Research Article

The RBF-FD and RBF-FDTD Methods for Solving Time-Domain Electrical Transient Problems in Power Systems

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Abstract

In this paper, the development and application of the radial basis function-finite difference (RBF-FD) method and the RBF-finite difference time domain (RBF-FDTD) method for solving electrical transient problems in power systems that are defined by the time-dependent ordinary differential equations (ODEs) and the time-dependent partial differential equations (PDEs), respectively, are presented. RBFs such as Gaussian (GA), Multiquadric (MQ), Inverse Quadric (IQ), and Inverse Multiquadric (IMQ) are used in these numerical methods to formulate the central finite difference approximations of the first- and second-order derivatives of a function. The algorithm of selecting "optimal" shape parameters for our basis functions is also applied, specifically to increase the accuracy of the suggested methods with regard to high accuracy needs. Finally, the accuracy, effectiveness, and applicability of our new approaches are evaluated through simulations of the switching transient voltages on a typical electrical circuit and a 220kV single-phase transmission line, lightning-induced voltages on a 110kV single-phase overhead distribution line, and transient voltages along two horizontal grounding electrodes excited by lightning impulse sources. The obtained numerical results demonstrate that our proposed RBF-based numerical approaches compare favorably to the traditional numerical methods.

1. Introduction

The transient waveforms of current and voltage that occur on power system elements such as overhead power lines and grounding systems as a result of direct or indirect lightning strikes, short-circuit faults, and circuit breaker switching operations [1] propagate along power lines to line-connected electric devices. These transient waveforms can significantly damage insulators and equipment and interrupt electricity, depending on the existing time and peak value. As a result, one of the primary tasks of overhead power line operation and economic insulation design is to accurately calculate these waveforms.

PDEs can express the variation of current and voltage during the transient state in either the frequency domain or the time domain. The latter, known as ODEs, is a commonly used form in which the variables of current and voltage depend only on time and are solved using classical methods such as Laplace or Fourier transforms [2], traveling wave methods such as the lattice-diagram method, the method of characteristics, and the state-space method [3–5]. Furthermore, for modeling electrical transient voltage waveforms, EMTP-ATP software, which is based on Bergeron’s method, has become popular.

Recently, several numerical methods [5] for solving ODEs and PDEs in engineering have been proposed and widely used. For example, in electromagnetic transient problems, the most popular time-domain discretization techniques, such as FDTD [6, 7], finite element time domain (FETD) [8, 9], the method of lines [10], transmission line
matrix [11], wavelet transforms [12, 13], and so on, directly solve transient equations. These methods are well known for their efficiency, power, and robustness in providing numerical solutions to time-dependent differential equations. The FDTD method, in particular, is the most widely used mathematically simple analysis technique for time-dependent electromagnetic applications. It approximates PDE derivatives in time and space at the same time and then solves PDEs using either a matrix analysis or an iterative method.

Over the last decade, RBF-based methods have been developed and successfully applied to science and engineering problems, as well as electromagnetic problems [14–18]. These methods produce solutions that are frequently far more accurate than traditional numerical methods. As a result, they have gained increasing attention from the scientific community interested in solving differential equations. RBF meshfree methods (global and local RBF meshfree method) in the time domain and use a difference scheme (called RBF-FD scheme) to compute the solutions on them is presented in [19]. This approach is similar to the FETD method. It means that at each time step, the spatial component is solved using the RBF meshfree method. On the other hand, new numerical methods combining RBFs and traditional approaches have been developed. These include RBF-DQ [20] and RBF-FD [21–30], which are used in 1-D and 2-D spatial domains, respectively. Unfortunately, the application of these improved methods to time-dependent ODEs and PDEs has not been widely adopted, particularly in the context of electrical transient problems.

In this paper, we develop and apply the RBF-FD method and the RBF-FDTD method, in which central RBF-FD approximations of the first- and second-order derivatives of the function are formulated in the matrix form (see Appendix A) using RBFs of GA, MQ, IQ, and IMQ types based on the theory presented in [22], to modeling the transient voltage waveforms of a typical electrical circuit and a 220 kV single-phase transmission line defined by time-dependent ODEs, lightning-induced voltages on a 110 kV single-phase distribution line, and lightning transient voltages along two horizontal grounding electrodes computed using this new method agree very well with those obtained by our traditional FDTD method and shown in [38].

The structure of this paper is as follows: the RBF-FD approximations derived using RBFs are shown in Section 2. In Section 3, the algorithm for determining the “optimal” shape parameter is introduced. The results of this work are presented in Section 4, where we apply the RBF-FD method, other traditional numerical methods, and EMTP-ATP software to the transient analysis of a typical electric circuit and a 220 kV single-phase transmission line, as well as the RBF-FDTD method to model lightning-induced voltages on a 110 kV single-phase overhead distribution line and lightning transient voltages along two horizontal grounding electrodes. Section 5 will include a remark and a conclusion.

2. RBF-FD Approximations

In general, because of its high effectiveness and simplicity, the traditional FD method, which expresses all FD approximations by using the Taylor series expansions of the first- and second-order derivatives of the function at an observed point [5], is recognized as the state-of-the-art numerical technique. In other words, this method is a highly effective computational tool. However, because the series is truncated with a second-order remainder, this method is usually of lower accuracy.

RBFs have received a lot of attention from the scientific community in the last decade because of their “meshfree” or “meshless” nature. As a result, RBFs are now used to devise derivative approximations instead of Taylor’s series. Some early versions of the RBF-FD method were applied [20–22]. We will present the formulae of the central RBF-FD approximations used for developing our proposed methods in Section 4 for solving time-dependent electrical transient problems.

Before we solve differential equations by using the RBFs, we recall that the $m$th-order derivative of a function $f$ at a point $x_i$ in the traditional finite differences is approximated by a linear combination of the values of the function $f$ at some adjacent points as follows [20–29]:

$$
\frac{d^m f}{dx^m}(x_i) \approx \sum_{j=1}^{N} w_{ij}^{(m)} f(x_j), \quad j = 1, \ldots, N, \tag{1}
$$

where $N$ is the total number of grid points, $f(x_i)$ is the value of $f$ at the grid point $x_i$, and $w_{ij}^{(m)}$ are the weighting coefficients associated with the $m$th-order derivative at the point $x_i$ and are computed using RBF interpolation.

Implementing (1) for the scheme of three equispaced points $N = 3$ of $x_i - \Delta x, x_i$, and $x_i + \Delta x$ in 1-D domain, the general expressions of RBF-FD approximations of the first- and second-order derivatives are written as follows:
\[
\frac{d\Phi}{dx}(x_i) = \alpha_1\Phi(x_i - \Delta x) + \alpha_2\Phi(x_i) + \alpha_3\Phi(x_i + \Delta x),
\]
\[
\frac{d^2\Phi}{dx^2}(x_i) = \beta_1\Phi(x_i - \Delta x) + \beta_2\Phi(x_i) + \beta_3\Phi(x_i + \Delta x),
\]
in which the function \(\Phi(\|x - xi\|_2)\) is the basis function that depends only on the distance norm from each center to some adjacent points and defined as in Table 1. The weighting coefficients \(\alpha_1, \alpha_2, \alpha_3\) and \(\beta_1, \beta_2, \beta_3\) are formulated as follows.

Two linear equations of (2) and (3) can be rewritten in following matrix forms as

\[A = \Psi\alpha,\]
\[B = \Psi\beta,\]
where \(A\) is a vector \([1 \times 3]\) that contains the elements of \(\Phi(\|x - xi\|_2)\) and \(B\) is also a vector \([1 \times 3]\) of \(\Phi(\|x - xi\|_2)\) elements. The system matrix \(\Psi [3 \times 3]\) has entries of \(\Psi_{i,j} = \Phi(\|x_i - x_j\|_2)\). Therefore, the unknown weighting coefficients in (4) and (5) are given by

\[\alpha = \Psi^{-1}A,\]
\[\beta = \Psi^{-1}B.\]

It can be noticed that using the GA, MQ, IQ, and IMQ for the function \(\Phi(\|x - xi\|_2)\) of the abovementioned procedure, we will find the unknown coefficients of \((\alpha_{GA}, \beta_{GA}), (\alpha_{MQ}, \beta_{MQ}), (\alpha_{IQ}, \beta_{IQ}),\) and \((\alpha_{IMQ}, \beta_{IMQ})\), respectively, in detail (see Appendix A).

### 3. Choosing “Optimal” Shape Parameters

The shape parameter \(c\) is critical in the theory and practice of the RBF-FD method, as presented in previous literature [20–29], and influences not only the accuracy of the solution but also its numerical stability. The accuracy of the RBF-FD method will be the same as that of the traditional FD method if this parameter \(c\) becomes large, which means \(c\) tends to infinity. Similarly, a small shape parameter causes numerical instability in the RBFs we use. Finding an “optimal” value of \(c\) that improves solution accuracy is thus an important task for researchers that is still an open issue (See [15]).

Many different methods for determining an “optimal” shape parameter of the RBF-FD method have been investigated and presented in numerous previous papers [21, 23–25]. Here, we will apply the algorithm proposed by Bayona et al. [23] for selecting “optimal” shape parameters to our RBFs, which can be described as follows.

Considering that the 1-D domain is divided by \(N\) scattered points, in which NI denotes the total number of interior points of the solution domain, thus, \(N\)-NI is two boundary points.

![Image](image-url)

**Table 1: List of some common RBFs.**

<table>
<thead>
<tr>
<th>Names of RBFs</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian (GA)</td>
<td>(e^{-r/c^2})</td>
</tr>
<tr>
<td>Multiquadrics (MQ)</td>
<td>(\sqrt{r^2 + c^2})</td>
</tr>
<tr>
<td>Inverse-quadrics (IQ)</td>
<td>(1/(r^2 + c^2))</td>
</tr>
<tr>
<td>Inverse multiquadrics (IMQ)</td>
<td>(1/\sqrt{r^2 + c^2})</td>
</tr>
</tbody>
</table>

where \(r = \|x - x_i\|_2\) and \(c > 0\) is the shape parameter.

The RBF-FD error is given by

\[E(c) = f - \hat{f}^h(c),\]
where \(\hat{f}^h = [f^h(x_1), \ldots, f^h(x_N, c)]^T\) is the RBF-FD approximation and \(f = [f(x_1), \ldots, f(x_N)]^T\) is the exact solution.

For finding the “optimal” shape parameter \(c^*\), the approximation error \(E(c)\) of (8) is minimized by the following expression as

\[\|E(c^*)\|_\infty = \min\|f - \hat{f}^h(c)\|_\infty, \quad \text{if } f \text{ is known},\]
\[\|E(c^*)\|_\infty = \min\|\Gamma^{-1}(c)e(c)\|_\infty, \quad \text{if } f \text{ is not known}.\]

In the case of practical problems where we do not know the exact solution, the RBF-FD error can be obtained by the following expression as

\[\|E(c^*)\|_\infty = \min\|\Gamma^{-1}(c)e(c)\|_\infty,\]
where \(\Gamma(c)\) is a NI \(\times\) NI sparse matrix that contains the weighting coefficients of (1) and \(e(c)\) is a vector of the local error of \(e_x(x_i, c)\).

In addition, we can also find an approximate value of \(c^*\) to the “optimal” shape parameter of \(c^*\) as

\[\|E(c^*)\|_\infty = \min\|\Gamma^{-1}(c)e_x(c)\|_\infty,\]
where \(e_x(c)\) is the estimated local error.

For a much more detailed presentation and discussion of this algorithm, we refer the readers to [23].

### 4. Numerical Results


As a first example of RBF-FD application to ODEs, we used this method to compute the transient voltage on the capacitor of the typical electric circuit shown in Figure 1.

Consider a switching operation in which the voltage source in the circuit of Figure 1 switched on at \(t = 0\), the voltage across the capacitor and the current satisfy the system of the following equations as

\[Ri_L(t) + L\frac{di_L(t)}{dt} + v_c(t) = 5,\]
\[i_L(t) = C\frac{dv_c(t)}{dt}.\]
We can obtain the time-dependent 2nd-order ODE of the capacitor voltage by first differentiating the current variable in (13) with respect to \( t \) and then substituting it into (12).

\[
LC \frac{d^2 v_c(t)}{dt^2} + RC \frac{dv_c(t)}{dt} + v_c(t) = 5. \tag{14}
\]

The analytical solution of the capacitor voltage of (14) obtained using the Laplace transform method is given by

\[
v_c(t) = 5 - \frac{\cos \left( \sqrt{3919/20} t + \arctan \left( -9/\sqrt{3919} \right) \right)}{\cos \left( \arctan \left( -9/\sqrt{3919} \right) \right)} 5e^{-0.45t}. \tag{15}
\]

To apply the RBF-FD method to (14), the unknown weighting coefficients of the RBF-FD approximations of the temporal derivatives are formulated the same as (6) and (7) by replacing \( \Delta x \) with \( \Delta t \), thus the RBF-FD equation of the capacitor voltage can be written as

\[
v^{n+1}_c = \frac{1}{RC\alpha_{1t} + LC\beta_{1t}} \left[ 5 - \left( RC\alpha_{2t} + LC\beta_{2t} + 1 \right) v^n_c - \left( RC\alpha_{1t} + LC\beta_{1t} \right) v^{n-1}_c \right]. \tag{16}
\]

We divided the time interval of ten seconds into 1000 to compare numerical solutions. Figure 2 shows the numerical results of the capacitor voltage calculated using the IMQ-FD method and many different approaches. This figure shows that all obtained solutions are well superimposed. We used two common error norms to evaluate the accuracy of all solutions: the root-mean-square error (RMS-error) and the infinfinite error \( \|E\|_{\infty} \). Table 2 shows the results of two error norms for all methods. It is clear that the IMQ-FD method is more accurate than all the rest of the methods.

Furthermore, it is critical to understand that the accuracy of the RBF-FD method is heavily influenced by the shape parameter \( c \). To demonstrate this, we applied the RBF-FD method to the algorithm [23] for determining “optimal” shape parameters presented in Section 3. The obtained results of two error norms are shown in Figure 3 and Table 3. The accuracy of the RBF-FD method is higher than that of the traditional FD method at the values of the “optimal” shape parameters, as shown in Table 3, and in this case, the IMQ-FD realizes the smallest error when compared to the MQ- and GA-FD versions. It is especially noticeable in Table 3. When the exact solution is unknown, Table 3 shows that the values of \( c \cdot e \) and \( \|E(c^*)\|_{\infty} \) obtained using (11) are equivalent to the values of \( c \cdot e \) and \( \|E(c^*)\|_{\infty} \). This algorithm will be used effectively in Case 2 and Case 4 of this section.

4.2 Case 2: RBF-FD Method Applied to the Single-Phase Transmission Line. In order to apply the RBF-FD method to the second example of ODEs, we examine the equivalent circuit diagram of the 220 kV short transmission line using...
a lumped-parameter model in Vietnam, which is depicted in Figure 5. These parameters are assigned to this line: \( R = 11.42 \Omega, L = 292.3 \text{ mH}, C = 1.54 \mu\text{F} \). This line is equivalent to a two-port model, allowing us to discover an analytical solution using the “lsim” function in MATLAB.

The single-phase voltage source of \( U_0 = 220\sqrt{2}\sin(100\pi t) \text{ kV} \) is switched on at \( t = 0 \) in the single-phase equivalent circuit of Figure 5, and the receiving-ended voltage and current must satisfy the following set of equations:

\[
\begin{align*}
    u_1 &= u_2 + Ri + L \frac{di}{dt}, & (17) \\
    i &= C \frac{du_2}{dt}. & (18)
\end{align*}
\]

First, by differentiating the current variable in (18) with respect to \( t \) and then substituting it into (17), the time-dependent 2nd-order ODE of the receiving end voltage can be obtained as follows:

\[
LC \frac{d^2 u_2}{dt^2} + RC \frac{du_2}{dt} + u_2 = U_0. \quad (19)
\]

Then, by applying the RBF-FD approximations of the first-and second-order derivatives in the temporal domain to the receiving end voltage in (19), we derive the RBF-FD equation as (20).

As shown in Figure 6, the solutions of the transient voltage at the receiving end of this line calculated in the time domain using the RBF-FD, conventional FD, wavelets, state variable, and TLM techniques, as well as the EMTP-ATP software, compare favorably with the analytical solution. It is noted in this figure that all the obtained solutions are well superimposed.

In general, the number of time steps used affects the accuracy of the solution in transient modeling. In this example, we divided the one-second time interval by 1000, which also means that the time step utilized in this case is \( 1.10^{-6} \). To evaluate the accuracy of solutions, we use the root-mean-square error (RMS-error) and the infinitive error (\( \|E]\infty \)) error norms to all techniques. Two error norms for all approaches are displayed in Table 4 where it can be seen that the IMQ-FD solution most closely matches the analytical one, while the EMTP-ATP error norm is the lowest. There is a very high level of agreement about accuracy and applicability to the RBF-FD method in practical application.

It is extremely important to be aware of the fact that the accuracy of RBF-FDM is significantly dependent on the shape parameter \( c \) that is selected, as is described in Section 3. In order to demonstrate this, we apply the algorithm for finding the "optimal" shape parameters of three RBFs (IMQ, MQ, and GA) that are presented in Section 3 to (20) to produce a better solution of the RBF-FDM for the transient voltage of the

![Figure 5: The single-phase equivalent circuit diagram of the 220 kV transmission line.](image)
Figure 6 illustrates the behavior of the RMS-error and \(\|E\|_\infty\) norms of the RBF-FD and FD methods for the transient voltage solutions while increasing the total number of time steps in order to demonstrate that the RBF-FDM is more efficient than the traditional FDM while maintaining similar accuracy. Again, using a few hundred points of time steps in the RBF-FDM, it is possible to find a solution that has the same level of accuracy as traditional numerical methods using more than a thousand points.

4.3 Case 3: RBF-FDTD Method Applied to the Single-Phase Distribution Line. As the first example of PDEs, to predict the transient behavior of the long overhead distribution line excited by external electromagnetic fields of nearby lightning strikes, the time-dependent PDEs used in our study to describe the lossless distribution line excited by lightning electromagnetic fields are the field-to-transmission line coupling equations proposed by Agrawal et al. in 1980 [31], as follows:

\[
\frac{\partial V_r^i(x, t)}{\partial x} + L \frac{\partial I^i(x, t)}{\partial t} = E_x^i(x, h, t),
\]

\[
\frac{\partial I(x, t)}{\partial x} + C \frac{\partial V(x, t)}{\partial t} = 0,
\]

in which \(L\) and \(C\) are the per-unit length inductance and capacitance, \(V_r^i(x, t)\) and \(I^i(x, t)\) are the scattered voltage and the total current along the line, respectively, and \(E_x^i(x, h, t)\) is the incident horizontal electric field along the \(x\) axis at the phase-conductor height.

The total induced voltage at any observation point along the line can be calculated using the Agrawal et al. model. This voltage can be derived from the scattered voltage and the finite integral of the incident vertical electric field, which is also referred to as the incident voltage [32–35, 39] using the following expression:

\[
V^T(x, t) = V^s(x, t) - \int_0^h E_z^i(x, h, t)dz = V^s(x, t) - V^i(x, t),
\]

where \(V^s(x, t) = \int_0^h E_z^i(x, h, t)dz \approx -hE_z^i(x, 0, t)\) is the incident voltage and the incident vertical electric field that does not vary in the height range of the line \(0 < z < h\).

In this study, we determined the scattered voltage on the overhead distribution line by applying Taylor’s series expansions with the second-order truncating term that was proposed by Paolone et al. [33] in the time domain. Hence, both the variables of the distributed voltage and the total current can be expressed by (23) and (24).

\[
V^s(x, t) = V^s(x, t_0) - \Delta tC^{-1}\frac{\partial I(x, t)}{\partial x} + \frac{\Delta t^2(LC)^{-1}}{2}\left(\frac{\partial^2 V^s(x, t)}{\partial x^2} - \frac{E_x^i(x, t)}{\partial x}\right) + O(\Delta t^3),
\]

\[
I(x, t) = I(x, t_0) - \Delta tL^{-1}\left(\frac{\partial V^s(x, t)}{\partial x} - E_x^i(x, t)\right) + \frac{\Delta t^2(LC)^{-1}}{2}\left(\frac{\partial^2 I(x, t)}{\partial x^2} - C\frac{\partial E_x^i(x, t)}{\partial t}\right) + O(\Delta t^3).
\]
The RBF-FDTD approach was utilized in this study in order to solve the equation system consisting of (23) and (24). This method requires that the 1-D approximations described in Section 2 be applied to each individual variable of $t$ and $x$. Hence, the RBF-FDTD scheme of the second order can be formulated as (25) and (26).
We denote here.

(i) $V^n_k = V^\ast[(k-1)\Delta x, n\Delta t]$
(ii) $I^n_k = I^\ast[(k-1)\Delta x, n\Delta t]$
(iii) $k$ is the $k$th-spatial step
(iv) $n$ is the $n$th-time step
(v) $\Delta x, \Delta t$ are the size of spatial and time steps, respectively
(vi) $Nx, NT$ are the number of spatial and time steps, respectively

The boundary conditions in terms of the scattered voltage and the total current with the loads at two line terminations are given by

\begin{align*}
V^n_i &= \int_0^b E^n_x(0,0,t)dz - Z_A I^n_i, \quad (27)
V^n_{N_x+1} &= \int_0^b E^n_x(L,0,t)dz + Z_B I^n_{N_x+1}, \quad (28)
\end{align*}

where $Z_A$ and $Z_B$ are chosen equal to the characteristic impedance of the line to avoid reflected waves at two line terminations.

We consider an example of the 110 kV single-phase distribution line as shown in Figure 9. The height of the line is 10 meters, the radius of the line is 5 millimeters, the length of the line is assumed to be one kilometer, the characteristic impedance is 500 $\Omega$, and the location of the lightning stroke is $y_0 = 50$ meters from the center of the line. The lightning current that is being utilized in this study is the subsequent return stroke. The peak value of the channel base current is 12 kA, the maximum time-derivative is going to be 50 kA/$\mu$s, and the velocity of the return stroke is equal to $3.10^8$ m/s. In this particular case study, all of the parameters are the same as those in [32–36].

The MQ, IMQ, and GA RBF-FDTD methods are utilized for the numerical implementation in order to calculate the lightning-induced voltage waveform on the line [36] that is described by the equation system consisting of (25) and (26). Figure 10 presents the lightning-induced voltage on the entire line over time that was obtained by utilizing the MQ-FDTD approach.

In this work, we can assume the solution of LIOV [37] as a benchmark one so that we can determine the “optimal” shape parameter and evaluate how accurate the results are. Figure 11 displays the RMS error that was produced by the MQ-FDTD technique inside the value ranges that were specified for both $c_s$ and $c_t$. From this finding, we are able to find the pair of “optimal” shape parameter values ($c_{s, opt}$, $c_{t, opt}$) that corresponds to the minimum RMS-error norm, and the lightning-induced voltage at the center of the line is illustrated in Figure 12. When compared to the FDTD solution, it has been demonstrated that the MQ-FDTD solution can be well superimposed with the LIOV one. In order to achieve these outcomes, the number of temporal steps selected for both approaches is 1000 and the number of spatial steps selected for both approaches is 200.

On the basis of this optimal algorithm of ($c_{s, opt}$, $c_{t, opt}$), the comparison of the RMS-error and infinitive error norms between the RBF-FDTD and FDTD methods is illustrated as in Table 6. Furthermore, Figure 13 is an illustration of the
The findings indicate that the RBF-FDTD method produces highly accurate solutions when the “optimal” shape parameters are chosen, and these solutions will converge to the conventional FDTD solution corresponding to larger values of $(c_x, c_t)$.

RMS-error norm versus $c_x$. The findings indicate that the RBF-FDTD method produces highly accurate solutions when the “optimal” shape parameters are chosen, and these solutions will converge to the conventional FDTD solution corresponding to larger values of $(c_x, c_t)$.

Figure 9: The geometrical model of modeling lightning-induced voltages on the single-phase distribution line [32, 33].

Figure 10: The lightning-induced voltage on the whole line is obtained using the MQ-FDTD method.

Figure 11: RMS-error norm of the MQ-FDTD method versus shape parameters $(c_x, c_t)$ for Case 3.

Table 6: Comparison of error norms between the RBF-FDTD and FDTD methods for Case 3.

| Methods     | RMS-error | $|E|_\infty$   |
|-------------|-----------|---------------|
| MQ-FDTD     | 7.294901e-04 | 3.440227e-03 |
| (c^*_x = 40, c^*_t = 2.858000e-08) | | |
| IMQ-FDTD    | 7.497511e-04 | 3.458755e-03 |
| (c^*_x = 62.73, c^*_t = 5.541600e-08) | | |
| GA-FDTD     | 7.536582e-04 | 3.458550e-03 |
| (c^*_x = 50.91, c^*_t = 4.691100e-08) | | |
| FDTD        | 2.300149e-03 | 3.768153e-03 |

Figure 12: Solutions of the lightning-induced voltage at the line center for Case 3.
A comparison of the RMS-error norm while increasing the total number of spatial steps in the MQ-FDTD and FDTD methods is illustrated in Figure 14 to evaluate the effectiveness of using the RBF-FDTD method for the practical problem of power systems defined by time-dependent PDEs. This is demonstrated by the fact that the RBF-FDTD method is significantly more efficient than the traditional FDTD method. This means that we can obtain the RBF-FDTD solution with only a few dozen points, which are still more accurate than the conventional FDTD solution with hundreds or thousands of points. As a result, the computational cost of the RBF-FDTD method is significantly reduced, particularly when applied to the transient analysis of long power lines or large electric networks.

4.4. Case 4: RBF-FDTD Method Applied to Two Horizontal Grounding Electrodes. In this example, we showed the use of the RBF-FDTD approach for simulating the lightning transient voltages along two 20 m and 100 m horizontal grounding electrodes buried at the depth of 0.5 m in the soil with \( \varepsilon_r = 50, \rho = 100, \) and \( \mu_r = 1 \) as in Figure 15. Every electrode is a copper conductor that is 15 millimeters in diameter.

The lightning impulse current source that is injected at the terminal end of the grounding electrode is expressed for the lightning transient analysis in the form of a double exponential function as

\[
i_e(t) = I_0(e^{-\theta_1 t} - e^{-\theta_2 t}),
\]

where \( I_0 = 12.935 \text{kA}, \theta_1 = 190099 \text{1/s}, \) and \( \theta_2 = 2922879 \text{1/s} \). It notices that all these parameters are chosen the same as that in [38] for comparing our RBF-FDTD and FDTD solutions with that shown in [38].

The nonuniform transmission line model that was used for modeling the transient voltage and current along the grounding electrode is illustrated as seen in Figure 16. The equations for the transient voltage \( v(x,t) \) and current \( i(x,t) \) distributions are expressed using the telegrapher’s equation system based on the transient electromagnetic theory as follows:

\[
-\frac{\partial v(x,t)}{\partial x} = R_e i(x,t) + L_e(x,t) \frac{\partial i(x,t)}{\partial t},
\]

\[
-\frac{\partial i(x,t)}{\partial x} = G_e(x,t) v(x,t) + C_e(x,t) \frac{\partial v(x,t)}{\partial t},
\]

in which, \( R_e \) is the per-unit length resistance. \( L_e(x,t), \) \( G_e(x,t), \) and \( C_e(x,t) \) are the per-unit length inductance, conductance, and capacitance, respectively, and are the functions of the position variable \( x \) and the temporal variable \( t \). For a much more detailed formulation and calculation of these parameters, we refer the readers to [38].

\[
e_{k}^{n+1/2} = \left[ \frac{1}{(L_{e,k}^{n+1/2}/\Delta t) + (R_{e}/2)} \right] \left[ \left( L_{e,k}^{n+1/2}/\Delta t - R_{e}/2 \right) e_{k}^{n-1/2} - \alpha_{1x} v_{k-1}^{n} - \alpha_{3x} v_{k+1}^{n} \right],
\]

\[
v_{k}^{n+1} = \left[ \frac{1}{(C_{e,k}^{n+1/2}/\Delta t) + (G_{e,k}^{n+1}/2)} \right] \left[ \left( C_{e,k}^{n+1}/\Delta t - G_{e,k}^{n+1}/2 \right) v_{k}^{n} - \alpha_{1x} i_{k-1}^{n} - \alpha_{3x} i_{k+1}^{n} \right].
\]

In this study, we proposed a different approach of the RBF-FDTD method for solving the equation system of (30) and (31), which is to apply the conventional FD approximation to the temporal derivative term and the RBF-FD approximation to the spatial derivative term. Thus, the RBF-FDTD equation systems for the lightning transient voltage and current are denoted as (32) and (33), respectively. Due to the fact that we only need to find the \( c_x \) shape parameter of the spatial derivative term, this approach is noticeably simpler and more cost-effective than the RBF-FDTD approach that was used in Case 3 and [40].

To select the \( c_x \) shape parameter for this engineering problem for which there is no an exact solution, we used expression (11) to compute the values of \( c_x^* \) and \( \|E(c_x^*)\|_{\infty} \), and the algorithm for this calculation is provided in Appendix B. The result for the 100 m grounding electrode is depicted in Figure 17. It has been demonstrated that the errors of four RBFs are minimized at four different \( c_x^* \) values. However, as \( c_x^* \) approaches \( 10^2 \), these errors will converge to the same constant value. It is worth noting that when \( c_x \) is chosen in the range of \([5, 100]\), the accuracy of the solution obtained by the RBF-FDTD method is always greater than that of the traditional FDTD method.

Based on the preceding discussion, the shape parameter of the RBFs is fixed at \( c_x = 50 \) in Case 4, the 20 m and 100 m long grounding electrodes are divided into 20 and 100 segments, respectively, and the number of time steps chosen is 900. The computed lightning transient voltages at three A, B, and C points along two horizontal grounding electrodes, with A corresponding to the left terminal end, B corresponding to the middle, and C corresponding to the right terminal end as shown in Figure 16, are presented in Figures 18 and 19, respectively. The transient waveforms simulated by the RBF-FDTD method outperform those obtained by our FDTD method and shown in [38]. The findings indicate that there is a good agreement, and the RBF-FDTD method works best with near-optimal shape parameters. These numerical results once again demonstrate the feasibility and efficiency of the proposed RBF-FDTD method.
Figure 14: RMS-error norm of the MQ-FD and FD methods versus the total number of spatial steps for Case 3.

Figure 15: Model of horizontal grounding electrodes buried in earth [38].

Figure 16: Equivalent circuit model of nonuniformly lumped parameters of the electrode for Case 4 [38].

Figure 17: Comparison of an infinitive error norm of the RBF-FDTD method versus the estimated shape parameter.
Here, we can remark some main points as follows:

(i) Generally, the RBF-based FD methods are more accurate than the conventional FD methods in the small value range of the shape parameter. Therefore, finding the “optimal” shape parameter is particularly critical for the application of RBF-based FD algorithms.

(ii) Applying the RBF-FD and RBF-FDTD methods to the time-dependent ODEs or PDEs, the coefficients of \((a_1, a_2, a_3)\) and \((\beta_1, \beta_2, \beta_3)\) of the RBF-FD approximations in the temporal domain can be easily formulated the same as that in the spatial domain presented in Section 2 by replacing \(\Delta x\) with \(\Delta t\).

(iii) There are two RBF-FDTD approaches, the first one is to use the RBF-FD approximation for both the temporal and spatial derivatives, and therefore, we must find an optimal value pair of \(c_t\) and \(c_x\) as shown in Figure 11. The second approach is to use the FD approximation for the temporal derivative and the RBF-FD approximation for the spatial derivative; thus, all that remains is to find the optimal value of \(c_x\). The second approach is simpler and more effective than the first, but it still has a high level of accuracy.

(iv) In some cases, if problems do not require too high accuracy, we can reduce the total number of temporal and spatial steps while still meeting the accuracy requirement. Therefore, the computational cost of the RBF-FD and RBF-FDTD methods will be significantly reduced (see Figures 4, 8, and 14). Thus, the RBF-based FD methods are more cost-effective than traditional FD methods.

(v) In particular, for every practical problem for which we do not have the exact solution, we first use the traditional FD method to solve it. The estimated shape parameter for the RBF-FD technique may then be found using the algorithm for determining the “optimal” shape parameter presented in Section 3 and \([22, 23]\). In our experience, the shape parameter may be chosen \(c_x \in [1, 100]\) while still ensuring that the RBF-FD approach is more stable and performs better than the standard FD method (see Figure 17 and Tables 3 and 5).

Finally, two RBF-FD and RBF-FDTD approaches for simulating the transient voltage of a typical electric circuit, lightning-induced voltages on the power distribution line, and lightning transient voltages along two horizontal grounding electrodes are presented. In addition, parameters such as the type of RBFs, the number of temporal and spatial steps, and the technique for selecting “optimal” shape parameters are used to increase the accuracy of RBF-FD and RBF-FDTD solutions. The numerical results obtained by the proposed methods are compared with the numerical results obtained by many other existing numerical methods, and it can be determined that (i) the first advantage is that, using the same number of discretization points, our proposed methods are more accurate than other numerical methods when choosing the “optimal” shape parameter. It should be noted that the computation cost of the RBF-based FD method, in general, is much higher than that of the traditional methods; (ii) the second advantage is that we can reduce the number of discretization steps in the RBF-based FD approach while still obtaining the same level of accuracy as with the conventional FD method that requires more discretization steps. In this case, the computation cost can be
significantly reduced; (iii) our proposed methods can be applied to any practical transient problem without the need for an exact solution, such as lossless, uniform, nonuniform lumped, and distributed parameter lines, as well as complex electrical networks in power systems, by using the algorithm for estimating the shape parameter. In the near future, this will all be done.

Appendix

A. The Weighting Coefficients of RBF Approximations

We consider the GA RBF as presented in Table 1, the first- and second-order derivatives with respect to the variable \( x \) are presented as

\[
\begin{bmatrix}
-\frac{2\Delta x}{c^2} e^{-\Delta x^2/c^2} \\
0 \\
\frac{2\Delta x}{c^2} e^{-\Delta x^2/c^2}
\end{bmatrix}
\begin{bmatrix}
\frac{2}{c^2} \left( \frac{2\Delta x^2}{c^2} - 1 \right) e^{-\Delta x^2/c^2} \\
-\frac{2}{c^2} \\
\frac{2}{c^2} \left( \frac{2\Delta x^2}{c^2} - 1 \right) e^{-\Delta x^2/c^2}
\end{bmatrix}
\begin{bmatrix}
\alpha_{1,GA} \\
\alpha_{2,GA} \\
\alpha_{3,GA}
\end{bmatrix},
\]

We can reformulate (A.3) and (A.4) as follows:

\[
A_{GA} = \Psi_{GA} \alpha_{GA},
\]

and

\[
B_{GA} = \Psi_{GA} \beta_{GA}.
\]

The unknown weighting coefficients in (A.5) and (A.6) can be calculated by

\[
\alpha_{GA} = \Psi_{GA}^{-1} A_{GA},
\]

\[
\beta_{GA} = \Psi_{GA}^{-1} B_{GA}.
\]

It can be noticed that applying the abovementioned procedure of (A.1) to (A.8) to the MQ and IMQ RBFs in the same way, we can also find the unknown weighting coefficients of (\( \alpha^{MQ}, \beta^{MQ} \)), (\( \alpha^{IQ}, \beta^{IQ} \)), and (\( \alpha^{IMQ}, \beta^{IMQ} \)), respectively.

B. The Optimal Estimated Shape Parameter

We consider the infinitive error norm (11) of the numerical solution of the transient voltage \( v(x, t) \) in Case 4, and the estimated local error is formulated as

\[
\epsilon_x(c) = v_{\text{RBF}} - v, \quad (B.1)
\]

Applying the RBF approximation of (2) to (B.1), we get

\[
\epsilon_x(c) = \left[ \alpha_{1x} v(x_i - \Delta x) + \alpha_{3x} v(x_i + \Delta x) \right] - v(x_i). \quad (B.2)
\]

It is easy to see that \( v(x_i - \Delta x) = -v(x_i + \Delta x) \) and \( \alpha_{1x} = -\alpha_{3x} \). So, (B.2) can be rewritten as follows:
\[ \mathbf{E}_c (c) = 2\alpha_3x_v (x_i + \Delta x) - v' (x_i). \]  

(B.3)

Expanding Taylor's series to the transient voltage, we have

\[ v (x_i + \Delta x) = \frac{\Delta x}{1!} v' (x_i) + \frac{\Delta x^2}{3!} v'' (x_i) \]

\[ + \frac{\Delta x^3}{5!} v''' (x_i) + \ldots \]  

(B.4)

Substituting (B.4) to (B.3), we obtain the following:

\[ \mathbf{E}_c (c) \approx 2\alpha_3x \left[ \left( \Delta x - \frac{1}{2\alpha_3x} \right) v' (x_i) \right. \]

\[ + \frac{\Delta x^3}{3!} v'' (x_i) + \frac{\Delta x^4}{5!} v''' (x_i) \].  

(B.5)

It can be noticed that three derivatives of \( v' (x_i) \), \( v'' (x_i) \), and \( v''' (x_i) \) are calculated based on the equation system of (30) and (31) where that is solved using the conventional FDTD method.

Substituting (B.5) to (11), we obtain the RBF-FD error norm and the optimal value \( c^*_c \) that is presented as in Figure 17.

Applying the abovementioned procedure of (B.1) to (B.5) to the transient voltage \( v_r (t) \) in Case 1 and Case 2 in the same way by replacing \( x \) with \( t \), we can also find the values of \( c^*_c \) and \( \| \mathbf{E} (c^*_c) \|_{\infty} \) as presented in Tables 3 and 5.

Data Availability

The parameter data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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