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Using the Modal Method in Rotor Dynamic Systems

Eduard Malenovský
Brno University of Technology, Faculty of Mechanical Engineering, Brno, Czech Republic

This article deals with computational modeling of nonlinear rotor dynamic systems. The theoretical basis of the method of dynamic compliances and the modal method, supplemented by the method of trigonometric collocation, are presented. The main analysis is focused on the solutions of the eigenvalue problem and steady-state and transient responses. The algorithms for solving this range of problems are presented. The finite element method, the method of dynamic compliances, and the modal method are supplemented by the trigonometric collocation method. The theoretical analysis is supplemented by the solution of a model task, which is focused on the application of the trigonometric collocation method. The solution of a technical application, which is a pump, is presented in this article.

Keywords Eigenvalue problem, Method of dynamic compliances, Method of trigonometric collocation, Modal method, Rotor dynamic systems, Steady-state response, Transient response

Coupled mechanical systems are common in many technical applications. They may have the form of several in-line one-rotor units in series (e.g., engine gearbox-driven mechanism). Using the system of coaxial rotors is well known in jet engines such as those that use one or two coaxial shafts (Kamenický et al., 2000). The systems with one shaft and an elastic nonrotating stator part are analyzed in this article.

The nonlinear couplings between rotating and nonrotating parts, together with the elastic stator part, may have a dominant influence on the behavior of the whole dynamic system. One of the ways of including nonlinear couplings in the dynamic system is to include them on the right side of the equation of motion, such as nonlinear forces. Another way is to include them on the left side of the equation of motion, such as additional mass, damping, and stiffness tensors. This possibility is more suitable when applying the modal method, which is the main focus of this article. One of the methods of determining the additional effects is by using the Taylor series. In this case the members of additional matrixes are determined by using partial derivation (El-Shafei, 1995). The second possible method is to separate the nonlinear couplings and the shaft parts from each other and determine the additional tensors of additional effects (Malenovský and Pochylý, 2001).

A variety of computational methods can be used for the analysis of dynamic behavior of rotor dynamic systems. The finite element method (FEM) (Krämer, 1993; Nelson and McVaugh, 1976; Zorzi and Nelson, 1997), is used most commonly, but its use for complicated systems leads to a higher order of global matrixes, which complicates the solution. It is possible to use many methods to decrease its order, for example, modal reduction of the problem (Dupal, 1998; Ehrich, 1999; Zeman, 2002).

It is also possible to use modal reduction for determining the steady-state response and for determining the forced response. The approach of using mode synthesis for determining the forced response is described in the work of Nelson and colleagues (1983). In all previously presented reductions it is very difficult to set a suitable number of modes, which are necessary to include in the solution. Choosing a higher number of modes leads to a higher order of matrixes, and the reduction is not as efficient. The modal method presented in this article is based on a reduction in frequency domain, so choosing a higher number of modes does not lead to a higher order of matrixes. The reduction in frequency range leads to using the method of dynamic compliances (MDC) or the modal method (MM).

Part of this article deals with the application of the method of trigonometric collocation (MTC). Nataraj and Nelson (1989) presented a general approach to this method. It describes the use of modal reduction to decrease the matrix order. The nonlinear support subsystems, such as forces and Lagrange multipliers, are included in the rotor dynamic system. This approach is one of the ways of including the support conditions. The knowledge

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Address correspondence to Eduard Malenovský, Brno University of Technology, Faculty of Mechanical Engineering, Technická 2, 61669 Brno, Czech Republic. E-mail: malenov@umtn.fme.vutbr.cz

of the tensors of nonlinear supports presented in this article is preferred. Using the MTC generally leads to a higher order of matrixes, especially when including a higher set of multiples. That is the general disadvantage of this method. Jean and Nelson (1990) presented a special approach to order reduction. Also, in this case the order of matrixes depends directly on the set of multiples in the Fourier series. The approach presented in this article, which is based on the reduction in frequency domain, doesn't increase the order of matrixes as rapidly.

The complex approach to the analysis of complicated nonlinear rotor dynamic systems is understood as the synthesis of

- The elastic mechanical shaft part,
- The elastic mechanical stator part, and
- The nonlinear parts between the rotating and nonrotating parts.

The basic dynamic properties that are determined by computational modeling using the MDC and the MM are the

- Eigenvalue problem
- Campbell diagram (which includes the stability problem)
- Steady-state response
- Transient response
- Cascade diagram.

It is possible to use a lot of excitations, including kinematics, for solving the transient response.

Computational modeling uses a direct shaft element with 4 dof in the node (two displacements and two rotations in plane perpendicular to the theoretical shaft axis). The local mass, stiffness, and gyroscopic effect matrixes from Nelson and McVaugh (1976) and Zorzi and Nelson (1997) were used; discrete elements from Krämer (1993) or Gasch and Pfützner (1980) were added.

The theoretical basis of the MDC for solving the frequency modal behavior and steady-state response is presented in, for example, Malenovský (1999). The main advantage of this method is the possibility of including in the solution the elastic stator part and data from experimental analysis. It also makes it possible to reduce the order of the task.

It is necessary to know the frequency of the modal behavior of the *separated* stator part (without shaft and couplings but with real couplings to the frame). It is possible to determine this behavior by using a suitable program system or an experimental method; we often use experimental modal analysis for its determination. The results are used for the composition of the matrix of dynamic compliances of the stator part. The number of couplings between the rotor and stator parts gives the minimum order of this matrix.

Figure 1 shows a generic scheme of the rotor system with one shaft, which includes the couplings between both parts. Both the rotor and the stator can be forced.

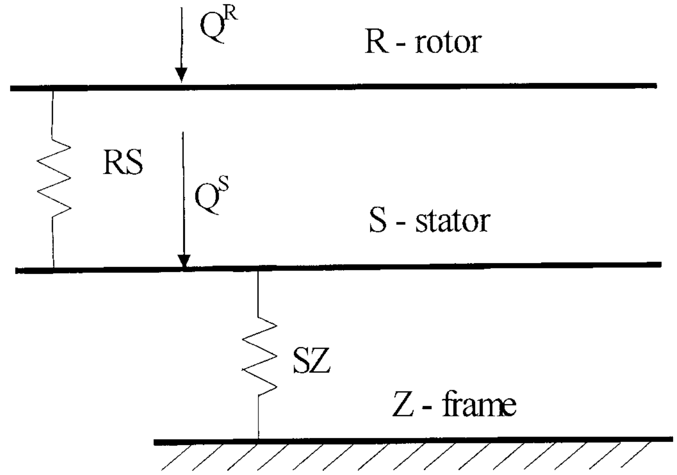

FIGURE 1
Scheme of the rotor system. Q^R, external excitation impact the shaft; Q^S, external excitation impact the stator; RS, couplings between rotor and stator; SZ, couplings between stator and frame.

SOLUTION OF THE STEADY-STATE RESPONSE

The equation of motion of linear rotor systems and harmonic excitation has the form

$$\mathbf{M}^R\ddot{\mathbf{q}} + \mathbf{B}^R(\omega)\dot{\mathbf{q}} + \mathbf{K}^R\mathbf{q} = \mathbf{Q}_0 e^{i\omega t} \qquad [1]$$

For the amplitude of response it is possible to write

$$\mathbf{q}_0 = \sum_{i=1}^{2n} \frac{\mathbf{v}_i^R \mathbf{w}_i^{R^T}}{i\omega - \lambda_i}\mathbf{Q}_0 = \mathbf{G}^R\mathbf{Q}_0 \qquad [2]$$

By partitioning the matrix of dynamic compliance into submatrixes whose orders are given by the numbers of nodes with and without couplings, the following is obtained:

$$\begin{bmatrix} \mathbf{q}^R \\ \mathbf{q}^{RS} \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{11}^R & \mathbf{G}_{12}^R \\ \mathbf{G}_{21}^R & \mathbf{G}_{22}^R \end{bmatrix} \begin{bmatrix} \mathbf{Q}^R \\ -\mathbf{Q}^C \end{bmatrix} \qquad [3]$$

If it is assumed that the responses of a *free* rotor ($\mathbf{Q}^C = \mathbf{0}$) are solved, then it is possible to write

$$\begin{aligned} \mathbf{q}_f^R &= \mathbf{G}_{11}^R\mathbf{Q}^R \\ \mathbf{q}_f^{RS} &= \mathbf{G}_{21}^R\mathbf{Q}^R \end{aligned} \qquad [4]$$

Then it is possible to rewrite Equation (3) in the form

$$\begin{aligned} \mathbf{q}^R &= \mathbf{q}_f^R - \mathbf{G}_{12}^R\mathbf{Q}^C \\ \mathbf{q}^{RS} &= \mathbf{q}_f^{RS} - \mathbf{G}_{22}^R\mathbf{Q}^C \end{aligned} \qquad [5]$$

where $\mathbf{G}_{ij}^R$ are the matrixes of dynamic compliance of the free rotor part.

If the left- and right-side eigenvectors of the stator are assumed to be the same, the matrix of dynamic compliance has the form

$$\mathbf{G}^S = \sum_{i=1}^{2n} \frac{\mathbf{v}_i^S \mathbf{v}_i^{S^T}}{i\omega - \lambda_i} \quad [6]$$

Because the stator can also be forced, the response of the stator in positions with connection to the rotor can be expressed as

$$\mathbf{q}^{SR} = \mathbf{q}_f^{SR} + \mathbf{G}^S \mathbf{Q}^C \quad [7]$$

Using the equilibrium of forces and displacement continuity for coupling elements, it is possible to write

$$\mathbf{q}^{RS} - \mathbf{q}^{SR} = \mathbf{G}^C \mathbf{Q}^C \quad [8]$$

where $\mathbf{G}^C$ is the matrix of dynamic compliance of nonlinear couplings. One of the possibilities is to write this matrix as the sum of mass, damping, and stiffness tensors, which has the form

$$\mathbf{G}^C = (-\omega^2 \mathbf{M}^C + \mathrm{i}\omega \mathbf{B}^C + \mathbf{K}^C)^{-1} \quad [9]$$

where the matrixes $\mathbf{M}^C$, $\mathbf{B}^C$, and $\mathbf{K}^C$ are the nonlinear functions of response ($\mathbf{q}^{RS} - \mathbf{q}^{SR}$). Equation (9) can be determined using the Taylor series of nonlinear coupling forces and is valid for only one harmonic part. The general form of the solution for the steady-state response, which is obtained from Equations (5), (7), and (8) is

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{12}^R \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{G}_{22}^R \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{G}^S \\ \mathbf{0} & \mathbf{I} & -\mathbf{I} & -\mathbf{G}^C \end{bmatrix} \begin{bmatrix} \mathbf{q}^R \\ \mathbf{q}^{RS} \\ \mathbf{q}^{SR} \\ \mathbf{Q}^C \end{bmatrix} = \begin{bmatrix} \mathbf{q}_f^R \\ \mathbf{q}_f^{RS} \\ \mathbf{q}_f^{SR} \\ \mathbf{0} \end{bmatrix} \quad [10]$$

or, in shorter form,

$$\mathbf{Ax} = \mathbf{b} \quad [11]$$

where $\mathbf{A}$ in Equation (11) is set together by the matrices of dynamic compliances for the rotor, stator, nonlinear couplings, units, and zero matrices. The right-hand side of this equality represents the responses of the free rotor and stator. Because of the nonlinear dynamic compliance of the couplings, Equation (11) is nonlinear, and it is possible to use the Aitken or Newton-Raphson method to attain the solution (as tested by the author).

THE EIGENVALUE PROBLEM

Unknown generalized coupling forces excite only a free rotor. Using the force equilibrium and displacement continuity for responses of the rotor and stator, it is possible to write

$$\begin{aligned} \mathbf{q}^{RS} &= -\mathbf{G}^R \mathbf{Q}^C \\ \mathbf{q}^{SR} &= \mathbf{G}^S \mathbf{Q}^C \\ \mathbf{q}^{RS} - \mathbf{q}^{SR} &= \mathbf{G}^C \mathbf{Q}^C \end{aligned} \quad [12]$$

The number of couplings between the rotor and stator parts gives the order of the matrixes. From Equation (12), it follows that

$$(\mathbf{G}^R + \mathbf{G}^S + \mathbf{G}^C)\mathbf{Q}^C = \mathbf{O} \quad [13]$$

or, in simple form,

$$\mathbf{Ax} = 0 \quad [14]$$

For a nontrivial solution of Equation (14), the complex value of the determinant of matrix $\mathbf{A}$ must be zero. It is necessary to note that it is difficult to talk about eigenvalues for a nonlinear system.

SOLUTION OF TRANSIENT RESPONSE

The equation to describe the motion of linear rotor systems (especially the free shaft part for future analysis) has the form

$$\mathbf{M}^R \ddot{\mathbf{q}} + \mathbf{B}^R(\omega)\dot{\mathbf{q}} + \mathbf{K}^R \mathbf{q} = \mathbf{Q}(t) \quad [15]$$

The solution of this equation in time step t, including initial conditions, may be written in matrix form (Yang, 1996)

$$\begin{aligned} \mathbf{q} = \sum_{i=1}^{2n} e^{\lambda_i t} \mathbf{v}_i^R \mathbf{w}_i^{R^T} \Big[\int_0^t \mathbf{Q}(\tau) e^{-\lambda_i \tau} \mathrm{d}\tau + \mathbf{M}^R \dot{\mathbf{q}}(0) \\ + \mathbf{B}^R \mathbf{q}(0) + \lambda_i \mathbf{M}^R \mathbf{q}(0) \Big] \end{aligned} \quad [16]$$

or in shorter form

$$\begin{aligned} \begin{bmatrix} \mathbf{q}^R \\ \mathbf{q}^{RS} \end{bmatrix} = \sum_{i=1}^{2n} e^{\lambda_i t} \begin{bmatrix} \mathbf{U}_{11}^R & \mathbf{U}_{12}^R \\ \mathbf{U}_{21}^R & \mathbf{U}_{22}^R \end{bmatrix}_i \Bigg\{ \int_0^t \begin{bmatrix} \mathbf{Q}^R(\tau) \\ -\mathbf{Q}^C(\tau) \end{bmatrix} e^{-\lambda_i \tau} \mathrm{d}\tau \\ + \begin{bmatrix} \mathbf{a}^R \\ \mathbf{a}^{RS} \end{bmatrix} + \lambda_i \begin{bmatrix} \mathbf{b}^R \\ \mathbf{b}^{RS} \end{bmatrix} \Bigg\} \end{aligned} \quad [17]$$

where $\mathbf{U}_{ij}^R$ are the submatrixes obtained using the dyadic multiplication of the right- and left-side eigenvectors, and $\mathbf{a}^R$, $\mathbf{a}^{RS}$, $\mathbf{b}^R$, and $\mathbf{b}^{RS}$ are the matrixes with the initial conditions. If a free rotor is assumed, then it is possible to write Equation (17) in the form

$$\begin{aligned} \mathbf{q}_f^R &= \sum_{i=1}^{2n} e^{\lambda_i t} \Big[\mathbf{U}_{11i}^R \int_0^t \mathbf{Q}^R(\tau) e^{-\lambda_i \tau} \mathrm{d}\tau + \mathbf{U}_{11i}^R \mathbf{a}^R \\ &\quad + \mathbf{U}_{12i}^R \mathbf{a}^{RS} + \lambda_i \mathbf{U}_{11i}^R \mathbf{b}^R + \lambda_i \mathbf{U}_{12i}^R \mathbf{b}^{RS} \Big] \\ \mathbf{q}_f^{RS} &= \sum_{i=1}^{2n} e^{\lambda_i t} \Big[\mathbf{U}_{21i}^R \int_0^t \mathbf{Q}^R(\tau) e^{-\lambda_i \tau} \mathrm{d}\tau + \mathbf{U}_{21i}^R \mathbf{a}^R \\ &\quad + \mathbf{U}_{22i}^R \mathbf{a}^{RS} + \lambda_i \mathbf{U}_{21i}^R \mathbf{b}^R + \lambda_i \mathbf{U}_{22i}^R \mathbf{b}^{RS} \Big] \end{aligned} \quad [18]$$

After substituting Equation (18) into Equation (17), the following is obtained:

$$\begin{aligned}\mathbf{q}^R &= \mathbf{q}_f^R - \sum_{i=1}^{2n} \mathbf{U}_{12i}^R \int_0^t \mathbf{Q}^C(\tau) e^{\lambda_i(t-\tau)} \mathrm{d}\tau \\ \mathbf{q}^{RS} &= \mathbf{q}_f^{RS} - \sum_{i=1}^{2n} \mathbf{U}_{22i}^R \int_0^t \mathbf{Q}^C(\tau) e^{\lambda_i(t-\tau)} \mathrm{d}\tau\end{aligned} \tag{19}$$

The coupling forces are the unknown values in the convolutory integral. It is assumed that these forces are constant during the time step. In time j, the convolution integral for the first equation above has the form

$$\begin{aligned}&\sum_{i=1}^{2n} \mathbf{U}_{12i}^R \int_0^{t_j} \mathbf{Q}^C(\tau) e^{\lambda_i(t_j-\tau)} \mathrm{d}\tau \\ &\quad = \mathbf{G}_{12}^R \mathbf{Q}_j^C + \sum_{i=1}^{2n} \mathbf{G}_{12i}^R \sum_{k=1}^{j-1} \mathbf{Q}_k^C e^{\lambda_i(t_j-t_k)}\end{aligned} \tag{20}$$

where the force subscripts mark the calculation time $t_k = k\Delta t$, $k = 1$ to j. Matrix $\mathbf{G}^R$ is determined from modal parameters, such as eigenvalues and eigenvectors (from which arises the name of this method). For example, the submatrix $\mathbf{G}_{12}^R$ is given by the expressions

$$\mathbf{G}_{12}^R = \sum_{i=1}^{2n} \frac{\mathbf{U}_{12i}^R}{\lambda_i} (e^{\lambda_i \Delta t} - 1), \quad \mathbf{G}_{12i}^R = \frac{\mathbf{U}_{12i}^R}{\lambda_i} (e^{\lambda_i \Delta t} - 1) \tag{21}$$

It is necessary to solve the convolution integral numerically in Equation (20), where increasing the time steps causes an increase in the calculation time. The recurrent formula for its calculation will be presented later. The arbitrary convolutory integral in time t_j can be rewritten as

$$\mathbf{I}_j = \sum_{i=1}^{2n} \mathbf{G}_i^R \sum_{k=1}^{j-1} \mathbf{Q}_k^C e^{\lambda_i(t_j-t_k)} + \sum_{i=1}^{2n} \mathbf{G}_i^R \mathbf{Q}_j^C = \mathbf{q}_c + \mathbf{G}^R \mathbf{Q}_j^C \tag{22}$$

After arrangement, Equation (22) has the form

$$\mathbf{I}_j = \sum_{i=1}^{2n} \mathbf{G}_i^R \left(\sum_{k=1}^{j-1} \mathbf{Q}_k^C e^{\lambda_i(t_j-t_k)} + \mathbf{Q}_j^C \right) = \sum_{i=1}^{2n} \mathbf{G}_i^R \mathbf{J}_j \tag{23}$$

where $\mathbf{J}_j$ marks the vector of the convolutory integral in step j. In the next time step, $j+1$, it takes the form of

$$\begin{aligned}&\sum_{k=1}^{j} \mathbf{Q}_k^C e^{\lambda_i(t_{j+1}-t_k)} + \mathbf{Q}_{j+1}^C \\ &\quad = e^{\lambda_i \Delta t} \left(\sum_{k=1}^{j-1} \mathbf{Q}_k^C e^{\lambda_i(t_j-t_k)} + \mathbf{Q}_j^C \right) + \mathbf{Q}_{j+1}^C\end{aligned} \tag{24}$$

which is the recurrent formula and in simple form is

$$\mathbf{J}_{j+1} = e^{\lambda_i \Delta t} \mathbf{J}_j + \mathbf{Q}_{j+1}^C \tag{25}$$

Then, Equation (19) it is possible to rewrite to the form

$$\begin{aligned}\mathbf{q}^R &= \mathbf{q}_f^R - \mathbf{q}_c^R - \mathbf{G}_{12}^R \mathbf{Q}^C \\ \mathbf{q}^{RS} &= \mathbf{q}_f^{RS} - \mathbf{q}_c^{RS} - \mathbf{G}_{22}^R \mathbf{Q}^C\end{aligned} \tag{26}$$

where the unknown values are the responses on the rotor $\mathbf{q}^R$, $\mathbf{q}^{RS}$ and coupling forces $\mathbf{Q}^C$.

It is possible to write the response on a separated stator in places connected to the rotor as follows:

$$\mathbf{q}^{SR} = \mathbf{q}_f^{SR} + \mathbf{q}_c^{SR} + \mathbf{G}^S \mathbf{Q}^C \tag{27}$$

where $\mathbf{G}^S$ is the modal matrix of the separated stator.

The nonlinear couplings between the rotor and stator generally depend on kinematics values. One of the ways of expressing nonlinear force equilibrium at coupling points leads to

$$\begin{aligned}\mathbf{Q}^C &= \mathbf{K}^C(\mathbf{q}^{RS} - \mathbf{q}^{SR}) + \mathbf{B}^C(\mathbf{q}^{RS\cdot} - \mathbf{q}^{SR\cdot}) \\ &\quad + \mathbf{M}^C(\mathbf{q}^{RS\cdot\cdot} - \mathbf{q}^{SR\cdot\cdot})\end{aligned} \tag{28}$$

Matrixes $\mathbf{K}^C$, $\mathbf{B}^C$, and $\mathbf{M}^C$ are generally dependent on the displacement and velocities. Using the differential method, it is possible to write the following for velocity and acceleration at time step j

$$\mathbf{q}_j^{\cdot} = \frac{\mathbf{q}_j - \mathbf{q}_{j-1}}{\Delta t}, \quad \mathbf{q}_j^{\cdot\cdot} = \frac{\mathbf{q}_j - 2\mathbf{q}_{j-1} + \mathbf{q}_{j-2}}{\Delta t^2} \tag{29}$$

After substituting Equation (29) into Equation (28) and rearranging terms, the following is obtained:

$$\mathbf{q}^{RS} - \mathbf{q}^{SR} - \mathbf{G}^C \mathbf{Q}^C = \mathbf{q}_c^{RS} \tag{30}$$

where

$$\begin{aligned}\mathbf{q}_c^{RS} = \mathbf{G}^C \Bigg[&\left(\frac{\mathbf{B}^C}{\Delta t} + \frac{2\mathbf{M}^C}{\Delta t^2} \right) \left(\mathbf{q}_{j-1}^{RS} - \mathbf{q}_{j-1}^{SR} \right) \\ &- \frac{\mathbf{M}^C}{\Delta t^2} \left(\mathbf{q}_{j-2}^{RS} - \mathbf{q}_{j-2}^{SR} \right) \Bigg]\end{aligned} \tag{31}$$

and

$$\mathbf{G}^C = \left(\mathbf{K}^C + \frac{\mathbf{B}^C}{\Delta t} + \frac{\mathbf{M}^C}{\Delta t^2} \right)^{-1} \tag{32}$$

Rewriting Equations (26), (27), and (30) into matrix form leads to

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{G}_{12}^R \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{G}_{22}^R \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{G}^S \\ \mathbf{0} & \mathbf{I} & -\mathbf{I} & -\mathbf{G}^C \end{bmatrix} \begin{bmatrix} \mathbf{q}^R \\ \mathbf{q}^{RS} \\ \mathbf{q}^{SR} \\ \mathbf{Q}^C \end{bmatrix} = \begin{bmatrix} \mathbf{q}_f^R - \mathbf{q}_c^R \\ \mathbf{q}_f^{RS} - \mathbf{q}_c^{RS} \\ \mathbf{q}_f^{SR} + \mathbf{q}_c^{SR} \\ \mathbf{q}_c^C \end{bmatrix} \tag{33}$$

This equation is nonlinear, and it is possible to use the Aitken or Newton-Raphson method for the solution (as tested by the author).

METHOD OF TRIGONOMETRIC COLLOCATION

For solving the steady-state response, it is possible to assume the excitation and response as multiples of the rotor speed, especially for the nonlinear system in which the acquired response can be subharmonic, ultraharmonic, or subultraharmonic multiples of the excitation frequency.

The nonlinear function can be included on the right or left side of the equation of motion. Especially in cases in which the tensors of additional effects such as mass, damping, or stiffness are known, it is more suitable to put them on the left side of the equation of motion. This approach is more suitable when using the MDC or MM, where it is necessary to put together the matrixes of dynamic compliances or modal matrixes.

Combining the MTC and FEM

It is possible to describe the excitation in the real or complex domain. Analysis in the real domain is presented here. A periodic excitation with known multiples is assumed, and the shaft vibrates around the known static equilibrium position. It is possible to assume the response as a set ν^q (superscript q means that the value concerns a response) of multiples ν_j^q in the form

$$\nu^q = \{\nu_j^q\} = \left\{1, 2, 3, \ldots, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{4}{3}, \frac{5}{3}, \ldots, \frac{k}{l}, \ldots, p\right\} \qquad [34]$$

where $j = 1$ to n. The arithmetic rule for determining k and l depends on the type of nonlinear couplings and spectra frequencies of excitation. The excitation force can be written in the following form (superscript Q means that the value concerns excitation):

$$\mathbf{Q}(t) = \sum_{j=1}^{m} \left[\mathbf{Q}_{s_j} \sin\left(\nu_j^Q \omega t\right) + \mathbf{Q}_{c_j} \cos\left(\nu_j^Q \omega t\right)\right] \qquad [35]$$

where the subscripts s and c mean sine and cosine parts. It is assumed that the response is in the same form as the excitation:

$$\mathbf{q} = \mathbf{q}(t) = \sum_{j=1}^{n} \left[\mathbf{q}_{s_j} \sin\left(\nu_j^q \omega t\right) + \mathbf{q}_{c_j} \cos\left(\nu_j^q \omega t\right)\right] \qquad [36]$$

The choice of the number and type of multiples depends on the type of nonlinearity of the coupling elements. One of the possibilities is to write the nonlinear equation of motion with respect to the application of the MTC in the following form:

$$\mathbf{M}(\mathbf{q}, \dot{\mathbf{q}})\ddot{\mathbf{q}} + \mathbf{B}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{K}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{q} = \mathbf{Q}(t) \qquad [37]$$

From Equation (36), the velocity and acceleration can be obtained as

$$\begin{aligned} \dot{\mathbf{q}} &= \sum_{j=1}^{n} \nu_j \omega\left[\mathbf{q}_{s_j} \cos\left(\nu_j^q \omega t\right) - \mathbf{q}_{c_j} \sin\left(\nu_j^q \omega t\right)\right] \\ \ddot{\mathbf{q}} &= \sum_{j=1}^{n} \nu_j^2 \omega^2\left[-\mathbf{q}_{s_j} \sin\left(\nu_j^q \omega t\right) - \mathbf{q}_{c_j} \cos\left(\nu_j^q \omega t\right)\right] \end{aligned} \qquad [38]$$

Substituting Equations (35), (36), and (38) into Equation (37) yields

$$\begin{aligned} &\sum_{j=1}^{n} \left[-\mathbf{M}\nu_j^{q^2} \omega^2 \sin\left(\nu_j^q \omega t\right) + \mathbf{B}\nu_j^q \omega \cos\left(\nu_j^q \omega t\right)\right. \\ &\quad \left.+ \mathbf{K} \sin\left(\nu_j^q \omega t\right)\right]\mathbf{q}_{s_j} + \sum_{j=1}^{n} \left[-\mathbf{M}\nu_j^{q^2} \omega^2 \cos\left(\nu_j^q \omega t\right)\right. \\ &\quad \left.- \mathbf{B}\nu_j^q \omega \sin\left(\nu_j^q \omega t\right) + \mathbf{K} \cos\left(\nu_j^q \omega t\right)\right]\mathbf{q}_{c_j} \\ &\quad = \sum_{j=1}^{m} \left[\mathbf{Q}_{s_j} \sin\left(\nu_j^Q \omega t\right) + \mathbf{Q}_{c_j} \cos\left(\nu_j^Q \omega t\right)\right] \end{aligned} \qquad [39]$$

where the nonlinear dependence of mass, damping, and stiffness matrices on the shaft center position and velocity are not explicitly written.

Because there are $2n$ unknowns in Equation (39), it must be written for at least $2n+1$ collocation times (it is necessary, taking into account the static equilibrium position). For a given time and multiple of time, the goniometric functions have constant value. By dividing the largest period of the response spectrum into a finite number of points, the collocation times are obtained. Then the equation of motion for the kth time step has the form

$$\sum_{j=1}^{n} \mathbf{C}_{k_j}(\mathbf{q}_k, \dot{\mathbf{q}}_k)\mathbf{q}_j = \sum_{j=1}^{m} \mathbf{Q}_{k_j} \qquad [40]$$

where the response $\mathbf{q}_k$ and velocity $\dot{\mathbf{q}}_k$ are the corresponding values in time t_k. In the above equation, the matrix $\mathbf{C}_{k_j}(\mathbf{q}_k, \dot{\mathbf{q}}_k)$ has the form

$$\begin{aligned} &\mathbf{C}_{k_j}(\mathbf{q}_k, \dot{\mathbf{q}}_k) \\ &\quad = -\mathbf{M}(\mathbf{q}_k, \dot{\mathbf{q}}_k)\nu_j^{q^2} \omega^2 \sin\left(\nu_j^q \omega t\right) + \mathbf{B}(\mathbf{q}_k, \dot{\mathbf{q}}_k)\nu_j^q \omega \cos\left(\nu_j^q \omega t\right) \\ &\quad + \mathbf{K}(\mathbf{q}_k, \dot{\mathbf{q}}_k) \sin\left(\nu_j^q \omega t\right) - \mathbf{M}(\mathbf{q}_k, \dot{\mathbf{q}}_k)\nu_j^{q^2} \omega^2 \cos\left(\nu_j^q \omega t\right) \\ &\quad - \mathbf{B}(\mathbf{q}_k, \dot{\mathbf{q}}_k)\nu_j^q \omega \sin\left(\nu_j \omega t\right) + \mathbf{K}(\mathbf{q}_k, \dot{\mathbf{q}}_k) \cos\left(\nu_j^q \omega t\right) \end{aligned} \qquad [41]$$

It is possible to use Equation (40) for a suitable number of collocation times to obtain a system of nonlinear algebraic equations.

It is possible to use, for describing nonlinear functions, the same expressions for the solution of the transient response. For a given collocation time, the shaft position and velocity are calculated and then the mass, damping, and stiffness tensors of the nonlinear couplings. Being able to use the time-dependent nonlinear functions is a great advantage of this method. It is possible because the nonlinear tensors are set for a specific time t_k and specific values of shaft position $\mathbf{q}_k$ and velocity $\dot{\mathbf{q}}_k$. They are more general when compared to expressions for solving the steady-state response, where the nonlinear functions depend only on amplitude. The large order of the resulting matrix, especially for a large number of multiples, is a disadvantage of this method. Another problem is with matrix conditionality.

Combining the MTC and MDC

The analysis will be made in the complex domain with respect to the application of the MDC. It is assumed that the free rotor and the separated stator systems are linear and that the coupling elements are nonlinear. It is also assumed that the shaft vibrates around the known static equilibrium position. Thus, the excitation and response can be given in the form

$$\begin{aligned} \mathbf{Q}(t) &= \sum_{j=1}^{m} \left(\mathbf{Q}_{j_{FP}} e^{\mathrm{i}\nu_j^Q \omega t} + \mathbf{Q}_{j_{BP}} e^{-\mathrm{i}\nu_j^Q \omega t} \right) \\ \mathbf{q}(t) &= \sum_{j=1}^{n} \left(\mathbf{q}_{j_{FP}} e^{\mathrm{i}\nu_j^q \omega t} + \mathbf{q}_{j_{BP}} e^{-\mathrm{i}\nu_j^q \omega t} \right) \end{aligned} \tag{42}$$

It is assumed that the left- and right-hand eigenvectors are the same for the stator. The dynamic compliance for n shapes of vibration and the jth part of the set of multiples of excitation frequency ω are given by

$$\mathbf{G}_j^S = \sum_{i=1}^{2n} \frac{\mathbf{v}_i^S \mathbf{v}_i^{S^T}}{i\nu_j^q \omega - \lambda_i} \tag{43}$$

The dynamic compliance of the free rotor system for the jth part of the set of the frequency multiples of excitation frequency and forward precession are given by

$$\mathbf{G}_j^R = \sum_{i=1}^{2n} \frac{\mathbf{v}_i^R \mathbf{w}_i^{R^T}}{i\nu_j^q \omega - \lambda_i} \tag{44}$$

By partitioning the matrix of dynamic compliance into submatrixes whose orders are given by the numbers of nodes with and without couplings for the jth part of the set of frequency multiples is

$$\begin{bmatrix} \mathbf{q}_j^R \\ \mathbf{q}_j^{RS} \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{j_{11}}^R & \mathbf{G}_{j_{12}}^R \\ \mathbf{G}_{j_{21}}^R & \mathbf{G}_{j_{22}}^R \end{bmatrix} \begin{bmatrix} \mathbf{Q}_j^R \\ -\mathbf{Q}_j^C \end{bmatrix} \tag{45}$$

For the free rotor, the response of the rotor is written as

$$\begin{aligned} \mathbf{q}_{f_j}^R &= \mathbf{G}_{j_{11}}^R \mathbf{Q}_j^R \\ \mathbf{q}_{f_j}^{RS} &= \mathbf{G}_{j_{21}}^R \mathbf{Q}_j^R \end{aligned} \tag{46}$$

and for all multiples as

$$\begin{aligned} \begin{bmatrix} \mathbf{q}_{f_1}^R \\ \mathbf{q}_{f_2}^R \\ \cdots \\ \mathbf{q}_{f_n}^R \end{bmatrix} &= \begin{bmatrix} \mathbf{G}_{1_{11}}^R & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{2_{11}}^R & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{G}_{n_{11}}^R \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1^R \\ \mathbf{Q}_2^R \\ \cdots \\ \mathbf{Q}_n^R \end{bmatrix} \\ \begin{bmatrix} \mathbf{q}_{f_1}^{RS} \\ \mathbf{q}_{f_2}^{RS} \\ \cdots \\ \mathbf{q}_{f_n}^{RS} \end{bmatrix} &= \begin{bmatrix} \mathbf{G}_{1_{21}}^R & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_{2_{21}}^R & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{G}_{n_{21}}^R \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1^R \\ \mathbf{Q}_2^R \\ \cdots \\ \mathbf{Q}_n^R \end{bmatrix} \end{aligned} \tag{47}$$

and in shorter form as

$$\begin{aligned} \mathbf{q}_f^R &= \mathbf{G}_{11}^R \mathbf{Q}^R \\ \mathbf{q}_f^{RS} &= \mathbf{G}_{21}^R \mathbf{Q}^R \end{aligned} \tag{48}$$

For the response of the separated stator in a position in which it is connected to the rotor, it is possible to write

$$\begin{bmatrix} \mathbf{q}_1^{SR} \\ \mathbf{q}_2^{SR} \\ \cdots \\ \mathbf{q}_n^{SR} \end{bmatrix} = \begin{bmatrix} \mathbf{q}_{f_1}^{SR} \\ \mathbf{q}_{f_2}^{SR} \\ \cdots \\ \mathbf{q}_{f_n}^{SR} \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1^S & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2^S & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{G}_n^S \end{bmatrix} \begin{bmatrix} \mathbf{Q}_1^C \\ \mathbf{Q}_2^C \\ \cdots \\ \mathbf{Q}_n^C \end{bmatrix} \tag{49}$$

or in shorter form,

$$\mathbf{q}^{SR} = \mathbf{q}_f^{SR} + \mathbf{G}^S \mathbf{Q}^C \tag{50}$$

Coupling elements are presented as dynamic subsystems. Using the conditions of force equilibrium and displacement continuity, it is possible to write (only for the forward precession) the equation of motion for this subsystem in the form

$$\begin{aligned} &\sum_{j=1}^{n} \left[\left(\mathbf{K}^C - \nu_j^{q^2} \omega^2 \mathbf{M}^C \right) + \mathrm{i}\nu_j^q \omega \mathbf{B}^C \right] \left(\mathbf{q}_j^{RS} - \mathbf{q}_j^{SR} \right) \mathrm{e}^{\mathrm{i}\nu_j^q \omega t} \\ &\quad = \sum_{j=1}^{n} \mathbf{Q}_j^C \mathrm{e}^{\mathrm{i}\nu_j^q \omega t} \end{aligned} \tag{51}$$

where matrixes $\mathbf{M}^C(\mathbf{q}, \dot{\mathbf{q}})$, $\mathbf{B}^C(\mathbf{q}, \dot{\mathbf{q}})$, $\mathbf{K}^C(\mathbf{q}, \dot{\mathbf{q}})$ for couplings are, in general, the nonlinear functions of displacement ($\mathbf{q} = \mathbf{q}^{RS} - \mathbf{q}^{SR}$) and velocity ($\dot{\mathbf{q}} = \dot{\mathbf{q}}^{RS} - \dot{\mathbf{q}}^{SR}$). In shorter form, it is possible to write this equation for the kth collocation time as

$$\sum_{j=1}^{n} \mathbf{C}_{k_j}^C \left(\mathbf{q}_j^{RS} - \mathbf{q}_j^{SR} \right) = \sum_{j=1}^{n} \mathbf{Q}_j^C \ \mathrm{e}^{\mathrm{i}\nu_j^q \omega t_k} \tag{52}$$

where

$$\mathbf{C}_{k_j}^C = \left[\left(\mathbf{K}^C - \nu_j^{q^2} \omega^2 \mathbf{M}^C \right) + \mathrm{i}\nu_j^q \omega \mathbf{B}^C \right]_k e^{\mathrm{i}\nu_j^q \omega t_k} \tag{53}$$

Equation (52) for l collocation times and forward precession has the form

$$\begin{bmatrix} \mathbf{C}^C_{1_1} & \mathbf{C}^C_{1_2} & \dots & \mathbf{C}^C_{1_n} \\ \mathbf{C}^C_{2_1} & \mathbf{C}^C_{2_2} & \dots & \mathbf{C}^C_{2_n} \\ \dots & \dots & \dots & \dots \\ \mathbf{C}^C_{l_1} & \mathbf{C}^C_{l_2} & \dots & \mathbf{C}^C_{l_n} \end{bmatrix} \begin{bmatrix} \mathbf{q}^{RS}_1 - \mathbf{q}^{SR}_1 \\ \mathbf{q}^{RS}_2 - \mathbf{q}^{SR}_2 \\ \dots \\ \mathbf{q}^{RS}_n - \mathbf{q}^{SR}_n \end{bmatrix}$$

$$= \begin{bmatrix} e^{i\nu^q_1 \omega t_1}\mathbf{I} & e^{i\nu^q_2 \omega t_1}\mathbf{I} & \dots & e^{i\nu^q_n \omega t_1}\mathbf{I} \\ e^{i\nu^q_1 \omega t_2}\mathbf{I} & e^{i\nu^q_2 \omega t_2}\mathbf{I} & \dots & e^{i\nu^q_n \omega t_2}\mathbf{I} \\ \dots & \dots & \dots & \dots \\ e^{i\nu^q_1 \omega t_l}\mathbf{I} & e^{i\nu^q_2 \omega t_l}\mathbf{I} & \dots & e^{i\nu^q_n \omega t_l}\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{Q}^C_1 \\ \mathbf{Q}^C_2 \\ \dots \\ \mathbf{Q}^C_n \end{bmatrix} \qquad [54]$$

It is possible to write Equation (54) for both precessions as

$$\mathbf{C}^C(\mathbf{q}^{RS} - \mathbf{q}^{SR}) = \mathbf{D}\mathbf{Q}^C \qquad [55]$$

The terms of rectangular matrix $\mathbf{D}$ are for specific collocation time constants. After manipulation, Equation (55) has the form

$$(\mathbf{q}^{RS} - \mathbf{q}^{SR}) = \mathbf{G}^C\mathbf{Q}^C \qquad [56]$$

where

$$\mathbf{G}^C = \mathbf{C}^{C^+}\mathbf{D} \qquad [57]$$

Rewriting Equations (48), (50), and (57) into matrix form leads to

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{G}^R_{12} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{G}^R_{22} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & -\mathbf{G}^S \\ \mathbf{0} & \mathbf{I} & -\mathbf{I} & -\mathbf{G}^C \end{bmatrix} \begin{bmatrix} \mathbf{q}^R \\ \mathbf{q}^{RS} \\ \mathbf{q}^{SR} \\ \mathbf{Q}^C \end{bmatrix} = \begin{bmatrix} \mathbf{q}^R_f \\ \mathbf{q}^{RS}_f \\ \mathbf{q}^{SR}_f \\ \mathbf{0} \end{bmatrix} \qquad [58]$$

or, in shorter form, as

$$\mathbf{A}\mathbf{x} = \mathbf{b} \qquad [59]$$

which is a system of nonlinear algebraic equations. It is important that Equation (58) be similar to Equation (10).

It is important to note that the algorithm for solving the response using the classical MDC or in combination with the MTC is as follows:

- Calculation of the modal behavior of the free rotor system
- Calculation or experimental determination of the modal behavior of the separated stator system
- Calculation of the response of the free rotor system
- Setting the general matrix and right-side vector for solving a nonlinear system of equations. It is necessary to calculate the general displacement and velocity in each iteration step. Nonlinear mass, damping, and stiffness tensors for nonlinear couplings are calculated using these values.

MODEL SAMPLE

The scheme of this sample is shown in Figure 2. The main goal of this sample is to compare the results obtained from computational simulation, such as the solution of transient response. In this comparison, the following are used: FEM (method 1); a combination of MTC and FEM (method 2); a combination of MTC and MDC (method 3); and the classical solution of the steady-state response in combination with the classical FEM (method 4). Only the first force multiple (due to unbalance) and corresponding response multiples are used in this sample for method 4. It is necessary to note that this sample is only for illustration and for comparing the results of different methods, which are presented in this article.

The model of a rotor system includes a shaft with a diameter of 0.09 m, which is divided into four finite elements. The journal bearings are assumed to be short and cavitated, with a

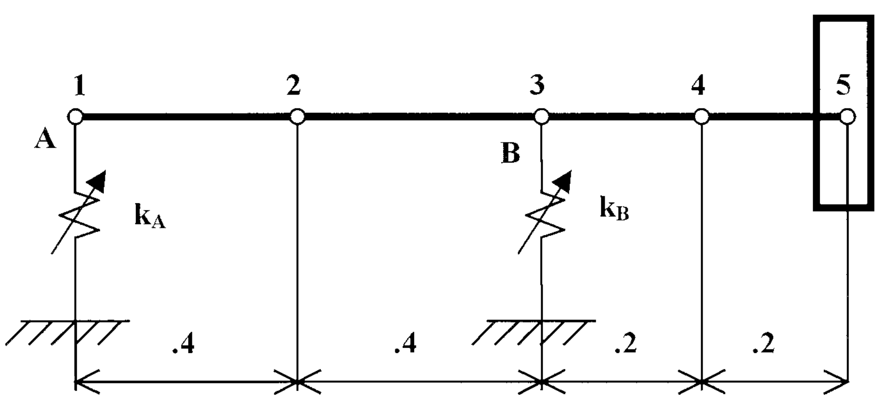

FIGURE 2
Scheme of a model sample.

diameter of 0.12 m, a length of 0.15 m, a radial clearance of 0.0002 m, and a dynamic viscosity of 0.004 Pas. Bearing A is loaded by a static force of 12,000 N and bearing B by 22,000 N. Expressions for radial and tangential forces for transient response are shown in the appendix. At node 5, a disk is set with a mass of 32 kg, a diameter moment of inertia 0.12 kg^2, and polar moment of inertia of 0.24 kg^2. At the same node, an unbalance of 0.006 kg is set. In addition, the coefficients of proportional damping ($\mathbf{B} = \alpha \mathbf{M}^R + \beta \mathbf{K}^R$) $\alpha = 4$ and $\beta = 0$ will be taken into account.

Expressions for nonlinear dynamic forces in the journal bearings were taken into account according to the work of Krämer (1993). The tensors of additional effects were calculated from these forces using the first term of the Taylor series.

The first simulation of response passing through the resonance state was calculated only for the analysis of the frequency tuning of this system. The calculation was made for an rpm range from 50 to 11400 1/min. The unbalance was 0.001 kg. The calculation time was 0–7 sec, with a number of steps of 50,000. The increase of rpm was caused by constant angular acceleration. Figure 3 shows the response in bearing B. It is evident from this figure that in the calculated rpm range, a resonance state occurred. The resonance state occurred at 5000 1/min, where the adjacent resonant peaks occurred.

The analysis of response was executed for constant rpm 7200 1/min (120 Hz), which is over the resonance frequency. The unbalance was 0.006 kg.

The overview of the methods used for obtaining the solution is as follows:

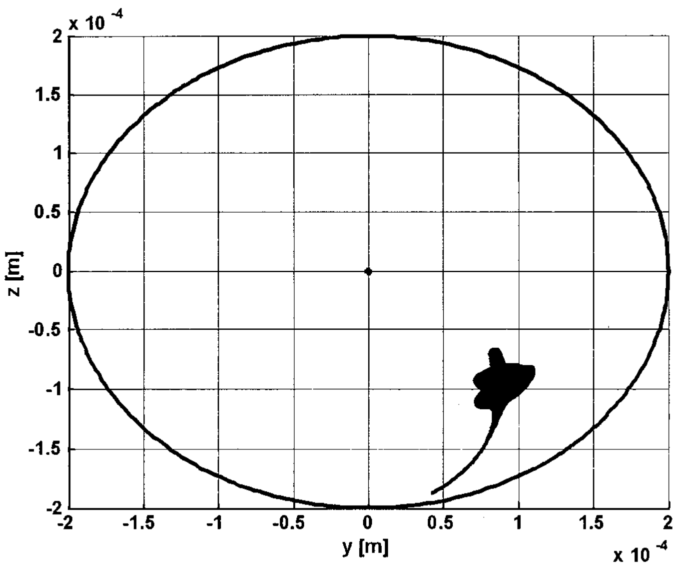

FIGURE 3
Computational simulation of a response passing through the resonance.

TABLE 1
Amplitudes of particular frequencies

Method	Amplitude × 10^6 m							
	f = 120 Hz		f = 240 Hz		f = 360 Hz		f = 480 Hz	
	y	z	y	z	y	z	y	z
Method 1	28.24	20.67	6.74	2.95	1.78	7.45	0.27	0.51
Method 2	29.27	21.75	6.90	4.13	2.07	1.12	0.37	0.18
Method 3	29.13	21.60	6.83	4.08	2.04	1.11	0.36	0.18
Method 4	28.83	17.72						

Method 1: Computational simulation, such as solving the transient response using the FEM. Calculation time 0–0.8 sec, with a general number of time steps of 12,000.

Method 2: A combination of the MTC and FEM. The following five multiples of rotor speed were taken into account: 120, 240, 360, 480, and 600 Hz.

Method 3: A combination of the MTC and MDC. The following five multiples of rotor speeds were taken into account: 120, 240, 360, 480, and 600 Hz.

Method 4: A classical solution of steady-state response and the FEM. Only the first multiple of rotor speed was taken into account—120 Hz.

Only the results from node 3, where bearing B was placed, are presented here (Figures 4 through 7). It is possible to determine, by method 1, the amplitude of the Fourier spectra of the steady part of the solution. For the other methods, it is possible to determine the amplitude only for the fixed frequency. Five values of amplitude are determined by methods 2 and 3, and only one value by method 4. Figure 4 shows the amplitudes of the Fourier spectra in the y direction; Figure 5 shows the amplitude of the Fourier spectra in the z direction, both only for method 1. The orbits, including the static equilibrium shaft position, are shown in Figure 6. Figure 7 shows these orbits relative to the static equilibrium position. Table 1 presents the amplitudes of choice frequencies and for all methods.

In all, it is possible to submit a good agreement of vibration amplitude values. The dynamic responses using methods 2 and 3 are almost the same. Good agreement is also evident for amplitudes of all multiples in both directions.

TECHNICAL APPLICATION

The technical application is a water feed pump with 10 runners. Two journal bearings of the same type are located out of the operational pump area. The journal bearings are assumed to be short and cavitated. The operating rpm is 2980 1/min, the length of the shaft is 3.24 m, and its mass is 544 kg. The diameter of the shaft is about 0.125 m, and the distance between bearings is 2.92 m. The schematic of this pump is shown in Figure 8; the corresponding computational model is shown in Figure 9. The

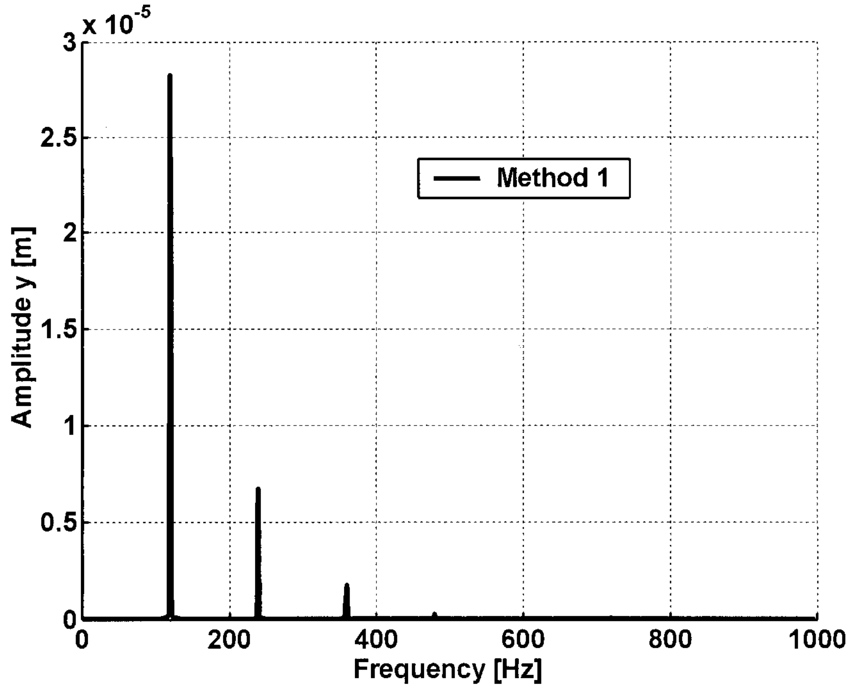

FIGURE 4
Amplitude spectra of response in direction y.

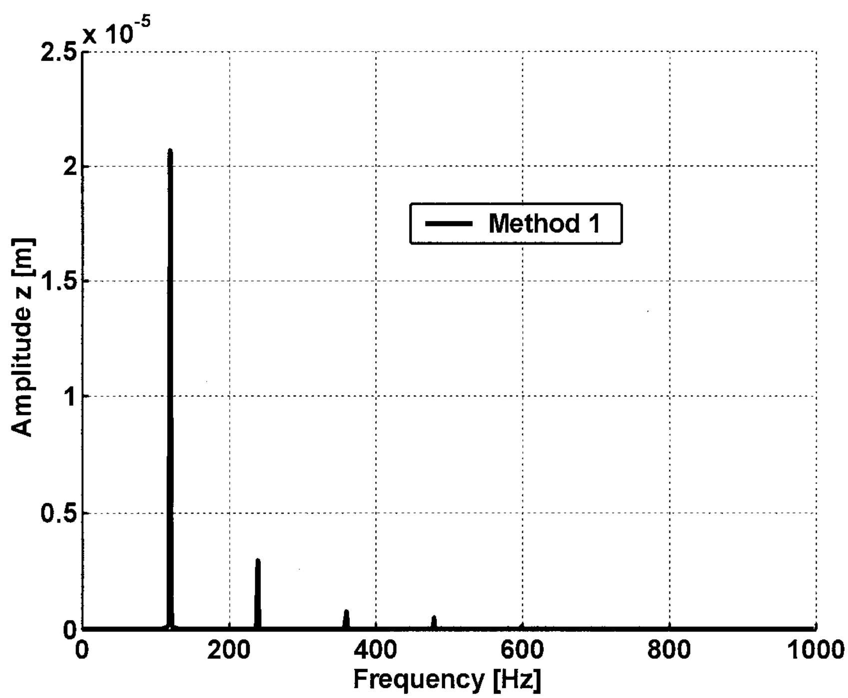

FIGURE 5
Amplitude spectra of response in direction z.

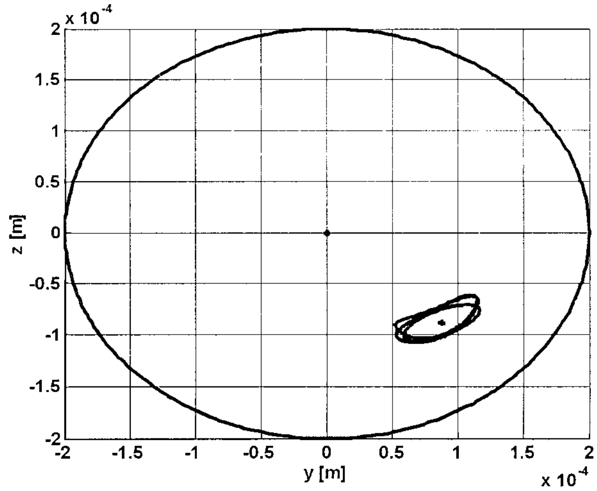

FIGURE 6
Global orbit in node 3.

computational analysis is focused on the operating state with nominal flow volume.

The main aims of the analysis are the frequency-tuning evaluation, the Campbell diagram, and the solution of the steady-state response. The computational model includes a liquid volume, which is closed by runners; the dynamic behavior of journal bearings; the dynamic compliances of journal bearing supports; and the influence of seals. Runners, bearing ring seals and shaft seals, are set at nodes 7–16.

The working medium has an influence on the rotor dynamic behavior (runners and seals). Two model types are specified: a dry rotor, without the influence of liquid, and a wet rotor, with the influence of liquid on runners and seals.

The program systems, based on the application of the FEM and the MDC, were used for the analysis of dynamic behavior.

Analysis of Dynamic Behavior of Seals

Additional mass, damping, and stiffness tensors are dependent on the rotor's rpm (Amoser, 1993, 1995; Childs, 1983). The computational rotor model includes two seals (a wearing ring seal and a shaft seal) in each runner and two seals in balancing pistons in nodes 17 and 18. All seals are taken into account as being long. The input parameters for the calculation were length; radial clearance; rpm and pressure difference in the wearing ring seal and shaft seal parts; liquid density; and viscosity. All tensor elements have a polynomial dependence on rpm. Figure 9 shows the additional effects of seals marked as k_{SEi}.

Analysis of Dynamic Behavior of Journal Bearings

The journal bearings are located in nodes 3 and 19, as shown in Figure 9. The stiffness and damping coefficients were taken

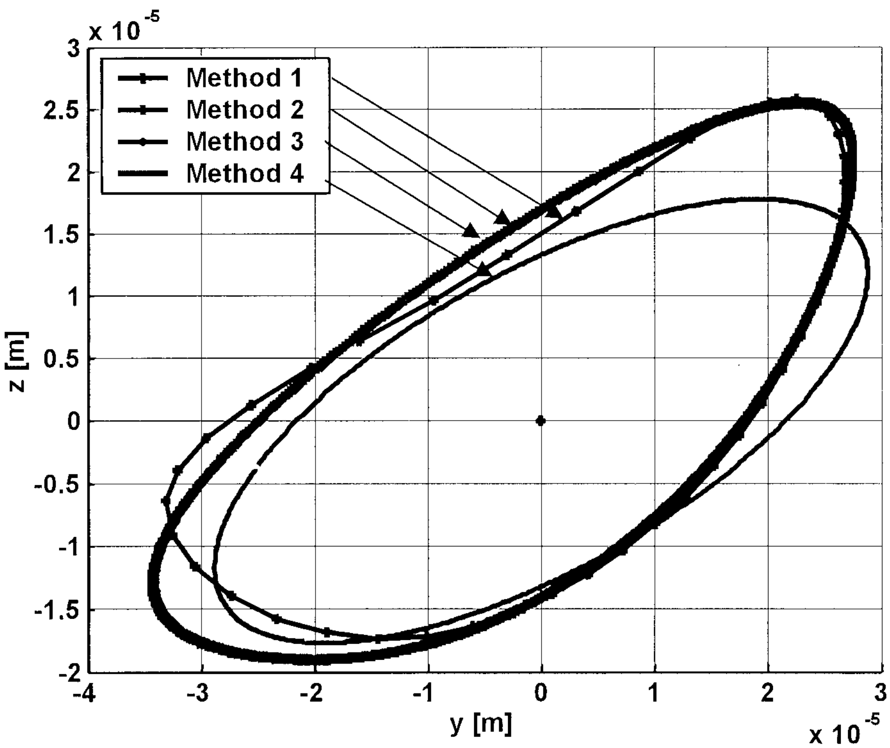

FIGURE 7
Detail of orbit in node 3.

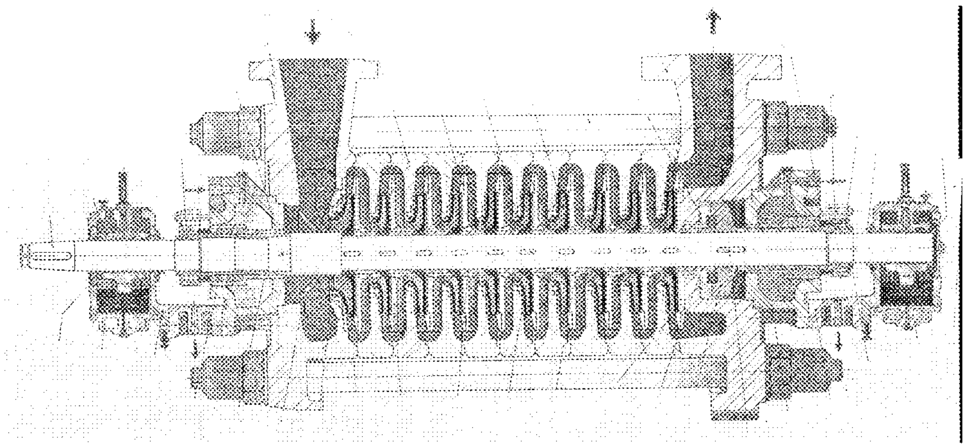

FIGURE 8
Scheme of the pump.

into account according to Krämer (1993) for the steady-state response. The static equilibrium position was calculated for a given static loading. The input parameters for the calculation were length, 85 mm; diameter, 90 mm; radial clearance, 0.15 mm; dynamic viscosity, 0.0083 Pas; oil density, 830 kg/m^3; and static loading at node 3 is 3483 N and at node 19 is 3557 N. Figure 9 shows the stiffness and damping of journal bearing marked k_{B1}, k_{B2}, and k_{BF}, which mark the dynamic stiffness of the journal bearing supports between bearing and frame, henceforth referred to as *support*.

Analysis of Journal Bearing Support

One of the possibilities of identifying support parameters is presented in Vázquez and colleagues (2001) or in Edwards and colleagues (2000). Both were used in identifying the dynamic behavior of the support from the harmonic excitation of an electromechanical shaker. In our case the frequency modal behavior of the separated support was determined experimentally and also calculated using the program system ANSYS 5.0 A. As an experiment, the experimental modal analysis was used. Experimental analysis was performed on the separated part of the pump, which consists of the support and suction parts. Screws joined both parts.

The axisymmetrical cylinder and cone parts with two opening parts were approximated using the computational model of support. Geometrical boundary conditions were specified, such as zero displacements in nodes joined to other parts of the pump. The output values from the analysis are the eigenfrequency of transversal vibration and eigenvectors in nodes joined to the journal bearings.

Two computational models were taken into account:

1. Model 1: rather elastic couplings with the frame of the pump. The geometrical boundary conditions are prescribed only for the nodes on the inner ring of the contact surface.

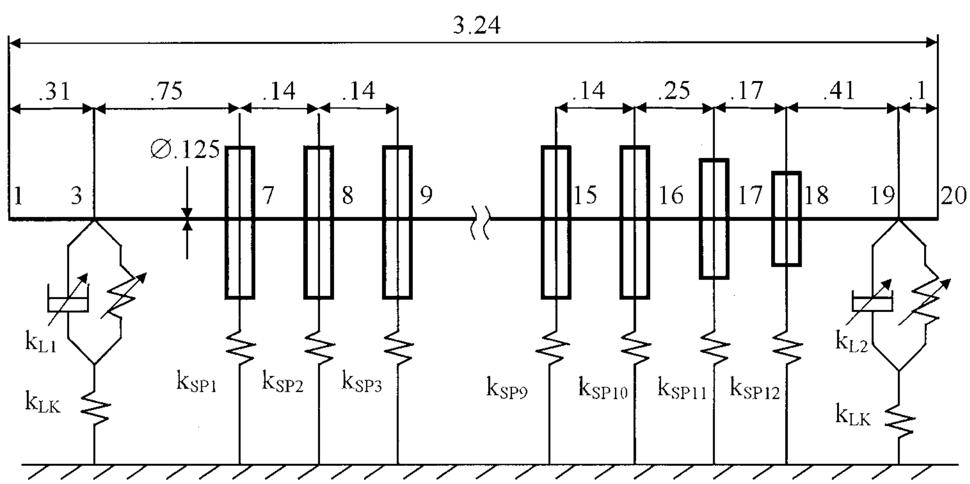

FIGURE 9
Computational model of the pump.

TABLE 2
Eigenfrequencies of journal-bearing supports

	Eigenfrequency (rad/sec)			
	Shape of vibrations		Shape of vibrations	
Method	Direction y	Direction z	Direction y	Direction z
Experiment	6163	7100	12,422	14,640
Calculation, Model 1	6118	6631	15,579	15,621
Calculation, Model 2	5399	5912	15,017	15,270

2. Model 2: rather stiff couplings with the frame of the pump. Geometrical boundary conditions are prescribed for all nodes on the contact surface.

Only the first two shapes of vibrations were taken into account with regard to operational rpm. There was good agreement for model 2 between experiment and calculation-presented stiffer couplings.

The experiment (experimental modal analysis) was performed in 2 directions in 6 points. In Table 2, the frequencies obtained from experiment and calculation are presented. Especially for the first eigenfrequency, a good agreement between the experiment and calculation for model 1 is evident. There was also evidence of good agreement in the shapes of vibration. This is a very important conclusion for the choice of boundary conditions by computational modeling of elastic supports, which is connected by screws. It is possible to simulate the screw connection that only the nodes on the inner ring of the contact surface have zero displacements.

Complex Dynamic Analysis of Rotor Systems

For comparison, the influence of liquid, journal bearings, compliance of support, and rpm were analyzed using several pump models. Individual models are specified in Table 3.

Model 1 represents a rotor system without liquid but with stiff bearings and stiff supports. Model 2 is the same as Model 1, where only the influence of gyroscopic effects was observed. Model 3 is the same as Model 2, where only the influence of liquid in runners was added, but without influence of seals. Model 4 is the same as Model 2, but instead of stiff bearings, journal bearings are included. Model 5 is the same as Model 4, where only the influence of liquid in runners was included. Model 6 is the same as Model 5, where only the influence of seals was included. Model 7 is the same as Model 6; it is a complex one and includes compliance supports.

The first two shapes of vibration for Model 1 are drawn in Figures 10 and 11, but only for illustration. It is evident from these figures that the shapes of vibrations are elastic.

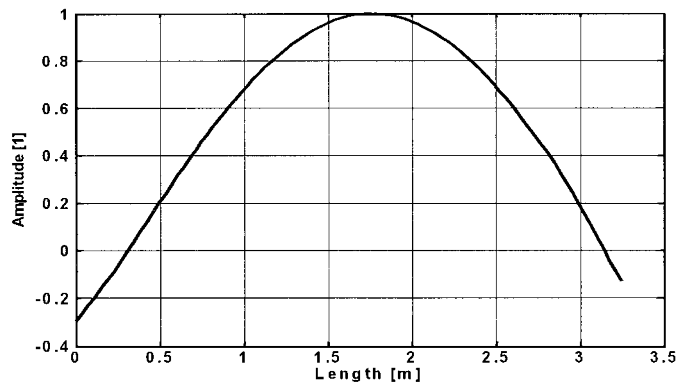

FIGURE 10
First shape of the vibration in Model 1.

The Campbell diagram was calculated for Model 7. Relatively higher values of additional damping in seals were calculated. That is why the two models were taken into account. The first one is without damping but with mass and stiffness, and the second one includes damping. The Campbell diagrams for both models are shown in Figures 12 and 13; only the imaginary part of the complex value is used. It is evident from these figures that the damping in seals has a substantial influence on eigenvalues. The model "without seal damping" has a resonant state of approximately 400 rpm. A slightly different situation occurred in the model with seal influence, where the crossing of eigenvalues occurred. For both models, the dependence of additional effects on rpm is evident. The correlation of eigenfrequencies with rotor speed is especially evident in Figure 13. Overdamping of some shapes of vibrations occurred in the model with seal influence.

It is evident from Figure 13 that the resonant state can occur during the analysis of the steady-state response. The unbalances in node 7 (0.96 kg) and in node 13 (1.321 kg) were taken into account for the calculation. Figure 14 shows the amplitude frequency response in the y direction (horizontal direction) in node

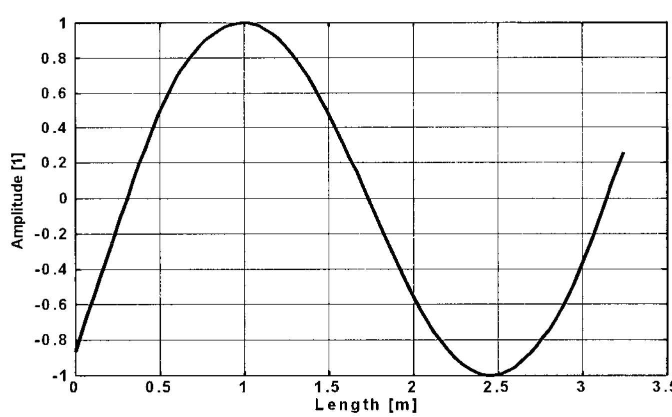

FIGURE 11
Second shape of the vibration in Model 1.

TABLE 3
Summary of eigenfrequencies for various models

Model		Eigenfrequency (1/min)			
Method	Specification	Shape			
1. FEM	Dry rotor; stiff bearings	1,254.3	5,097.0	11,425.0	18,633.0
(linear)	Without the influence seals				
0 1/min	Stiff supports				
2. FEM	Dry rotor; stiff bearings	1,251.1	5,074.7	11,362.0	18,594.0
(linear)	Without the influence seals	1,255.0	5,119.2	11,488.0	18,671.0
2980 1/min	Stiff supports				
3. FEM	Wet rotor; stiff bearings	1,186.9	4,857.6	10,977.0	18,146.0
(linear)	Without the influence seals	1,193.4	4,909.8	11,128.0	18,263.0
2980 1/min	Stiff supports				
4. FEM	Dry rotor; journal bearings	1,209.2	2,079.7	4,945.5	11,405.0
(nonlinear)	Without the influence seals	1,242.2	2,187.4	5,069.5	11,441.0
2980 1/min	Stiff supports				
5. FEM	Wet rotor, journal bearings	927.5	2,098.1	3,732.5	9,224.7
(nonlinear)	Without the influence seals	947.0	2,251.5	3,937.5	9,228.5
2980 1/min	Stiff supports				
6. FEM	Wet rotor; journal bearings	1,899.0	3,737.0	7,726.1	15,035.0
(nonlinear)	With the influence seals	2,134.0	3,782.7	7,786.0	15,159.0
2980 1/min	Stiff supports				
7. MDC	Wet rotor; journal bearings	1,856.4	3,713.7	6,827.7	8,794.9
(nonlinear)	With the influence seals	2,108.3	3,765.3	6,932.3	9,537.8
2980 1/min	Compliance supports				

12 for Model 7. The response in the second direction has almost the same dependence. It is possible to make the conclusion that with respect to higher damping values in seals, the expressive excitation of any shape of vibration does not occur.

CONCLUSION

This article is especially focused on rotor dynamic problems, but it is possible to use the presented methods in any other area of mechanics, too. The solution of frequency modal

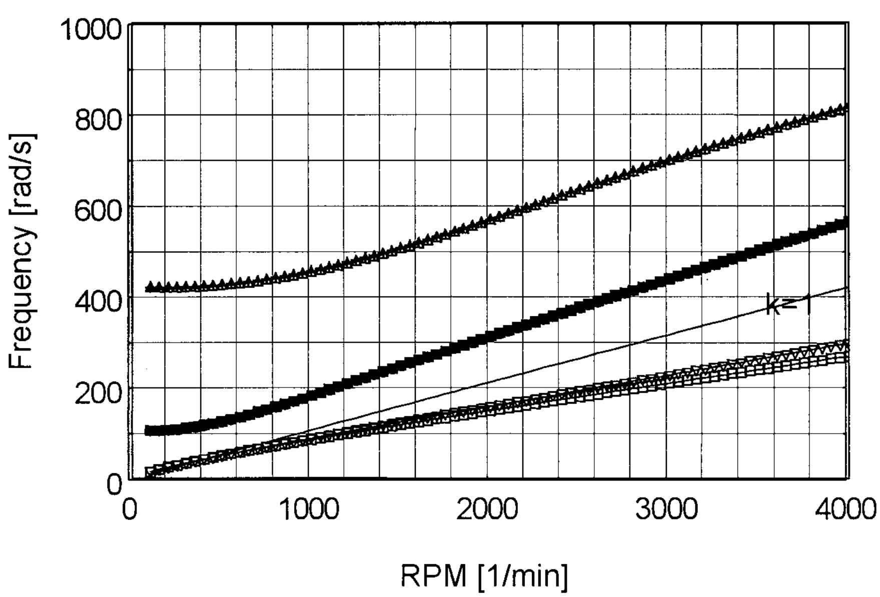

FIGURE 12
Campbell diagram without seal damping.

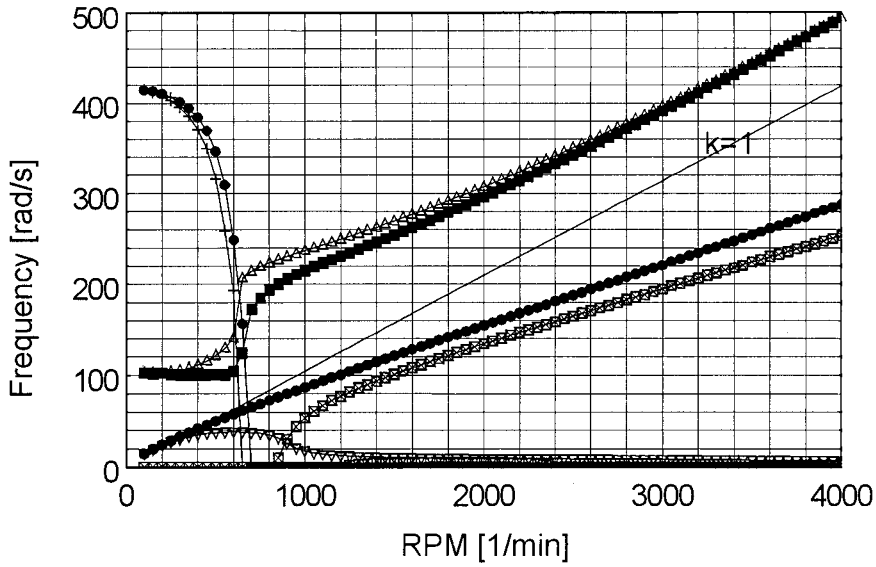

FIGURE 13
Campbell diagram with seal damping.

behavior, steady-state response, and transient response (computational simulation) were described. The method of dynamic compliances and modal method are presented. Using these methods it is possible to make a synthesis of rotating and nonrotating parts and of a lot of nonlinear coupling elements. It is also possible to include experimental data in the solution, especially the dynamic compliance of the stator, which is possible to include in the analysis of rotor dynamic systems. The next advantage is the possibility of including the nonlinear dynamic compliance (stiffness) of coupling elements. It is necessary to know the dynamic behavior of a free rotor part, especially the modal frequency behaviors such as eigenfrequency and eigenvectors, for this method. It is assumed in many cases that the free rotor part is conservative (i.e., without damping), which can be a great limitation. There are problems with the numerical solution of free nonconservative (i.e., with damping) rotor parts. It is necessary to use a very sensitive approach to this analysis. A new approach for solving the transient response, which is based on computational modeling using the modal synthesis method, is presented in this article. It is also necessary to note that using these methods it is also possible to decrease the order of composition.

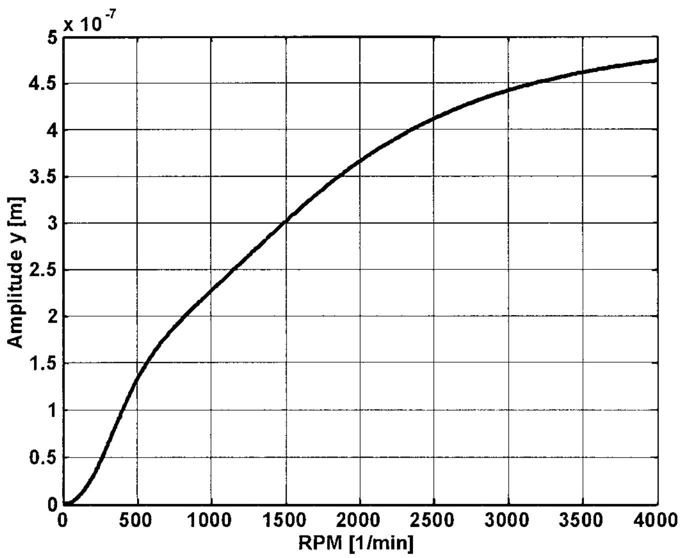

FIGURE 14
Amplitude response y in node 12.

The method of trigonometric collocation connected the advantages of solving the steady-state response and the transient response. It is possible to include the nonlinear time-dependent couplings in the solution. The steady-state response is solved, including the time-dependent nonlinear coupling (support) elements. It is possible to include the following in the solution: dynamic forces, or additional mass, and damping, or stiffness tensors. This is the biggest advantage of this method. The method is also suitable for polyharmonically forced rotor dynamic systems and for the analysis of subharmonic and ultraharmonic multiples. It is also possible to determine the response for an arbitrary set of multiples in a relatively short frequency range.

There is a problem with determining the set of response multiples of an assumed solution. The solution can be unknown, especially for evolutive dynamic systems. It is possible in this case to create a vector with more unknowns in a set of multiples, but it causes a higher order of matrixes. This approach was very time-consuming and also can cause problems with the numerical stability of the solution. A new approach to the solution of the steady-state response is presented; it is based on computer modeling using the modal synthesis and trigonometric collocation methods.

The theoretical analysis presented in this article was performed using Fortran or Matlab program codes. Many program systems were used for the analysis. The first of them is based on the application of the classical FEM; the second one is based on the application of MDC or MM and is suitable for the analysis of one shaft. The third one is based on the application of MDC or MM and is suitable for the analysis of two coaxial shafts. It is possible to include in the solution a lot of nonlinear coupling elements, such as journal bearings, SQUEEZE film dampers, couplings due to magnetic field, seals, tilt pad bearings, ball bearings, and contact problems. It is possible to determine the mass, damping, or stiffness tensors of nonlinear couplings using the Taylor series, which is more suitable for the methods presented in this article. The present program systems were tested and the results were compared to the exact solution or experiment. The systems were used for solving many technical applications; only one of them is presented.

NOMENCLATURE

A	matrix of a system of equations
B	damping matrix (Ns/m, Nms/rad)
b	vector of right side
C	matrix of dynamic stiffness (N/m, Nm/rad)
G	matrix of dynamic compliances or modal matrix (m/N, rad/Nm)
i	imaginary unit
I	identity matrix
K	stiffness matrix (N/m, Nm/rad)
k_B	dynamic bearing stiffness (N/m)
k_{BF}	dynamic stiffness of elastic bearing support (N/m)
k_{SE}	dynamic seal stiffness (N/m)
L	length of bearing (m)
M	mass matrix (1)
O, x, y, z	fixed coordinate system
O	zero matrix (1)
q	generalized vector of response (m, rad)
Q	generalized vector of excitation or forces (N, Nm)
r	radius of bearing (m)
t	time (sec)
Δt	time step (sec)
v, **w**	right- and left-side eigenvectors (1)
x	vector of unknowns

Greek Letters

α	coefficient of proportional damping mass matrix
β	coefficient of proportional damping derived from stiffness matrix
γ	circumferential angle (rad)
δ	radial clearance (m)
ε	relative shaft position (1)
η	dynamic viscosity (Pas)
λ_i	ith eigenvalue (rad/sec)
τ	time (sec)
ω	rotor speed (rad/sec)
ν	set of multiples of force frequency (1)

Superscripts

R	the places on the rotor without couplings (without connections with stator), or that the value is concerning the rotor
RS	the places on rotor with couplings (where the rotor is connected with stator)
SR	the places on stator with couplings (where the stator is connected with rotor)
S	the value is concerning stator
C	the value is concerning coupling element
$+$	pseudoinverse
$\cdot$	time derivation

Subscripts

f	free or separated rotor or stator parts
c	convolutory integral
o	amplitude or static values
s	sine part
c	cosine part
T	transpose
FP	forward-whirl precession
BP	backward-whirl precession

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APPENDIX

Radial and tangential forces of short-cavity journal bearings

$$F_r(\varepsilon, \varepsilon^{\cdot}, \gamma^{\cdot}) = \frac{\eta L^3 \omega r}{2\delta^2}\left[\left(1 - \frac{2\gamma^{\cdot}}{\omega^2}\right)\frac{2\varepsilon^2}{(1-\varepsilon^2)^2} + \pi\frac{\varepsilon^{\cdot}}{\omega}\frac{1+2\varepsilon^2}{(1-\varepsilon^2)^{5/2}}\right]$$

$$F_t(\varepsilon, \varepsilon^{\cdot}, \gamma^{\cdot}) = \frac{\eta L^3 \omega r}{2\delta^2}\left[-\frac{\pi}{2}\left(1 - \frac{2\gamma^{\cdot}}{\omega^2}\right)\frac{\varepsilon}{(1-\varepsilon^2)^{3/2}} - \frac{\varepsilon^{\cdot}}{\omega}\frac{4\varepsilon}{(1-\varepsilon^2)^2}\right]$$