

## Research Article

# A Fast Holomorphic Embedding Power Flow Approach for Meshed Distribution Networks

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This paper proposes a novel holomorphic embedding approach for solving the nonlinear power flow equation for meshed electric distribution networks with ZIP load model. In the proposed approach, bus voltages are modelled as holomorphic functions in the constant power injections and then expanded using Maclaurin series. The Z-bus matrix is implicitly used to calculate Maclaurin series coefficients for the expanded voltage functions in a recursive manner. The necessary and sufficient conditions for the convergence of expanded voltage functions are found. Performance evaluations show that the proposed approach solves the nonlinear power flow equations faster than the existing approaches when applied to 18-, 33-, 69-, 141-, 3239-, 5701-, and 6921-bus distribution network test cases.

## 1. Introduction

The increased penetration levels of Distributed Energy Resources (DERs) and Electric Vehicles (EVs) have changed distribution systems to become more active. Further, the increased penetration levels require an increased level of reliability and resiliency that can be achieved by having meshed distribution networks. Solving the nonlinear power flow equations for meshed distribution systems in shorter times can enable better planning and operation decisions, especially with the increased penetration levels of DERs and EVs. Planning decisions with increased penetration levels of DERs and EVs require running multiple scenarios over several years to cover various possible future realisations while considering demand and DERs uncertainties. Operation decisions with increased penetration levels of DERs and EVs require faster computations for maintaining system security and achieving optimal system operation in near real time.

Conventionally, the nonlinear power flow equations are solved iteratively. Iterative approaches such as, Backward-Forward Sweep (BFS) [1], network reduction [2], direct [3], loop impedance [4], and compensation-based [5], are traditionally used for passive and radial distribution systems.

Using these approaches for meshed distribution systems with high penetration levels of DERs and EVs can cause convergence issues. Thus, full Newton–Raphson (NR)-based approaches with full Jacobian matrix inverse at each iteration are usually used to solve the nonlinear power flow equations for meshed distribution systems (i.e., similar to transmission systems) [6]. Fast NR-based approaches with partial Jacobian matrix inverse cannot be applied directly in distribution systems due to the small X/R ratio and require the application of complex per unit normalization technique [7, 8].

In 2012, Dr. Antonio Trias remarkably proposed a recursive approach for solving the nonlinear power flow equations using Holomorphic Voltage Functions (HVF) for PQ and slack buses only, namely, the Holomorphic Embedding Load Flow Method (HELM). HELM core concept is explained by modelling bus voltages as holomorphic functions in a scaling parameter chosen to reduce computational effort at the first recursion and then expanding these holomorphic voltage functions using Taylor expansion [9, 10]. Then, the expanded HVFs are then approximated using Padé Approximation (PA) to extend the radius of convergence. The main advantage of HELM is that its convergence does not depend on the initial conditions, and it

is guaranteed to reach the solution if it exists. Furthermore, the tremendous work in [11] expanded benefits from the formulation of the original HELM to include PV buses. Nevertheless, a comparison between NR and HELM shows that HELM main disadvantage is that it requires more computational time than NR when high precision solutions are desired [12–14]. This is mainly due to the application of the time intensive PA. To reduce computational time and increase the benefits from the original HELM, the authors in [15, 16] astonishingly exploited the radial and weakly meshed topology of the traditional distribution systems. Thus, the reduction in computational time is not achievable for meshed distribution systems. In [17, 18], the authors interestingly proposed a modified holomorphic embedding method for hybrid meshed AC-DC microgrid and for remote voltage control, respectively; however, the time intensive PA is still applied. The authors in [19, 20] showed the benefits from utilizing the HELM approach for three-phase unbalanced optimal power flow and for probabilistic power flow for small distribution test networks; however, the long time needed to solve the power flow equations impeded extending the application to larger distribution test networks. In [21, 22], the authors notably used static approximate Newton–Raphson–Sidel (ANRS) and the developed down-hill (DDH) algorithm methods to derive a reliable and efficient holomorphic approach to evaluate dynamic available transfer capability and proposed distributed slack bus model formulation for the holomorphic embedding load flow method, respectively; however, Padé Approximation is still used.

In this paper, a novel approach for power flow in active and meshed AC and DC electric distribution systems with ZIP load model based on HVFs, namely, Implicit Z-bus Recursive (IZR) power flow method is proposed. The proposed approach entails modelling bus voltages as holomorphic functions in the constant power injections and calculating the radius of convergence at each recursion instead of applying analytic continuation methods, such as the time intensive PA, which resulted in significantly shorter computational time. The performance of the proposed approach is evaluated and compared with that of existing approaches using 18-, 33-, 69-, 141-, 3239-, 5701-, and 6921-bus distribution system test cases. The comparison shows that the proposed approach is faster than HELM, NR, IZG, and BFS under the tested conditions and cases. The novelties in the proposed approach are as follows:

- (1) A novel approach for power flow in active and meshed AC and DC electric distribution systems with ZIP load model and constant power DER model is proposed. In the proposed approach, bus voltages are modelled as holomorphic functions in the constant power injections scaling parameter. This is different from the approaches in [9, 15, 16], where bus voltages are modelled as holomorphic functions in a scaling parameter chosen to reduce computational effort at the first recursion only; thus, the proposed approach is more intuitive than the previous approaches. The approach proposed in this paper allows solving the power flow equations in one recursion if constant power injections are not present while the approaches

in [9, 15, 16] require more than one recursion; thus, the proposed approach has better computational efficiency than the previous approaches.

- (2) The radius of convergence is checked at each recursion instead of applying analytic continuation methods such as PA. This is different from the approaches in [9, 15, 16], where analytic continuation methods are used for all recursions resulting in higher computational efforts. Checking the radius of convergence at each recursion avoids the computational efforts needed for the application of analytic continuation methods and ensures the robustness of the proposed method under various operating conditions. However, it cannot guarantee a solution when operating very close to the voltage collapse point. In practice, control actions (e.g., switching reactive devices and changing transformers tap positions) are usually applied to maintain the voltage levels of the distributions systems above the minimum voltage level and far from the voltage collapse point. Thus, the need to evaluate the holomorphic voltage very close to the voltage collapse point is significantly low for well-behaved distribution systems.
- (3) A sufficient condition for convergence after the first recursion is found for the holomorphic embedding approach. This is provided for the first time, up to the author’s knowledge, in this work. Satisfying the proposed condition reduces the computational efforts needed for subsequent recursions. This is provided for the first time, up to the author’s knowledge, in this work.
- (4) The performance of the proposed approach is evaluated and compared with existing approaches using 18-, 33-, 69-, 141-, 3239-, 5701-, and 6921-bus distribution system test cases. This is different from the approaches in [9, 16], where performance evaluation and comparison were performed using smaller test cases.

The paper is organized as follows: Section II models bus voltages as holomorphic functions and shows Maclurin series expansions for the holomorphic voltage functions. Section III presents the traditional HELM approach. Section IV presents the proposed power flow approach with the necessary and sufficient conditions for convergence. Section V compares the performance of the proposed approach with the performance of HELM, NR, IZG, and BFS approaches. Finally, the conclusions are presented in Section V.

## 2. Modelling Bus Voltages as Holomorphic Functions

Holomorphic functions are complex-valued analytic functions which are infinitely complex differentiable around every point within their domain. These functions can be expanded around zero using Maclurin series [23]. The Maclurin Series Expansion (MSE) of a holomorphic voltage

function  $V(\alpha)$  is given in (1) and approximated for the first  $n$  terms in (2).

$$V(\alpha) = \sum_{k=0}^{\infty} V[k]\alpha^k = \sum_{k=0}^{\infty} \frac{V^{(k)}(\alpha)}{k!} \alpha^k \quad \forall \alpha < R, \quad (1)$$

$$V(\alpha) \approx \sum_{k=0}^n V[k]\alpha^k = \sum_{k=0}^n \frac{V^{(k)}(\alpha)}{k!} \alpha^k \quad \forall \alpha < R, \quad (2)$$

where  $\alpha$  is a complex-valued parameter,  $V^{(k)}(\alpha)$  is the complex-valued  $k^{\text{th}}$  derivative of  $V(\alpha)$ ,  $V[k]$  is the complex-valued  $k^{\text{th}}$  coefficient of the MSE of the voltage function, and  $R$  is the radius of convergence. In the next section,  $\alpha$  is shown to be a parameter for the constant power injections.

### 3. The Traditional Holomorphic Embedding Power Flow Approach

This section describes the traditional HELM approach as developed in [10, 11, 24], and the flow chart of slack and PQ buses is shown in Figure 1. Although the approach can be expanded to PV buses, it is reviewed here for slack and PQ buses only for simplicity. The traditional approach starts from the general nonlinear system of complex power flow equations for slack bus and PQ buses as shown in (3) for slack bus and in (4) for PQ buses.

$$V_1 = V_{Sl}, \quad (3)$$

$$\sum_{j=1}^N Y_{ij} V_j = \frac{S_{Li}^*}{V_i^*} + I_{Li} \quad \forall i \in N, \quad (4)$$

where bus 1 is selected to be the slack bus for simplicity and the total number of PQ buses is  $N$ . Index  $j$  is for all buses while index  $i$  is for PQ buses only. The slack bus fixed complex voltage,  $V_{Sl}$ , is assumed to be the reference bus voltage without loss of generality.  $Y_{ij}$  represents the  $ij$  complex element of the bus admittance matrix.  $I_{Li}$  and  $S_{Li}$  are the complex constant current and constant power injections at bus  $i$ , respectively.

The bus voltages in equations (3) and (4) are then modelled as holomorphic voltage functions in the complex embedding parameter,  $\alpha$ , as shown in (5) and (6) [11]. It can be shown that the voltage functions (i.e.,  $V_1(\alpha)$ ,  $V_j(\alpha)$ , and  $V_i^*(\alpha^*)$ ) satisfy Cauchy–Riemann equations with this embedding approach; thus, they are holomorphic functions [10, 11]. Also, this embedding approach is chosen to have an easy to solve system of equations at  $\alpha = 0$ , while the original system can be retrieved at  $\alpha = 1$ . However, in the next section, it will be shown that embedding the constant current injection, as in equation (6), is not necessary and results in increased computational time.

$$V_1(\alpha) = 1 + \alpha(V_{Sl} - 1) \quad (5)$$

$$\sum_{j=1}^N Y_{ijse} V_j(\alpha) = \frac{\alpha S_{Li}^*}{V_i^*(\alpha^*)} - \alpha Y_{ish} V_i(\alpha) + \alpha I_{Li} \quad \forall i \in N, \quad (6)$$

where  $Y_{ijse}$  and  $Y_{ish}$  are the series and shunt elements, respectively, of the bus admittance matrix for all buses except slack bus and  $I_{Li}$  is the constant current injection at bus  $i$ .

The next step is to substitute the holomorphic voltage functions by their Maclurin series expansions and setting  $\alpha = 0$  in (5) and (6) to find Maclurin series coefficients at  $k = 0$  (i.e.,  $V_1[0]$  and  $V_j[0]$ ) as shown in (7) and (8). Setting  $\alpha = 0$  indicates a power system with no injections and no shunt elements; thus, all bus voltages are equal to 1.

$$V_1[0] = 1, \quad (7)$$

$$\sum_{j=1}^N Y_{ijse} V_j[0] = 0 \Rightarrow V_j[0] = 1 \quad \forall i \in N. \quad (8)$$

The Maclurin series coefficients for  $k > 0$  are found by substituting  $V_1(\alpha)$ ,  $V_j(\alpha)$ , and  $V_i^*(\alpha^*)$  by their Maclurin series expansions in equations (3) and (4) resulting in the recursive equations in (9)–(12).

$$V_1[1] = V_{Sl} - 1, \quad (9)$$

$$V_1[k] = 0 \quad \forall k > 1, \quad (10)$$

$$\sum_{j=1}^N Y_{ijse} V_j = S_{Li}^* W_i^*[0] - Y_{ish} V_i[0] + I_{Li} \quad \forall i \in N, \quad (11)$$

$$\sum_{j=1}^N Y_{ijse} V_j[k] = S_{Li}^* W_i^*[k-1] - Y_{ish} V_i[k-1] \quad \forall i \in N, \forall k > 1, \quad (12)$$

where

$$W_i[k] = \begin{cases} 1 \\ V_i[0] \end{cases}, \quad \forall k = 0, \\ \frac{\sum_{u=0}^{k-1} W_i[u] V_i[k-u]}{V_i[0]}, \quad \forall k > 0, \quad (13)$$

The next step is to apply Padé approximant. Padé approximant for the expanded voltage function in (2) is a rational function expansion with two polynomials computed from the first  $n$  terms in Maclurin series expansion as in the following equation [25]:

$$V_{i(L/M)}(\alpha) \approx \frac{\sum_{l=0}^{L-1} a[l] \alpha^l}{\sum_{m=1}^{M-1} b[m] \alpha^m} \quad \forall i \in N, \quad (14)$$

where  $V_{i(L/M)}(\alpha)$  is the Padé approximant for the voltage function, which can be diagonal (i.e.,  $L = M = n/2$ ) when  $n$  is even or near diagonal (i.e.,  $|L - M| = 1$ ) when  $n$  is odd. It should be noted that  $V_{i(N/0)}(\alpha)$  corresponds to the Maclurin series expansion for the voltage function, and  $a[l]$  and  $b[m]$  are the Padé approximant coefficients, which are calculated from MSE coefficients  $V[k]$ , by solving the system of linear equations in (4), which is time intensive. The next section will show the proposed method based on checking the radius

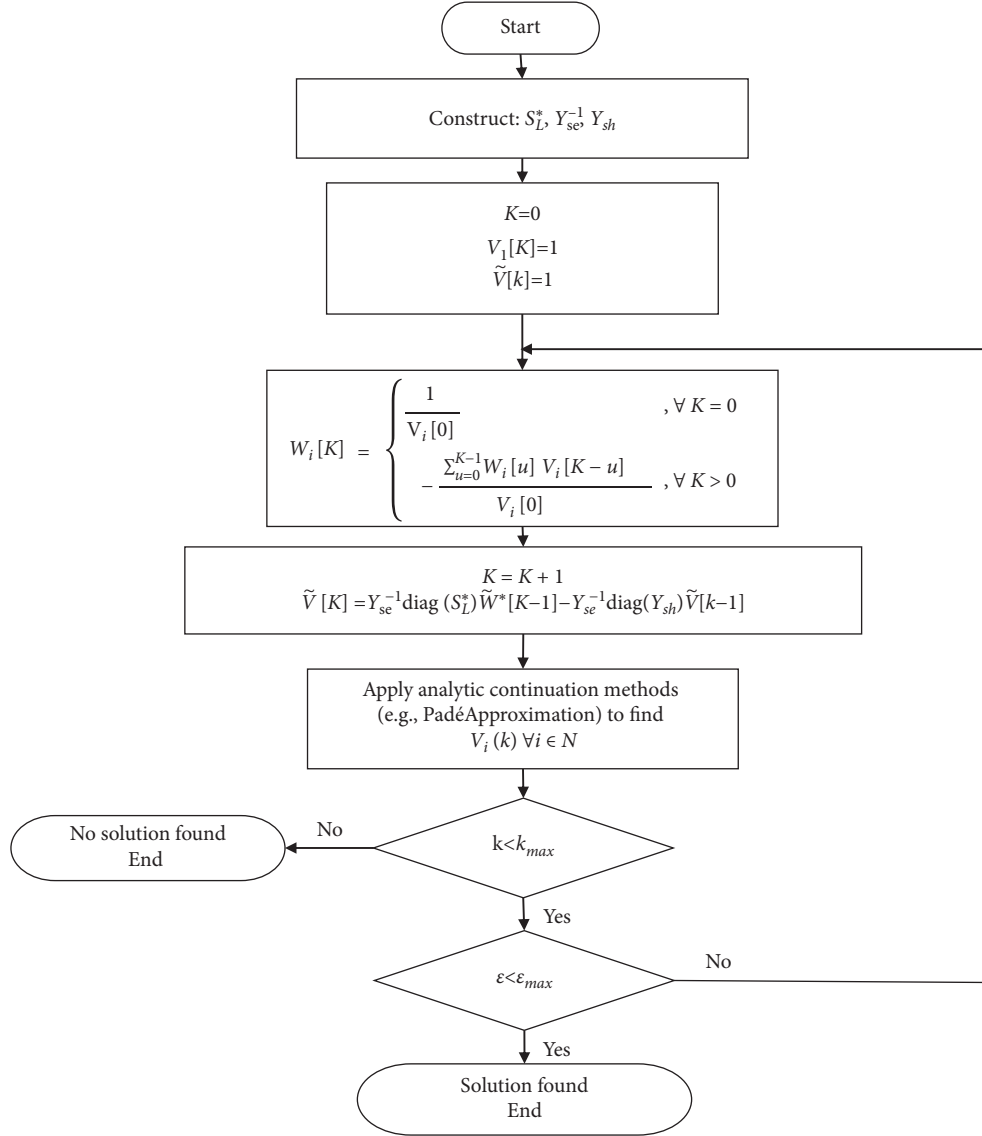


FIGURE 1: Flowchart for the traditional HELM approach.

of convergence at each recursion instead of applying Padé approximant resulting in less computational time.

$$b[0] = 1, \quad (15a)$$

$$a[0] = V[0]b[0], \quad (15b)$$

$$a[1] = V[1]b[0] + V[0]b[1], \quad (15c)$$

$$[L] = V[L]b[0] + V[L-1]b[1] + \dots + V[L-M]b[M], \quad (15d)$$

$$0 = V[L+1]b[0] + V[L]b[1] + \dots + V[L-M+1]b[M], \quad (15e)$$

$$0 = V[L+M]b[0] + V[L+M-1]b[1] + \dots + V[M]b[M]. \quad (15f)$$

#### 4. Proposed Implicit Z-Bus Recursive Power Flow Method

Starting from a general meshed AC distribution network consisting of linear elements, ZIP loads, and constant power DERs, this section describes the proposed approach shown in the flow chart (Figure 2 for slack and PQ buses). The general nonlinear complex power flow equations that describe the steady state of such distribution system are given in (16) and (17). Equation (16) sets the slack bus voltage, while equation (17) is Kirchhoff's current law current application to the remaining buses.

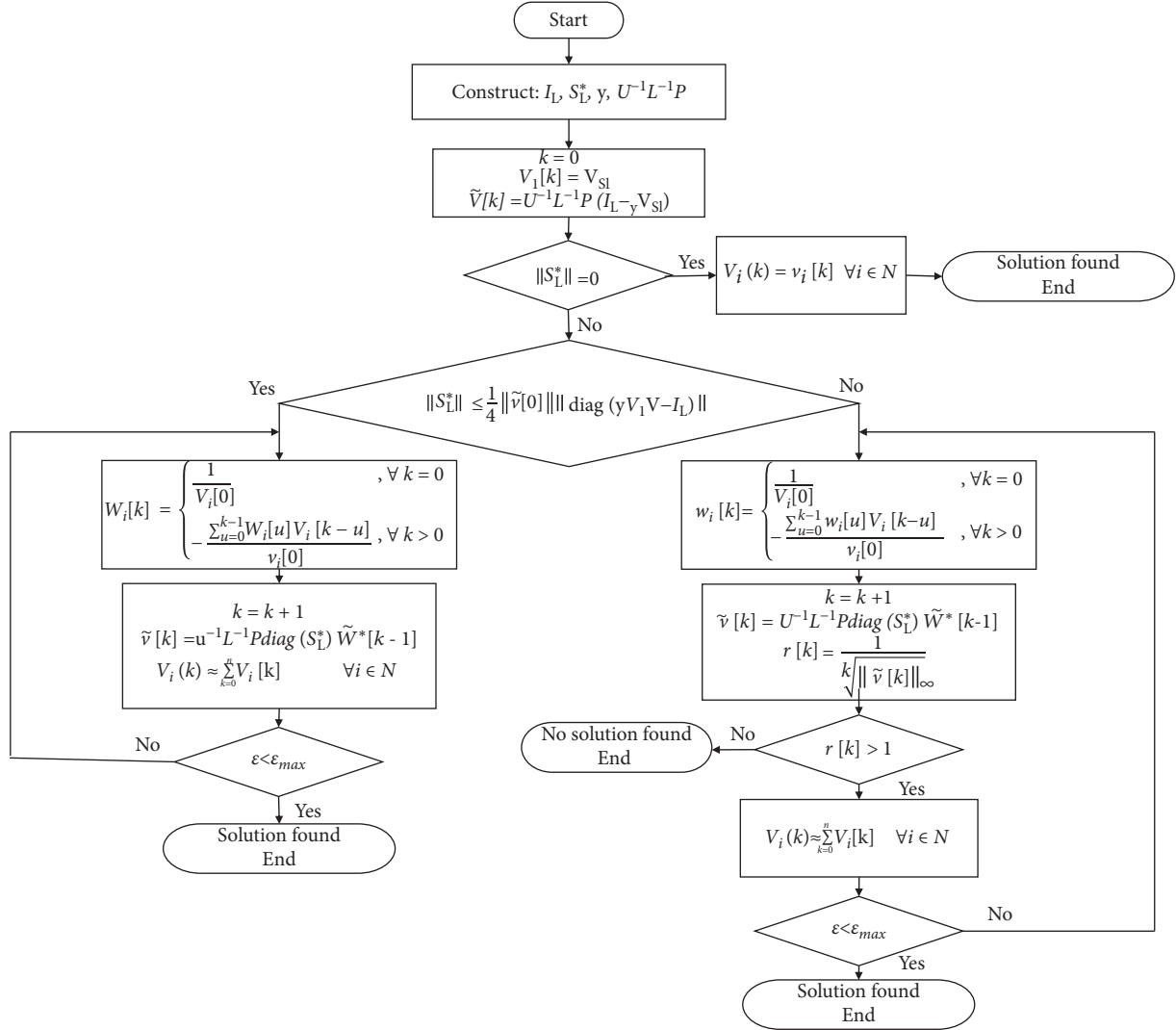


FIGURE 2: Flowchart for the proposed IZR approach.

$$V_1 = V_{sl}, \quad (16)$$

$$\sum_{j=1}^N Y_{ij} V_j = \frac{S_{Li}^*}{V_i^*} + I_{Li} - Y_{Li} V_i \quad \forall i \in N, \quad (17)$$

where bus 1 is selected to be the slack bus for simplicity, the total number of the remaining buses is  $N$ , and index  $j$  is for all buses while index  $i$  is for all buses except the slack bus. The slack bus fixed complex voltage,  $V_{sl}$ , is assumed to be the reference voltage without loss of generality.  $Y_{ij}$  represents the  $ij$  complex element of the bus admittance matrix,  $Y_{Li}$  represents bus  $i$  constant admittance load,  $I_{Li}$  is the complex constant current injection at bus,  $i$ , and  $S_{Li}$  is the complex constant power at bus  $i$ . The model in (16) and (17) can be used for DC distribution systems where  $Y_{ij}$  represents the conductance of the  $ij$  element of the bus admittance matrix,  $Y_{Li}$  represents bus  $i$  constant load conductance, and  $S_{Li}$  reduces to the constant DC active power injected at bus  $i$ . Furthermore, all voltages and currents have real components only.

Equation (17) is then rewritten in the matrix forms given in (18)–(20) to solve for bus voltages.

$$(\mathbf{y} \tilde{\mathbf{Y}}) \begin{pmatrix} V_1 \\ \tilde{\mathbf{V}} \end{pmatrix} + \text{diag}(\tilde{\mathbf{Y}}_L) \tilde{\mathbf{V}} = \text{diag}(\tilde{\mathbf{S}}^*) \tilde{\mathbf{W}}^* + \tilde{\mathbf{I}}_L, \quad (18)$$

$$\tilde{\mathbf{V}} = (\tilde{\mathbf{Y}} + \text{diag}(\tilde{\mathbf{Y}}_L))^{-1} (\text{diag}(\tilde{\mathbf{S}}^*) \tilde{\mathbf{W}}^* + \tilde{\mathbf{I}}_L - \mathbf{y} V_1), \quad (19)$$

$$\tilde{\mathbf{V}} = \tilde{\mathbf{U}}^{-1} \tilde{\mathbf{L}}^{-1} \tilde{\mathbf{P}} (\text{diag}(\tilde{\mathbf{S}}^*) \tilde{\mathbf{W}}^* + \tilde{\mathbf{I}}_L - \mathbf{y} V_1), \quad (20)$$

where  $\mathbf{y}$  is an  $N \times 1$  vector describing the mutual admittances between the slack bus (i.e., bus number 1) and the remaining buses,  $\tilde{\mathbf{V}}$  and  $\tilde{\mathbf{W}}$  are  $N \times 1$  vectors with  $V_i^*$  and  $V_i^{*-1}$  entries, respectively.  $\tilde{\mathbf{I}}_L$  is an  $N \times 1$  vector with  $\tilde{I}_{Li}$  entries.  $\tilde{\mathbf{Y}}^{-1}$  is an  $N \times N$  matrix implicitly indicating the bus

impedance matrix (i.e., Z-bus) for all buses except slack bus,  $\text{diag}(\tilde{\mathbf{Y}}_L)$  is an  $N \times N$  diagonal matrix with  $Y_{Li}$  entries, while  $\text{diag}(\tilde{\mathbf{S}}^*)$  is an  $N \times N$  diagonal matrix with  $S_i^*$  entries.  $\tilde{\mathbf{L}}$ ,  $\tilde{\mathbf{U}}$ , and  $\tilde{\mathbf{P}}$  are the lower, upper, and permutation matrices for the admittance matrix,  $(\tilde{\mathbf{Y}} + \text{diag}(\tilde{\mathbf{Y}}_L))$ .

Equation (20) can be solved for  $\tilde{\mathbf{V}}$  either iteratively or recursively. To solve the equation recursively, equations (16) and (20) are embedded with parameter,  $\alpha$  as shown in (21) and (22). This embedding implies that  $\alpha$  is a parameter for constant power injections. This embedding is simpler than the embedding method used in [10, 11, 24] as discussed in the previous section and results in faster computation as demonstrated in the next section. The voltage functions  $V_1(\alpha)$ ,  $\tilde{\mathbf{V}}(\alpha)$ , and  $\tilde{\mathbf{W}}^*(\alpha^*)$  satisfy Cauchy–Riemann equations and have continuous first partial derivatives; thus, they are holomorphic. It should be noted that the retrieval of the original constant power injections is done by setting  $\alpha = 1$ .

$$V_1(\alpha) = V_{Sl}, \quad (21)$$

$$\tilde{\mathbf{V}}(\alpha) = \tilde{\mathbf{U}}^{-1} \tilde{\mathbf{L}}^{-1} \tilde{\mathbf{P}} (\text{diag}(\alpha \tilde{\mathbf{S}}^*) \tilde{\mathbf{W}}^*(\alpha^*) + \tilde{\mathbf{I}}_L - \mathbf{y} V_1(\alpha)). \quad (22)$$

The next step is to set  $\alpha = 0$  in (2), (21), and (22) to find MSE coefficients at  $k = 0$  (i.e.,  $V_1[0]$  and  $\tilde{\mathbf{V}}[0]$ ) as shown in (23)–(25), which indicates that constant impedance and constant current loads are present, and constant power injections are not present (i.e.,  $V_1[0]$  and  $\tilde{\mathbf{V}}[0]$  give bus voltages under no constant power injections as shown in the flowchart in Figure 3).

$$V_1[0] = V_1(0) = V_{Sl}[0, 1], \quad (23)$$

$$V_1[k] = 0 \quad \forall k > 0, \quad (24)$$

$$\tilde{\mathbf{V}}[0] = \tilde{\mathbf{V}}(0) = \tilde{\mathbf{U}}^{-1} \tilde{\mathbf{L}}^{-1} \tilde{\mathbf{P}} (\tilde{\mathbf{I}}_L - \mathbf{y} V_{Sl}). \quad (25)$$

If constant power injections are present, the MSE coefficients for those buses, when  $k$  is greater than zero, are found by substituting (2) in (22), which can be shown to result in the recursive equations in (26) and (27). It should be noted that equations (2) and (23) indicate that for  $k$  greater than zero, the slack bus voltage has zero coefficients as in (24).

$$\tilde{\mathbf{V}}[k] = \tilde{\mathbf{U}}^{-1} \tilde{\mathbf{L}}^{-1} \tilde{\mathbf{P}} \text{diag}(\tilde{\mathbf{S}}^*) \tilde{\mathbf{W}}^*[k-1] \quad \forall k > 0, \quad (26)$$

where

$$W_i[k] = \begin{cases} \frac{1}{V_i[0]}, & \forall k = 0, \\ \frac{\sum_{u=0}^{k-1} W_i[u] V_i[k-u]}{V_i[0]}, & \forall k > 0. \end{cases} \quad (27)$$

Now, bus voltages are computed at each recursion,  $k$ , by setting  $\alpha = 1$  in (28) until the convergence tolerance in (30) is satisfied, where  $\varepsilon$  is the maximum convergence tolerance. The radius of convergence at the  $k^{\text{th}}$  recursion,  $r[k]$ , must be

greater than 1, to ensure that the radius of convergence,  $R$ , is greater than 1 as shown in (30) (i.e., the necessary and sufficient condition for convergence). The conditions in (30) are checked instead of applying the time intensive PA as in [10, 11, 24].

$$V_i(\alpha) \approx \sum_{k=0}^n V_i[k] \alpha^k \quad \forall i \in N \ \& \ \alpha < R, \quad (28)$$

$$\varepsilon_{\max} = \max_i \left( \sum_{j=2}^N Y_{ij} V_j V_i^* + Y_{Li} |V_i|^2 + (Y_{i1} V_1 - I_{Li}) V_i^* - S_{Li}^* \right) \leq \varepsilon, \quad (29)$$

$$\begin{aligned} \text{If } r[k] &= \frac{1}{\sqrt[k]{\tilde{\mathbf{V}}[k]_{\infty}}} > 1 \ \forall k > 0 \Leftrightarrow R \\ &= \lim_{k \rightarrow \infty} r[k] > 1. \end{aligned} \quad (30)$$

Another sufficient condition for convergence is found by rewriting equation (17) as in (31) and rearranging in the quadratic matrix equation form shown in (32). From (32), the condition on the constant power injections,  $\mathbf{S}_L^*$ , in (33) is found. The condition in (33) is then rewritten as in (34) using equation (25). Thus, it is sufficient to confirm that the conditions in (34) is satisfied to ensure that the radius of convergence,  $R$ , is greater than 1. The benefit of using the condition in (34) is that it can indicate convergence from the first recursion thus avoiding the need to compute  $r[k]$  at every recursion.

$$\sum_{j=2}^N Y_{ij} V_j V_i^* + \tilde{Y}_{Li} |V_i|^2 + (Y_{i1} V_1 - \tilde{I}_{Li}) V_i^* - S_{Li}^* = 0 \ \forall i, \quad (31)$$

$$\begin{aligned} \text{diag}(\tilde{\mathbf{V}}^*) \tilde{\mathbf{Y}} \tilde{\mathbf{V}} + \text{diag}(\tilde{\mathbf{Y}}_L^*) |\tilde{\mathbf{V}}|^2 \\ + \text{diag}(\mathbf{y} V_1 - \tilde{\mathbf{I}}_L) \tilde{\mathbf{V}}^* - \mathbf{S}_L^* = 0, \end{aligned} \quad (32)$$

$$\text{If } \mathbf{S}_L^* \leq \frac{1}{4} \tilde{\mathbf{U}}^{-1} \tilde{\mathbf{L}}^{-1} \tilde{\mathbf{P}} \text{diag}(\mathbf{y} V_1 - \tilde{\mathbf{I}}_L)^2 \Rightarrow R > 1, \quad (33)$$

$$\text{If } \mathbf{S}_L^* \leq \frac{1}{4} \tilde{\mathbf{V}}[0] \text{diag}(\mathbf{y} V_1 - \tilde{\mathbf{I}}_L) \Rightarrow R > 1. \quad (34)$$

## 5. Performance Evaluation

This section presents power flow analysis results on 18-, 33-, 69-, 141-, 3239-, 5701-, and 6921-bus distribution test systems described in [26–30], respectively, with initial loading levels and number of loops data given in Table 1. The performance of the proposed recursive method is evaluated and compared with the performance of HELM [9], NR, IZG [31], and BFS [5] approaches, under different loading levels and convergence tolerances. MATPOWER [32] toolbox for MATLAB on a laptop computer with Intel core i7 1.8 GHz 64 bit processor and 8 GB RAM is used.

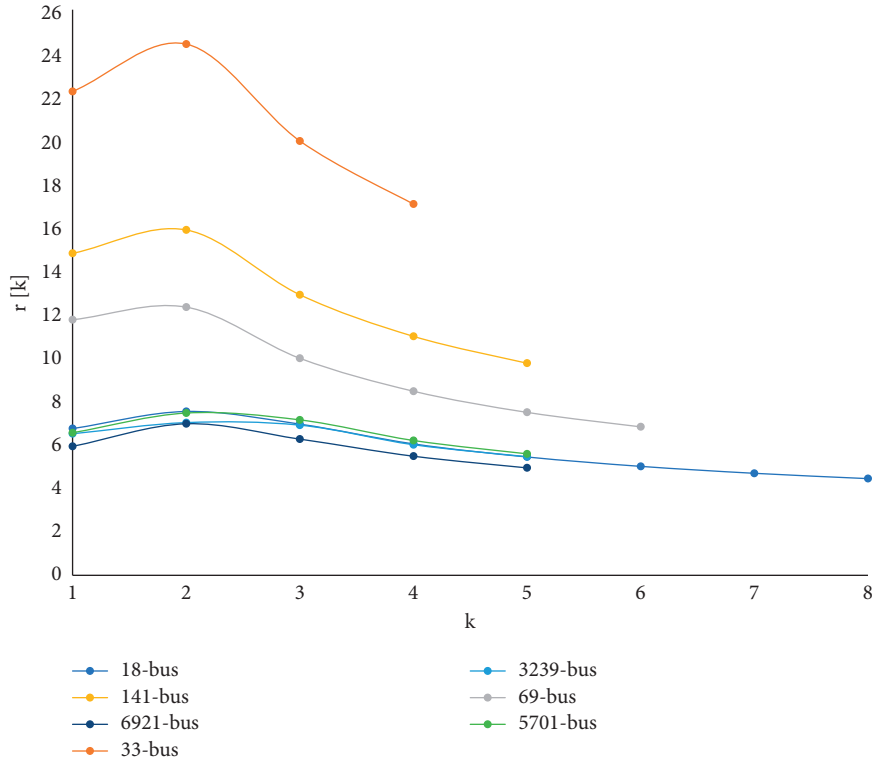


FIGURE 3: Radius of convergence at initial loading levels.

TABLE 1: Initial loading levels and number of loops.

Test case	Initial loading levels (MW)	Number of loops
<b>18</b>	11.6	0
<b>33</b>	37.2	5
<b>69</b>	38	11
<b>141</b>	119	0
<b>3239</b>	62.5	5
<b>5701</b>	74.4	0
<b>6921</b>	28.4	9

TABLE 2: Execution times, number of iterations, and number recursions to reach  $\varepsilon \leq 10^{-4}$  MVA at initial loading levels.

Test network	Time (s)					Number of iterations/recursions				
	IZR	HELM	NR	IZG	BFS	IZR	HELM	NR	IZG	BFS
<b>18</b>	0.0131	0.0418	0.8143	0.7961	0.8928	11	6	3	6	6
<b>33</b>	0.0124	0.0418	0.8173	0.8239	N/A	5	5	3	3	N/A
<b>69</b>	0.0115	0.0568	0.8649	0.8688	N/A	7	6	3	5	N/A
<b>141</b>	0.086	0.5403	0.9065	0.8905	0.9943	6	5	3	4	4
<b>3239</b>	0.0266	1.3456	1.4466	1.4318	N/A	6	5	3	4	N/A
<b>5701</b>	0.047	2.3476	1.9853	2.0059	2.1311	6	5	3	4	6
<b>6921</b>	0.0472	3.7286	2.3204	2.1893	N/A	6	6	3	4	N/A

TABLE 3: High loading levels.

Test case	High loading levels (MW)
<b>18</b>	19.5
<b>33</b>	76.26
<b>69</b>	47.5
<b>141</b>	148.75
<b>3239</b>	84.735
<b>5701</b>	104.16
<b>6921</b>	35.5

**5.1. Initial Loading Levels.** The power flow equations for all test networks under initial loading levels were solved to reach convergence tolerance less than  $10^{-4}$  MVA. All approaches were able to reach the specified convergence tolerance for all test networks. The sufficient condition for convergence in (34) is satisfied. Furthermore, Figure 3 shows that  $r[k]$  tends to converge and settle at a radius of convergence,  $R$ , greater than one for the proposed approach when applied to all test networks.

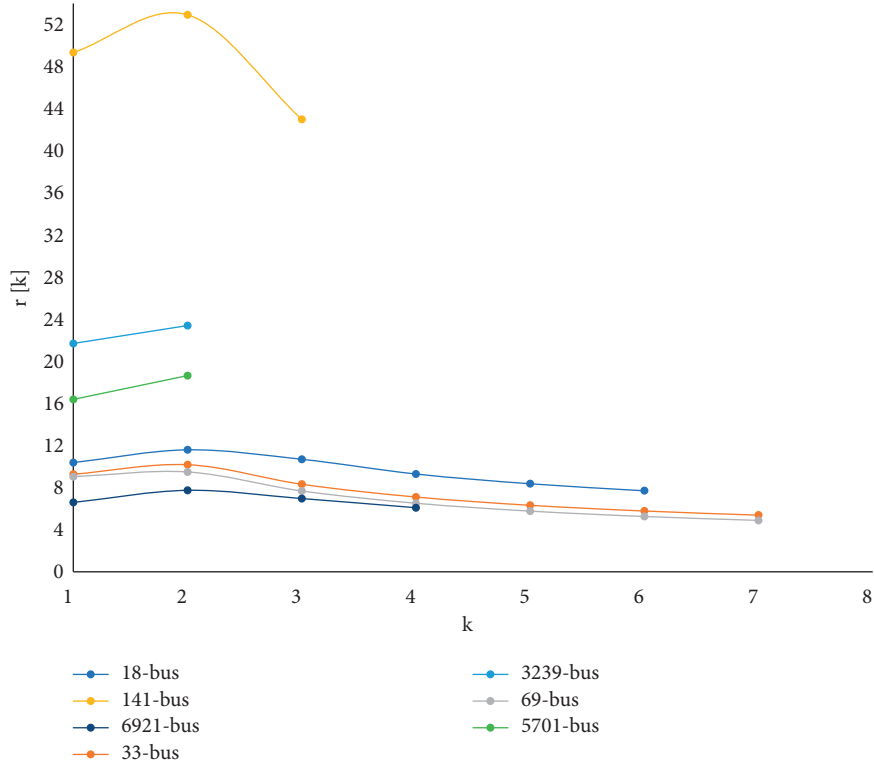


FIGURE 4: Radius of convergence at high DERs penetration levels.

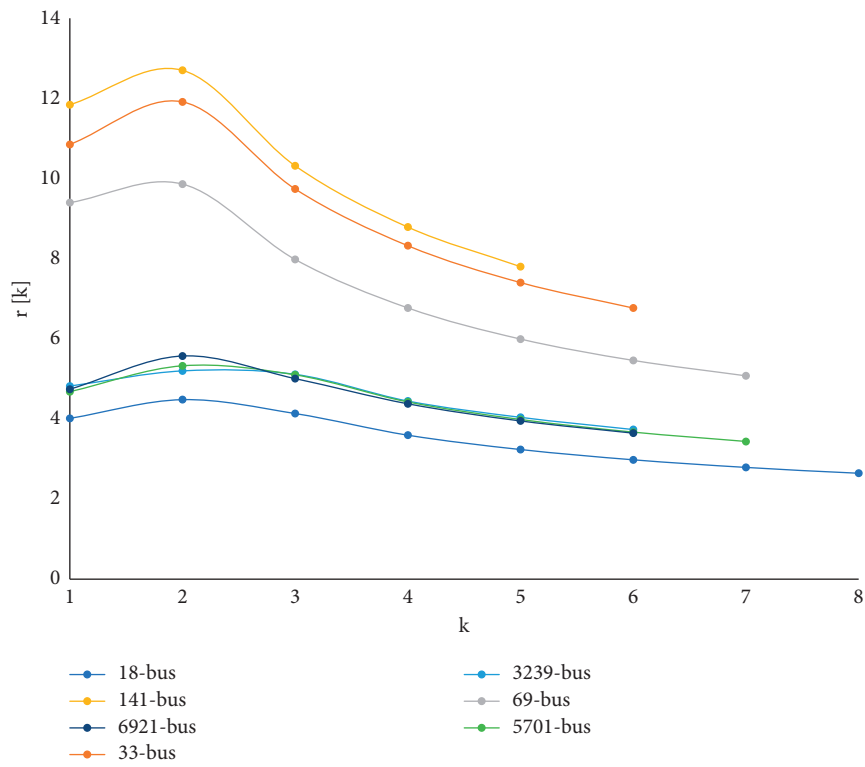


FIGURE 5: Radius of convergence at high loading levels.



TABLE 4: Execution times, number of iterations, and number recursions to reach  $\varepsilon \leq 10^{-4}$ MVA at high loading levels.

Test network	Time (s)					Number of iterations/recursions				
	IZR	HELM	NR	IZG	BFS	IZR	HELM	NR	IZG	BFS
<b>18</b>	0.0203	0.0731	0.8115	0.9747	0.9464	18	11	3	9	10
<b>33</b>	0.0174	0.0941	0.9583	0.8529	N/A	7	6	3	3	N/A
<b>69</b>	0.0156	0.1307	0.8847	0.8608	N/A	8	4	3	5	N/A
<b>141</b>	0.0098	0.1974	0.8708	0.8593	0.995	6	6	3	5	5
<b>3239</b>	0.0341	1.8295	1.4518	1.4025	N/A	7	6	3	5	N/A
<b>5701</b>	0.0439	3.9531	1.9859	1.8577	2.0783	8	7	3	4	8
<b>6921</b>	0.0386	3.6680	2.2196	2.0623	N/A	7	6	3	4	N/A

TABLE 5: High DERs penetration levels.

Test case	DERs penetration levels (MW)
<b>18</b>	4.06
<b>33</b>	126.48
<b>69</b>	87.4
<b>141</b>	154.7
<b>3239</b>	43.75
<b>5701</b>	44.64
<b>6921</b>	2.84

TABLE 6: Execution times, number of iterations, and number recursions to reach  $\varepsilon \leq 10^{-4}$ MVA at high DERs penetration levels.

Test network	Time (s)					Number of iterations/recursions				
	IZR	HELM	NR	IZG	BFS	IZR	HELM	NR	IZG	BFS
<b>18</b>	0.02	0.0958	0.9123	0.8618	0.9276	7	6	3	5	4
<b>33</b>	0.0155	0.1246	0.8879	0.8804	N/A	8	6	3	4	N/A
<b>69</b>	0.0288	0.1316	0.9238	0.8227	N/A	8	7	3	4	N/A
<b>141</b>	0.0075	0.1249	0.9174	0.9019	0.9947	4	4	2	2	2
<b>3239</b>	0.0108	0.512	1.446	1.4026	N/A	3	3	3	2	N/A
<b>5701</b>	0.0137	0.7864	1.9275	1.8949	2.1249	3	3	3	3	3
<b>6921</b>	0.0294	2.8637	2.2346	2.1428	N/A	5	5	3	4	N/A

The iterative approaches (i.e., NR, IZG, and BF) satisfied the convergence criterion slower than the proposed recursive IZR approach due to the need for matrix inversion at each iteration and due to sensitivities to initial power flow conditions (i.e., assuming flat-start initial conditions) as shown in Table 2. Furthermore, Table 2 shows that the recursive HELM approach satisfied the convergence criterion slower than the proposed recursive IZR. This is due to the increased computation time needed to perform Padé approximation for each recursion.

**5.2. High Loading Levels.** The test networks were stressed by increasing loading levels uniformly, assuming constant power factor for each load facility, until the minimum bus voltage limit of 0.9 pu was reached. The corresponding loading levels for each test network are shown in Table 3.  $r[k]$  converged at a radius of convergence greater than one for all test networks as shown in Figure 4, and the sufficient condition for convergence in (34) is satisfied. Furthermore, Figure 5 indicates that  $r[k]$  tends to converge at radius of

convergence lower than the corresponding one in Figure 3 for all test networks due to the increased loading levels. The proposed IZR approach was able to satisfy the convergence criterion faster than the remaining approaches under all test networks as shown in Table 4.

**5.3. High DERs Penetration Levels.** The DERs penetration levels were increased uniformly assuming the initial loading levels in Table 1 until the maximum bus voltage limit of 1.1 pu was reached. The corresponding DERs penetration levels for each test network are shown in Table 5. The sufficient condition for convergence was satisfied for all test networks and  $r[k]$  converged a  $R$  greater than one for all test networks as shown in Figure 4. Furthermore, Figure 4 indicates that  $r[k]$  tends to converge at a radius of convergence higher than the corresponding one in Figure 3 for all test networks due to the decreased loading levels. The proposed IZR approach was able to satisfy the convergence criterion faster than the remaining approaches under all test networks as shown in Table 6.

## 6. Conclusion and Future Work

This paper presented a novel and fast approach for active and meshed AC and DC distribution systems with the ZIP load model and constant power DER model, namely, Implicit Z-Bus Recursive Power Flow Method. In the proposed approach, bus voltages are modelled as holomorphic functions in the constant power injections scaling parameter and the time intensive Padé Approximation are not applied, which resulted in significant reduction in the computational time. Further, the proposed approach allows solving the power flow equations in one recursion if constant power injections are not present while the approaches in [7, 13, 14] require more than one recursion. However, the proposed approach is not guaranteed to reach solution when operating very close to the voltage collapse point. In practice, control actions are usually applied to maintain the voltage levels of the distributions systems above the minimum voltage level and far from the voltage collapse point. Thus, the need to evaluate the holomorphic voltage very close to the voltage collapse point is significantly low for well-behaved distribution systems. The performance of the proposed approach was validated and compared with that of existing approaches using 18-, 33-, 69-, 141-, 3239-, 5701-, and 6921-bus distribution test systems. The comparisons show that the proposed approach satisfies the convergence criterion faster than the other approaches. Future work will consider applying the proposed approach for unbalanced distribution networks with PV buses while considering loads and DERs uncertainties.

## Nomenclature

$\alpha$ :	Complex-valued constant power injections scaling parameter
$\epsilon$ :	Convergence tolerance
$\epsilon_{\max}$ :	Maximum convergence tolerance
$i$ :	Index for bus number except slack bus
$j$ :	Index for bus number
$k$ :	Index for recursion number
$N$ :	Total number of buses without the slack bus
$n$ :	Number of completed recursions
$r[k]$ :	Radius of convergence at the $k^{\text{th}}$ recursion
$R$ :	Radius of convergence
$I_{Li}$ :	Complex constant current injection at bus $i$
$S_{Li}$ :	Complex constant power injection at bus $i$
$Y_{Li}$ :	Constant admittance load at bus $i$
$Y_{ij}$ :	$ij$ complex element of the bus admittance matrix
$Y_{ijse}$ :	$ij$ complex series elements of the bus admittance matrix
$Y_{ijsh}$ :	$ij$ complex shunt elements of the bus admittance matrix
$V_i$ :	Complex bus voltage
$V_{Sl}$ :	Slack bus constant complex voltage
$V(\alpha)$ :	Holomorphic voltage function
$V_{L/M}(\alpha)$ :	Padé approximant for the voltage function
$V^{(k)}(\alpha)$ :	Complex-valued $k^{\text{th}}$ derivative of $V(\alpha)$
$V[k]$ :	Complex-valued $k^{\text{th}}$ coefficient of the maclurin series expansion of $V(\alpha)$

$*$ :	Subscript for denoting conjugate operator
$^{-1}$ :	Subscript for denoting inverse operator
$\text{diag}(\cdot)$ :	Diagonal operator
$\sim$ :	Subscript for denoting all buses except slack bus

### Vectors and Matrices

$\mathbf{I}_L$ :	$N \times 1$ vector with $I_{Li}$ entries
$\mathbf{S}_L^*$ :	$N \times 1$ vector with $S_{Li}^*$ entries
$\text{diag}(\mathbf{S}_L^*)$ :	$N \times N$ diagonal matrix with $S_{Li}^*$ entries
$\mathbf{y}$ :	$N \times 1$ vector describing the mutual admittances between the slack bus and the remaining buses
$\tilde{\mathbf{Y}}^{-1}$ :	$N \times N$ matrix implicitly indicating the bus impedance matrix (i.e., Z-bus) for all buses except slack bus
$\tilde{\mathbf{Y}}_{se}$ :	$N \times N$ matrix the includes the series elements of the bus admittance matrix for all buses except slack bus
$\tilde{\mathbf{Y}}_{sh}$ :	$N \times N$ matrix the includes the shunt elements of the bus admittance matrix for all buses except slack bus
$\text{diag}(\mathbf{Y}_L)$ :	$N \times N$ diagonal matrix with $Y_{Li}$ entries
$\tilde{\mathbf{V}}$ :	$N \times 1$ vector with $V_i^*$ entries
$\tilde{\mathbf{W}}$ :	$N \times 1$ vector with $V_i^{*-1}$ entries
$\mathbf{L}, \mathbf{U}, \mathbf{P}$ :	Lower, upper, and permutation matrices for LU decomposition.

## Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

## Conflicts of Interest

The author declares that there are no conflicts of interest.

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