & -t_3 & t_1 \end{bmatrix} + v \begin{bmatrix} 0 & 0 & -t_2 & -t_4 \\ 0 & 0 & t_1 & t_3 \\ t_2 & t_4 & 0 & 0 \\ -t_1 & -t_3 & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} a_1^{(m-1)} \\ a_2^{(m-1)} \\ b_1^{(m-1)} \\ b_2^{(m-1)} \end{bmatrix}. \tag{12}$$

Here the matrix elements are

$$\begin{aligned} t_1 &= \cos q_{m-1} s_{m-1} \cos q_m s_{m-1}, \\ t_2 &= \cos q_{m-1} s_{m-1} \sin q_m s_{m-1}, \\ t_3 &= \sin q_{m-1} s_{m-1} \cos q_m s_{m-1}, \\ t_4 &= \sin q_{m-1} s_{m-1} \sin q_m s_{m-1}. \end{aligned} \tag{13}$$

The matrix chain-relations can be written compactly by expressing the 4×4 matrices as outer products (denoted by the symbol $\otimes$) of two 2×2 matrices as

$$\begin{bmatrix} a_1^{(m)} \\ a_2^{(m)} \\ b_1^{(m)} \\ b_2^{(m)} \end{bmatrix} = \left\{ \left(\begin{bmatrix} t_1 & t_3 \\ t_2 & t_4 \end{bmatrix} + u \begin{bmatrix} t_4 & -t_2 \\ -t_3 & t_1 \end{bmatrix} \right) \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + v \begin{bmatrix} t_2 & t_4 \\ -t_1 & -t_3 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} \begin{bmatrix} a_1^{(m-1)} \\ a_2^{(m-1)} \\ b_1^{(m-1)} \\ b_2^{(m-1)} \end{bmatrix}. \tag{14}$$

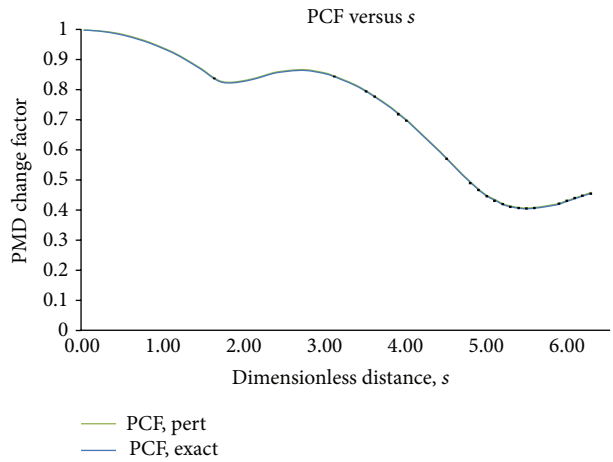

Figure 2: The PCF curves for a perturbative limit with $\Lambda = 1$ and $L_B = 12$.

2.3. Calculation of PMD Correction Factor (PCF). The sum of squares of the ω-differentiated amplitudes is similar to power and can be calculated by the following expression using expressions from Appendix A:

$$\begin{aligned}\frac{\left|A_{1\omega}^{(m)}(s)\right|^2+\left|A_{2\omega}^{(m)}(s)\right|^2}{(a_\omega/q)^2} &= \left(\frac{1}{2}\right)\left[\left(1-n^2\right)\right.\\ &\cdot\left\{\left(p_1^{(m)}\right)^2+\left(p_2^{(m)}\right)^2+\left(p_3^{(m)}\right)^2+\left(p_4^{(m)}\right)^2\right\}\\ &\left.+\left(p_5^{(m)}\right)^2+\left(p_6^{(m)}\right)^2+\left(p_7^{(m)}\right)^2+\left(p_8^{(m)}\right)^2\right]\\ &+\left(\frac{1}{2}\right)\left[\left(1-n^2\right)\right.\\ &\cdot\left\{\left(p_1^{(m)}\right)^2+\left(p_2^{(m)}\right)^2-\left(p_3^{(m)}\right)^2-\left(p_4^{(m)}\right)^2\right\}\\ &\left.+\left(p_5^{(m)}\right)^2+\left(p_6^{(m)}\right)^2-\left(p_7^{(m)}\right)^2-\left(p_8^{(m)}\right)^2\right]\\ &\cdot\cos 2qs+\left[\left(1-n^2\right)\left\{p_1^{(m)}p_3^{(m)}+p_2^{(m)}p_4^{(m)}\right\}\right.\\ &\left.+p_5^{(m)}p_7^{(m)}+p_6^{(m)}p_8^{(m)}\right]\sin 2qs.\end{aligned} \tag{15}$$

Here m $(=1,2,3)$ refers to segments in sequential manner.

For calculating the normalized PCF we need a similar expression for unspun-fiber given below:

$$\frac{\left[\left|A_{1\omega}(s)\right|^2+\left|A_{2\omega}(s)\right|^2\right]_{\text{unspun-fiber}}}{(a_\omega/q)^2}=(qs)^2. \tag{16}$$

Then the expression for the PCF becomes

$$\mathrm{PCF}^{(m)}(s) = \left[\frac{\left|A_{1\omega}^{(m)}(s)\right|^2+\left|A_{2\omega}^{(m)}(s)\right|^2}{\left[\left|A_{1\omega}(s)\right|^2+\left|A_{2\omega}(s)\right|^2\right]_{\text{unspun-fiber}}}\right]^{1/2}. \tag{17}$$

The LHS is a function of parameters n and q and argument s. In general the expressions are quiet complicated, but for the first segment, the PCF is easily calculated and is given by

$$\mathrm{PCF}^{(1)}(s)=\sqrt{1-n^2\left\{1-\left(\frac{\sin qs}{qs}\right)^2\right\}}. \tag{18}$$

3. Numerical Results

The physical constants $((\Delta\beta,\alpha_0,\eta)$ or equivalently $(L_B,l_0,\Lambda))$ and the parameters (n,q) appearing in the PCF expressions are related by

$$\begin{aligned}q&=\left(\frac{2\Lambda}{\pi l_0}\right)\left[1+\left(\frac{\pi l_0}{4L_B}\right)\right]^{1/2},\\ n&=\left[1+\left(\frac{\pi l_0}{4L_B}\right)\right]^{-1/2}.\end{aligned} \tag{19}$$

We show results for sets of parameters in two extreme limits to emphasize the difference between the exact and perturbative calculations.

The Small-q Limit ($\Lambda < L_B$). In this limit two sets of parameters were chosen to get small-q-values (less than 1). This corresponds to beat length being much larger than the spin period.

The resulting plots are given in Figures 2 and 3.

It is seen that the curves in Figure 2 for exact and perturbative calculations for small-q approximation are almost identical.

The curves in Figure 3 for exact and perturbative calculations are almost identical. Note that after $s = 5$ the two curves start diverging a little.

The Large-q Limit ($\Lambda > L_B$). In this limit two sets of parameters were chosen to get large-q-values (much larger than 1). This corresponds to beat length being smaller than spin period.

The resulting plots are given in Figures 4 and 5.

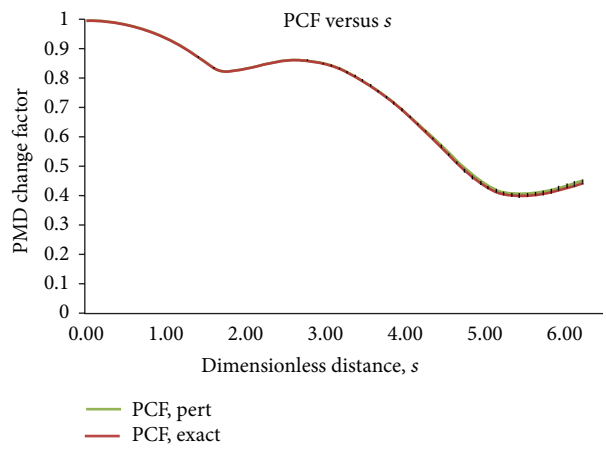

Figure 3: The PCF curve for a perturbative limit with $\Lambda = 1$ and $L_B = 5$.

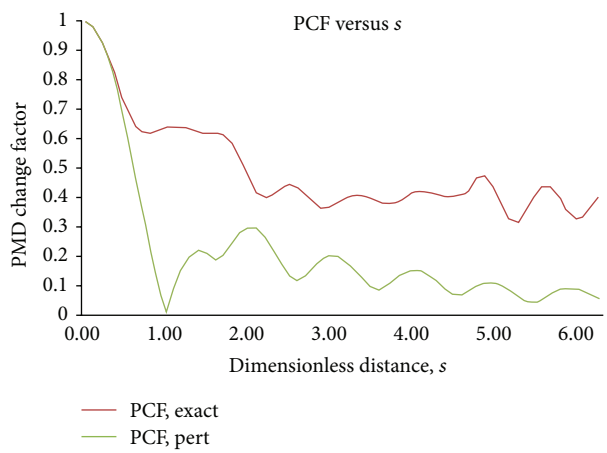

Figure 4: The PCF curve for a nonperturbative limit with $\Lambda = 5$ and $L_B = 1$.

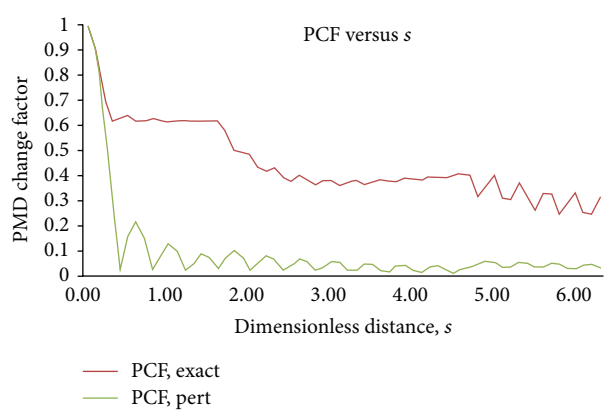

Figure 5: The PCF curves for a nonperturbative limit with $\Lambda = 12$ and $L_B = 1$.

The top and bottom curves in Figure 4 show exact and perturbative calculations, respectively. It is seen that perturbative approximation underestimates the PCF in this regime. The two start diverging significantly for value of s a little less than 1.

The top and bottom curves in Figure 5 show exact and perturbative calculations, respectively. It is seen that perturbative approximation underestimates the PCF in this regime. The two start diverging significantly for value of s a little beyond zero.

Table 1: PCF versus z plots.

Parameters: Λ, L_B, l_0 (in meters)	Values (n, q)	Comments
$(1, 12, 1)$	$(0.9978, 0.6379)$	$\Lambda \ll L_B$
$(1, 5, 1)$	$(0.9879, 0.6444)$	$\Lambda < L_B$

Table 2: PCF versus z plots.

Parameters: Λ, L_B, l_0 (in meters)	Values (n, q)	Comments
$(5, 1, 1)$	$(0.7864, 4.0475)$	$\Lambda > L_B$ (physical nonperturbative limit)
$(12, 1, 1)$	$(0.7864, 9.7139)$	$\Lambda \gg L_B$ (physical very nonperturbative limit)

4. Conclusions

The sine-function spin profile can be approximated in general by any number of segments. In this work a 3-segment approximation was chosen and analytical results for the PCF function were obtained. The PCF calculations were also repeated under the assumptions of the perturbative approximation made in [1]. As expected, it was shown that the perturbative approximation has limited validity compared to an exact calculation.

The 3-segment approximation given here can be extended to any number of segments for the spin function. The analytical results become very complicated very soon but they will approach the exact results as the number of segments increases. The method is also generalizable to an arbitrary spin function, which can be approximated by linear segments. This applies to almost all practically realizable spin functions. The exact analytic expressions for segment-approximated spin function and approximate numerical calculation of the exact spin function should complement one another to enhance our understanding of the underlying physics (Tables 1 and 2).

Appendix

A. Exact Calculation for Segments

A.1. The Specific 3-Segment Solutions. The details about solutions for 3 segments follow.

Segment I $(0 \le s \le \pi/2)$. The equations are

$$\begin{bmatrix} A_{1s}^{(1)}(s) \\ A_{2s}^{(1)}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2i\theta_1(s)} \\ e^{-2i\theta_1(s)} & 0 \end{bmatrix} \begin{bmatrix} A_1^{(1)}(s) \\ A_2^{(1)}(s) \end{bmatrix}. \quad \text{(A.1)}$$

The boundary conditions are

$$\begin{aligned} \left[A_1^{(1)}(s=0)\right] &= 1, \\ \left[A_{1s}^{(1)}(s=0)\right] &= 0, \end{aligned} \quad \text{(A.2a)}$$

$$\begin{aligned} \left[A_2^{(1)}(s=0)\right] &= 0, \\ \left[A_{2s}^{(1)}(s=0)\right] &= ia. \end{aligned} \quad \text{(A.2b)}$$

Let

$$n = \left(\frac{\tilde{c}}{q}\right) = \left[1 + \left(\frac{\pi l_0}{4L_B}\right)\right]^{-1/2}, \quad \text{(A.3)}$$

and then the analytical solutions are similar to those given in Section 2.2. Consider

$$\begin{bmatrix} e^{-i\tilde{c}s} A_1^{(1)}(s) \\ \left(\frac{q}{a}\right) e^{i\tilde{c}s} A_2^{(1)}(s) \end{bmatrix} = \begin{bmatrix} 1 & -in \\ 0 & i \end{bmatrix} \begin{bmatrix} \cos qs \\ \sin qs \end{bmatrix}. \quad \text{(A.4)}$$

Comparison with general expression gives the following coefficients:

$$\begin{aligned} a_1^{(1)} &= 1, \\ b_1^{(1)} &= 0, \\ a_2^{(1)} &= 0, \\ b_2^{(1)} &= -n. \end{aligned} \quad \text{(A.5)}$$

For calculating PCF, the amplitudes have to be differentiated with respect to ω, which will be denoted by subscript ω. Some useful relations needed for this are

$$\begin{aligned} \frac{d}{d\omega}\left(\frac{a}{q}\right) &= n^2\left(\frac{a_\omega}{q}\right), \\ a_\omega &= \frac{da}{d\omega} = \frac{\gamma}{2\eta}, \quad \gamma = \frac{d(\Delta\beta)}{d\omega}, \\ n_\omega &= -n\left(\frac{a}{q}\right)\left(\frac{a_\omega}{q}\right), \\ q_\omega &= a\left(\frac{a_\omega}{q}\right). \end{aligned} \quad \text{(A.6)}$$

Then we can write

$$\begin{aligned} &\begin{bmatrix} \left(\frac{q}{a}\right) e^{-i\tilde{c}s} A_{1\omega}^{(1)}(s) \\ e^{i\tilde{c}s} A_{2\omega}^{(1)}(s) \end{bmatrix} \\ &\quad = \left(\frac{a_\omega}{q}\right) \begin{bmatrix} p_1^{(1)} + ip_2^{(1)} & p_3^{(1)} + ip_4^{(1)} \\ p_5^{(1)} + ip_6^{(1)} & p_7^{(1)} + ip_8^{(1)} \end{bmatrix} \begin{bmatrix} \cos qs \\ \sin qs \end{bmatrix}, \\ &p_1^{(1)} = 0, \\ &p_2^{(1)} = -nqs, \\ &p_3^{(1)} = -qs, \end{aligned}$$

$$
\begin{aligned}
p_4^{(1)} &= n,\\
p_5^{(1)} &= 0,\\
p_6^{(1)} &= \left(1-n^2\right) qs,\\
p_7^{(1)} &= 0,\\
p_8^{(1)} &= n^2.
\end{aligned}
\tag{A.7}
$$

Some interesting relations are found as

$$
\begin{aligned}
\Delta\beta &= \left(\frac{4\pi q}{\Lambda}\right)\sqrt{1-n^2},\\
z &= \left(\frac{\Lambda}{2\pi}\right)s,\\
\alpha_0 &= \left(\frac{2\pi^2 q^2}{\Lambda}\right) n\sqrt{1-n^2}.
\end{aligned}
\tag{A.8}
$$

Segment II ($\pi/2 \le s \le 3\pi/2$). The equations are

$$
\begin{bmatrix} A_{1s}^{(2)}(s) \\ A_{2s}^{(2)}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2i\theta_2(s)} \\ e^{-2i\theta_2(s)} & 0 \end{bmatrix} \begin{bmatrix} A_1^{(2)}(s) \\ A_2^{(2)}(s) \end{bmatrix}. \tag{A.9}
$$

The boundary conditions are

$$
\begin{aligned}
\left[A_1^{(1)}\left(s=\frac{\pi}{2}\right)\right] &= \left[A_1^{(2)}\left(s=\frac{\pi}{2}\right)\right],\\
\left[A_{1s}^{(1)}\left(s=\frac{\pi}{2}\right)\right] &= \left[A_{1s}^{(2)}\left(s=\frac{\pi}{2}\right)\right].
\end{aligned}
\tag{A.10}
$$

Similar expressions exist for $A_2^{(2)}(s)$. Using the chain-relations with $n = 2$, the analytical solutions are obtained:

$$
\begin{bmatrix} e^{-i(-\tilde{c}s+2c)} A_1^{(2)}(s) \\ \left(\frac{q}{a}\right) e^{i(-\tilde{c}s+2c)} A_2^{(2)}(s) \end{bmatrix} = \begin{bmatrix} 1-n^2+n^2\cos\pi q - in\sin\pi q & n\left(n\sin\pi q + i\cos\pi q\right) \\ -n\left(1-\cos\pi q\right) & n\sin\pi q + i \end{bmatrix} \begin{bmatrix} \cos qs \\ \sin qs \end{bmatrix}. \tag{A.11}
$$

The ω-differentiated amplitudes are found as

$$
\begin{aligned}
&\begin{bmatrix} \left(\frac{q}{a}\right) e^{-i(\tilde{c}s-2c)} A_{1\omega}^{(2)}(s) \\ e^{i(\tilde{c}s-2c)} A_{2\omega}^{(2)}(s) \end{bmatrix} = \left(\frac{a_\omega}{q}\right)\\
&\quad\cdot \begin{bmatrix} p_1^{(2)} + ip_2^{(2)} & p_3^{(2)} + ip_4^{(2)} \\ p_5^{(2)} + ip_6^{(2)} & p_7^{(2)} + ip_8^{(2)} \end{bmatrix} \begin{bmatrix} \cos qs \\ \sin qs \end{bmatrix},\\
&p_1^{(2)} = n^2\left\{2\left(1-\cos\pi q\right) - \pi q\sin\pi q + qs\sin\pi q\right\},\\
&p_2^{(2)} = n\left\{\sin\pi q - \pi q\cos\pi q + qs\cos\pi q\right\},\\
&p_3^{(2)} = n^2\left(-2\sin\pi q + \pi q\cos\pi q\right) - \left(1-n^2\right.\\
&\quad\left. + n^2\cos\pi q\right) qs,\\
&p_4^{(2)} = n\left\{-\left(\cos\pi q + \pi q\sin\pi q\right) + qs\sin\pi q\right\},\\
&p_5^{(2)} = n\left\{\left(1-2n^2\right)\left(1-\cos\pi q\right)\right.\\
&\quad\left. - \left(1-n^2\right)\pi q\sin\pi q + \left(1-n^2\right) qs\sin\pi q\right\},\\
&p_6^{(2)} = \left(1-n^2\right) qs,\\
&p_7^{(2)} = n\left\{-\left(1-2n^2\right)\sin\pi q + \left(1-n^2\right)\pi q\cos\pi q\right.\\
&\quad\left. + \left(1-n^2\right)\left(1-\cos\pi q\right) qs\right\},\\
&p_8^{(2)} = n^2.
\end{aligned}
\tag{A.12}
$$

Segment III ($3\pi/2 \le s \le 2\pi$). The equations are

$$
\begin{bmatrix} A_{1s}^{(3)}(s) \\ A_{2s}^{(3)}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2i\theta_3(s)} \\ e^{-2i\theta_3(s)} & 0 \end{bmatrix} \begin{bmatrix} A_1^{(3)}(s) \\ A_2^{(3)}(s) \end{bmatrix}. \tag{A.13}
$$

The boundary conditions are

$$
\begin{aligned}
\left[A_1^{(2)}\left(s=\frac{3\pi}{2}\right)\right] &= \left[A_1^{(3)}\left(s=\frac{3\pi}{2}\right)\right],\\
\left[A_{1s}^{(2)}\left(s=\frac{3\pi}{2}\right)\right] &= \left[A_{1s}^{(3)}\left(s=\frac{3\pi}{2}\right)\right].
\end{aligned}
\tag{A.14}
$$

Similar expressions exist for $A_2^{(3)}(s)$. Using the chain-relations with $n = 3$, the analytical solutions are obtained:

$$
\begin{aligned}
&\begin{bmatrix} e^{-i(\tilde{c}s-4c)} A_1^{(3)}(s) \\ \left(\frac{q}{a}\right) e^{i(\tilde{c}s-4c)} A_2^{(3)}(s) \end{bmatrix}\\
&= \begin{bmatrix} 1-n^2+n^2\cos\pi q + in\left\{n^2\sin 2\pi q + \left(1-n^2\right)\left(\sin 3\pi q - \sin\pi q\right)\right\} & n^2\sin 2\pi q - in\left\{n^2\cos 2\pi q + \left(1-n^2\right)\left(1+\cos 3\pi q - \cos\pi q\right)\right\} \\ \left[n\left(\cos\pi q - \cos 3\pi q\right) + in^2\left(\sin 3\pi q - \sin 2\pi q - \sin\pi q\right)\right] & \left[n\left(\sin\pi q - \sin 3\pi q\right) + i\left\{1-n^2+n^2\left(\cos\pi q + \cos 2\pi q - \cos 3\pi q\right)\right\}\right] \end{bmatrix} \begin{bmatrix} \cos qs \\ \sin qs \end{bmatrix}.
\end{aligned}
\tag{A.15}
$$

The ω-differentiated amplitudes are found as

$$\begin{bmatrix} \left(\frac{q}{a}\right) e^{-i(\tilde{c}s-4c)} A_{1\omega}{}^{(3)}(s) \\ e^{i(\tilde{c}s-4c)} A_{2\omega}{}^{(3)}(s) \end{bmatrix} = \left(\frac{a_\omega}{q}\right) \cdot \begin{bmatrix} p_1{}^{(3)} + ip_2{}^{(3)} & p_3{}^{(3)} + ip_4{}^{(3)} \\ p_5{}^{(3)} + ip_6{}^{(3)} & p_7{}^{(3)} + ip_8{}^{(3)} \end{bmatrix} \begin{bmatrix} \cos qs \\ \sin qs \end{bmatrix},$$

$$p_1{}^{(3)} = 2n^2 (1 - \cos 2\pi q - \pi q \sin 2\pi q) + n^2 \cdot qs \sin 2\pi q,$$

$$p_2{}^{(3)} = n\left[-3n^2 \sin 2\pi q - \left(1 - 3n^2\right)(\sin 3\pi q - \sin \pi q) + \pi q \left\{2n^2 \cos 2\pi q + \left(1 - n^2\right)(3\cos 3\pi q - \cos \pi q)\right\} - \left\{n^2 \cos 2\pi q + \left(1 - n^2\right)(1 - \cos \pi q + \cos 3\pi q)\right\} qs\right],$$

$$p_3{}^{(3)} = -2n^2 (\sin 2\pi q - \pi q \cos 2\pi q) - \left(1 - n^2 + n^2 \cdot \cos 2\pi q\right) qs,$$

$$p_4{}^{(3)} = n\left[3n^2 \cos 2\pi q + \left(1 - 3n^2\right)(1 - \cos \pi q + \cos 3\pi q) + \pi q \left\{2n^2 \sin 2\pi q + \left(1 - n^2\right)(3\sin 3\pi q - \sin \pi q)\right\} - \left\{n^2 \sin 2\pi q + \left(1 - n^2\right)(\sin 3\pi q - \sin \pi q)\right\} qs\right],$$

$$p_5{}^{(3)} = n\left(1 - 2n^2\right)(\cos 3\pi q - \cos \pi q) + n\left(1 - n^2\right) \cdot (\sin 3\pi q - \sin \pi q)\pi q + n\left(1 - n^2\right)(\sin \pi q - \sin 3\pi q) qs,$$

$$p_6{}^{(3)} = \left(1 - n^2\right) qs + n^2 \left[\left(2 - 3n^2\right)(\sin \pi q + \sin 2\pi q - \sin 3\pi q) + \left(1 - n^2\right)(3\cos 3\pi q - 2\cos 2\pi q - \cos \pi q)\pi q - \left(1 - n^2\right)(1 - \cos \pi q - \cos 2\pi q + \cos 3\pi q) qs\right],$$

$$p_7{}^{(3)} = n\left(1 - 2n^2\right)(\sin 3\pi q - \sin \pi q) + n\left(1 - n^2\right) \cdot (\cos \pi q - 3\cos 3\pi q)\pi q + n\left(1 - n^2\right)(\cos 3\pi q - \cos \pi q) qs,$$

$$p_8{}^{(3)} = n^2 + n^2 \left[\left(2 - 3n^2\right)(1 - \cos \pi q - \cos 2\pi q + \cos 3\pi q) + \left(1 - n^2\right)(3\sin 3\pi q - 2\sin 2\pi q - \sin \pi q)\pi q + \left(1 - n^2\right)(\sin \pi q + \sin 2\pi q - \sin 3\pi q) qs\right]. \tag{A.16}$$

B. Perturbative Calculation for Segments

The perturbative approach is based on the following assumptions:

(i) The coupling between the polarization states is so small that the equations become decoupled.

(ii) The top component is constant ($A_1{}^{(m)} = 1$, $m = 1, 2, 3$) and only the second component changes.

(iii) The boundary conditions remain unchanged.

Under these assumptions the dimensionless constant q becomes $\tilde{c}$, which is related to the physical lengths as

$$\tilde{c} = \frac{2}{\pi}\left(\frac{\Lambda}{l_0}\right). \tag{B.1}$$

The new equations and their solutions take the following form.

Segment I ($0 \le s \le \pi/2$). Perturbative equations are as follows:

$$\begin{bmatrix} A_{1s}{}^{(1)}(s) \\ A_{2s}{}^{(1)}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2i\tilde{c}s} \\ e^{-2i\tilde{c}s} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{B.2}$$

Solutions are as follows:

$$A_2{}^{(1)}(s) = \left(\frac{a}{\tilde{c}}\right) ie^{-i\tilde{c}s} \sin \tilde{c}s. \tag{B.3}$$

The sum of squares of the ω-differentiated amplitudes is as follows:

$$\left(\frac{\left|A_{1\omega}{}^{(1)}(s)\right|^2 + \left|A_{2\omega}{}^{(1)}(s)\right|^2}{\left(a_\omega/\tilde{c}\right)^2}\right)_{\text{pert}} = \frac{1}{2}(1 - \cos 2\tilde{c}s) = \sin^2 \tilde{c}s. \tag{B.4}$$

So

$$\text{PCF}^{(1)}(s)_{\text{pert}} = \left[\frac{\left(\left|A_{1\omega}{}^{(1)}(s)\right|^2 + \left|A_{2\omega}{}^{(1)}(s)\right|^2\right)_{\text{pert}}}{\left[\left|A_{1\omega}(s)\right|^2 + \left|A_{2\omega}(s)\right|^2\right]_{\text{unspun-fiber}}}\right]^{1/2} = \frac{\sin \tilde{c}s}{\tilde{c}s}. \tag{B.5}$$

Segment II ($\pi/2 \le s \le 3\pi/2$). Perturbative equations are as follows:

$$\begin{bmatrix} A_{1s}^{(2)}(s) \\ A_{2s}^{(2)}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2i(-\tilde{c}s+2c)} \\ e^{-2i(-\tilde{c}s+2c)} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad \text{(B.6)}$$

Solutions are as follows:

$$A_2^{(2)}(s) = e^{i(\tilde{c}s-2c)} \left(\frac{a}{\tilde{c}}\right) \cdot \left[-(1-\cos 2c)\cos\tilde{c}s + (\sin 2c + i)\sin\tilde{c}s\right]. \quad \text{(B.7)}$$

The sum of squares of the ω-differentiated amplitudes is as follows:

$$\left(\left|A_{1\omega}^{(2)}(s)\right|^2 + \left|A_{2\omega}^{(2)}(s)\right|^2\right)_{\text{pert}} = \frac{1}{2}\left(\frac{a_\omega}{\tilde{c}}\right)^2 \left\{(3-2\cos 2c) + (\cos 4c - 2\cos 2c)\cos 2\tilde{c}s + (\sin 4c - 2\sin 2c)\sin 2\tilde{c}s\right\}. \quad \text{(B.8)}$$

Expression for PCF is obtained as before.

Segment III ($3\pi/2 \le s \le 2\pi$). Perturbative equations are as follows:

$$\begin{bmatrix} A_{1s}^{(3)}(s) \\ A_{2s}^{(3)}(s) \end{bmatrix} = ia \begin{bmatrix} 0 & e^{2i(\tilde{c}s-4c)} \\ e^{-2i(\tilde{c}s-4c)} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad \text{(B.9)}$$

Solutions are as follows:

$$\begin{aligned} A_2^{(3)} = {} & e^{i(-\tilde{c}s+4c)} \left(\frac{a}{\tilde{c}}\right) \\ & \cdot \left[\{(-1+\cos 2c + \cos 4c - \cos 6c) \right. \\ & + i(-\sin 2c - \sin 4c + \sin 6c)\}\cos\tilde{c}s \\ & + \{(\sin 2c + \sin 4c - \sin 6c) \\ & \left. + i(1+\cos 2c + \cos 4c - \cos 6c)\}\sin\tilde{c}s\right]. \end{aligned} \quad \text{(B.10)}$$

The sum of squares of the ω-differentiated amplitudes is as follows:

$$\begin{aligned} & \left(\left|A_{1\omega}^{(3)}(s)\right|^2 + \left|A_{2\omega}^{(3)}(s)\right|^2\right)_{\text{pert}} \\ & = \frac{1}{2}\left(\frac{a_\omega}{\tilde{c}}\right)^2 \left\{(5-4\cos 4c) + (2\cos 10c - \cos 8c - 2\cos 6c)\cos 2\tilde{c}s + (2\sin 10c - \sin 8c - 2\sin 6c)\sin 2\tilde{c}s\right\}. \end{aligned} \quad \text{(B.11)}$$

The PCF can be calculated as before.

Competing Interests

The author declares that he has no competing interests.

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