THE HEAT RADIATION PROBLEM: THREE-DIMENSIONAL ANALYSIS FOR ARBITRARY ENCLOSURE GEOMETRIES

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Received 25 June 2003 and in revised form 22 February 2004

This paper gives very significant and up-to-date analytical and numerical results to the three-dimensional heat radiation problem governed by a boundary integral equation. There are two types of enclosure geometries to be considered: convex and nonconvex geometries. The properties of the integral operator of the radiosity equation have been thoroughly investigated and presented. The application of the Banach fixed point theorem proves the existence and the uniqueness of the solution of the radiosity equation. For a nonconvex enclosure geometries, the visibility function must be taken into account. For the numerical treatment of the radiosity equation, we use the boundary element method based on the Galerkin discretization scheme. As a numerical example, we implement the conjugate gradient algorithm with preconditioning to compute the outgoing flux for a three-dimensional nonconvex geometry. This has turned out to be the most efficient method to solve this type of problems.

1. Introduction

Heat radiation is a very important phenomenon in our modern technology. One of the factors that account for the importance of the thermal radiation in some applications is the manner in which radiant emission depends on temperature. For conduction and convection, the transfer of energy between two locations depends on the temperature difference of the locations. The transfer of energy by thermal radiation, however, depends on the differences of the individual absolute temperatures of the bodies, each raised to a power in the range of about 4 or 5. It is also evident that the importance of radiation becomes intensified at high absolute temperature levels. Consequently, radiation contributes substantially to the heat transfer in furnaces and combustion chambers and in the energy emission from a nuclear explosion. Also heat radiation must often be considered when calculating thermal effects in devices such as a rocket nozzle, a nuclear power plant, or a gaseous-core nuclear rocket. One of the most interesting features about transport of heat radiative energy between two points on the diffuse grey surface is its formulation as

Journal of Applied Mathematics 2004:4 (2004) 311–330
2000 Mathematics Subject Classification: 45B05, 65R20, 65F10, 65N38
URL: http://dx.doi.org/10.1155/S1110757X04306108

an integral equation. An important consequence of this fact is that the pencil of rays emitted at one point can impinge another point only if these two points can "see" each other, that is, the domain is convex. The presence of the shadow zones should also be taken into consideration in heat radiation analysis whenever the domain where the radiation heat transfer takes place is nonconvex. Shadow zones computation in some respect is not easy, but we were able to develop an efficient geometrical algorithm to determine the shadow function in the two-dimensional case for polygonal domains and then this algorithm was transformed to the three-dimensional case for an enclosure with polyhedral boundary [5, 8].

In [1, 2], a boundary element method was implemented for two-dimensional enclosures to obtain a direct numerical solution for the integral equation; however, this permits quite high error bounds. In [6], two-dimensional convex and nonconvex geometries have been considered and some solution methods for the discrete heat equation, for example, the conjugate gradient method, direct solvers, and multigrid methods, have been compared.

Our main concern in this paper is to focus on the analytical aspect of the radiosity equation and to show how the boundary element method based on the Bubnov-Galerkin discretization scheme can be used for the solution of the radiosity equation. Now we give a short overview of this paper.

In Section 2, we present a systematic derivation of the heat radiosity equation. This is preceded by thorough definitions of the quantities needed to derive this equation. In Section 3, we present some important analytical results concerning the integral operator of the radiosity equation. In Section 4, we prove with the help of the Banach fixed point theorem the existence and the uniqueness of the solution of the radiosity equation. In Section 5, we describe the Bubnov-Galerkin discretization scheme for the solution of the radiosity equation and present a numerical example for the calculation of the outgoing flux for a nonconvex enclosure.

2. The formulation of the heat radiation problem

We consider an enclosure $\Omega \subset \mathbb{R}^d$, $d = 2,3$, with boundary Γ. The boundary of the enclosure is composed of N elements as shown in Figure 2.1.

The heat balance for an element k with area dA_k reads as

$$Q_k = q_k dA_k = (q_{0,k} - q_{i,k}) dA_k, \tag{2.1}$$

where

(i) $q_{i,k}$ is the rate of incoming radiant energy per unit area on the element k,
(ii) $q_{0,k}$ is the rate of outgoing radiant energy per unit area on the element k,
(iii) dA_k is the area of element k,
(iv) q_k is the energy flux supplied to the element k by some means other than the radiation inside the enclosure to make up for the net radiation loss and maintain the specified inside surface temperature.

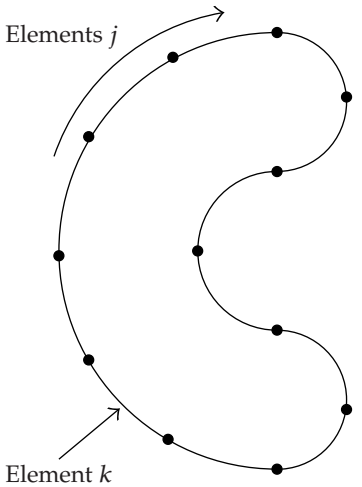

Figure 2.1. Boundary of the enclosure.

A second equation results from the fact that the energy flux leaving the surface is composed of emitted and reflected energy. This yields to

$$q_{0,k} = \varepsilon_k \sigma T_k^4 + \varrho_k q_{i,k}, \tag{2.2}$$

where

(i) ε_k is the emissivity coefficient $(0 < \varepsilon_k < 1)$,
(ii) σ_k is the Stefan-Boltzmann constant which has the value $5.669996 \cdot 10^{-8}\mathrm{W}/(\mathrm{m}^2 K^4)$,
(iii) ϱ_k is the reflection coefficient with the relation $\varrho_k = 1 - \varepsilon_k$ for opaque grey surfaces.

The incident flux $q_{i,k}$ is composed of the portions of the energy leaving the viewable surfaces of the enclosure and arriving at the kth surface. If the kth surface can view itself (is nonconvex), a portion of its outgoing flux will contribute directly to its incident flux. The incident energy is then equal to

$$\begin{aligned} dA_k q_{i,k} = {} & dA_1 q_{0,1} F_{1,k} \beta(1,k) + dA_2 q_{0,2} F_{2,k} \beta(2,k) + \cdots + dA_j q_{0,j} F_{j,k} \beta(j,k) \\ & + \cdots + dA_k q_{0,k} F_{k,k} \beta(k,k) + \cdots + dA_N q_{0,N} F_{N,k} \beta(N,k). \end{aligned} \tag{2.3}$$

From the view factor, reciprocity relation [10] follows:

$$\begin{aligned} dA_1 F_{1,k} \beta(1,k) &= dA_k F_{k,1} \beta(k,1), \\ dA_2 F_{2,k} \beta(2,k) &= dA_k F_{k,2} \beta(k,2), \\ &\vdots \\ dA_N F_{N,k} \beta(N,k) &= dA_k F_{k,N} \beta(k,N). \end{aligned} \tag{2.4}$$

Then (2.3) can be rewritten in such a way that the only area appearing is dA_k:

$$\begin{aligned} dA_k q_{i,k} = {} & dA_k F_{k,1}\beta(k,1)q_{0,1} + dA_k F_{k,2}\beta(k,2)q_{0,2} \\ & + \cdots + dA_k F_{k,j}\beta(k,j)q_{0,j} + \cdots + dA_k F_{k,k}\beta(k,k)q_{0,k} \\ & + \cdots + dA_k F_{k,N}\beta(k,N)q_{0,N}, \end{aligned} \tag{2.5}$$

so that the incident flux can be expressed as

$$q_{i,k} = \sum_{j=1}^{N} F_{k,j}\beta(k,j)q_{0,j}. \tag{2.6}$$

The visibility factor $\beta(k,j)$ is defined as (see, e.g., [6])

$$\beta(k,j) = \begin{cases} 1 & \text{when there is a heat exchange between the surface element } k \\ & \text{and the surface element } j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Substituting (2.6) into (2.2) and using the relation $\varrho_k = 1 - \varepsilon_k$, we finally get

$$q_{0,k} = \varepsilon_k \sigma T_k^4 + (1-\varepsilon_k)\sum_{j=1}^{N} F_{k,j}\beta(k,j)q_{0,j}. \tag{2.8}$$

2.1. The calculation of the view factor $F_{k,j}$. The total energy per unit time leaving the surface element dA_k and incident on the element dA_j is given through

$$Q_{k,j} = L_k dA_k \cos\theta_k d\omega_k, \tag{2.9}$$

where $d\omega_k$ is the solid angle subtended by dA_j when viewed from dA_k (see Figure 2.2) and L_k is the total intensity of a black body for the surface element dA_k.

The solid angle $d\omega_k$ is related to the projected area of dA_j and the distance $S_{k,j}$ between the elements dA_k and dA_j and can be calculated as

$$d\omega_k = \frac{dA_j \cos\theta_j}{S_{k,j}^2}, \tag{2.10}$$

where θ_j denotes the angle between the normal vector n_j and the distance vector $S_{k,j}$. Substituting (2.10) into (2.9) gives the following equation for the total energy per unit time leaving dA_k and arriving at dA_j:

$$Q_{k,j} = \frac{L_k dA_k \cos\theta_k dA_j \cos\theta_j}{S_{k,j}^2}. \tag{2.11}$$

In [10], we have the relation between the total intensity L_k and the total emissivity E_k of a black body, that is,

$$L_k = \frac{E_k}{\pi} = \frac{\sigma T_k^4}{\pi}, \tag{2.12}$$

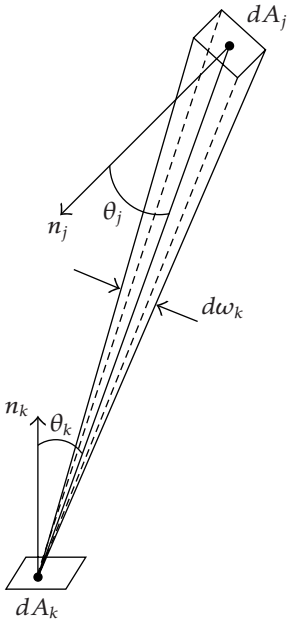

Figure 2.2. Calculation of the view factor.

and consequently (2.11) becomes

$$Q_{k,j} = \frac{\sigma T_k^4 \cos\theta_k \cos\theta_j dA_k dA_j}{\pi S_{k,j}^2}. \tag{2.13}$$

From the definition of the view factor $F_{k,j}$ (see [10]), together with (2.13), we get

$$F_{k,j} := \frac{Q_{k,j}}{\sigma T_k^4 dA_k} = \frac{\cos\theta_k \cos\theta_j dA_j}{\pi S_{k,j}^2}. \tag{2.14}$$

2.2. The boundary integral equation. Now we are able to derive the boundary integral equation describing the heat balance in a grey body. The substitution of (2.14) into (2.8) leads to

$$q_{0,k} = \varepsilon_k \sigma T_k^4 + (1-\varepsilon_k) \sum_{j=1}^{N} \frac{\cos\theta_k \cos\theta_j dA_j}{\pi S_{k,j}^2} \beta(k,j) q_{0,j}. \tag{2.15}$$

If the number of the area elements $N \to \infty$, then for all $x \in dA_k$, we obtain the boundary integral equation

$$q_0(x) = \varepsilon(x)\sigma T^4(x) + (1-\varepsilon(x)) \int_\Gamma G(x,y) q_0(y) d\Gamma_y \quad \text{for } x \in \Gamma, \tag{2.16}$$

where the kernel $G(x,y)$ denotes the view factor between the points x and y of Γ.

From the above considerations and for general enclosure geometries, $G(x,y)$ is given through

$$G(x,y) := G^*(x,y)\beta(x,y) := \frac{[n(y)\cdot(y-x)]\cdot[n(x)\cdot(x-y)]}{c_0|x-y|^{d+1}}\beta(x,y), \tag{2.17}$$

where $c_0 = 2$ for $d = 2$ and $c_0 = \pi$ for $d = 3$.

For convex enclosure geometries, $\beta(x,y) \equiv 1$. If the enclosure is not convex, then we have to take into account the visibility function $\beta(x,y)$,

$$\beta(x,y) = \begin{cases} 1 & \text{for } n(y)\cdot(y-x) > 0 \wedge n(x)\cdot(x-y) > 0 \wedge \vec{xy} \cap \Gamma = \varnothing, \\ 0 & \text{for } \vec{xy} \cap \Gamma \neq \varnothing, \end{cases} \tag{2.18}$$

where $\vec{xy}$ denotes the open straight segment between the points x and y. Definition (2.18) implies that $\beta(x,y) = \beta(y,x)$. Since $G^*(x,y)$ is symmetric, then $G(x,y)$ is also symmetric.

3. Properties of the integral operator

Equation (2.16) is a Fredholm boundary integral equation of the second kind. We introduce the integral operator $\widetilde{K} : L^\infty(\Gamma) \to L^\infty(\Gamma)$ with

$$\widetilde{K}q_0(x) := \int_\Gamma G(x,y)q_0(y)d\Gamma_y \quad \text{for } x \in \Gamma,\ q_0 \in L^\infty(\Gamma). \tag{3.1}$$

This integral operator has the following properties.

Lemma 3.1. *Let Γ be a Ljapunow surface in $C^{1,\delta}$ with $\delta \in [0,1)$. Then for any arbitrary point $x \in \Gamma$,*

$$\int_\Gamma G^*(x,y)d\Gamma_y = 1, \tag{3.2}$$

where $G^*(x,y)$ is given by (2.17).

Proof. First we choose a local coordinate system in the point $x \in \Gamma$ so that $x = (0,0,0)$ and the plane (ξ_1,ξ_2) is tangent to Γ in x. Furthermore, we choose $y = (\xi_1,\xi_2,f(\xi_1,\xi_2))$ in the neighbourhood of $\xi_1 = \xi_2 = 0$. Using the assumption that $\Gamma \in C^{1,\delta}$ with $\delta \in [0,1)$, together with the Taylor expansion of y in the local coordinate system and some trivial estimates (see [6]), we get the following inequalities:

$$\left|\frac{n(x)\cdot(y-x)}{|y-x|^2}\right| \le c_1|\xi_\alpha|^{\delta-1}, \qquad \left|\frac{n(y)\cdot(x-y)}{|x-y|^2}\right| \le c_2|\xi_\alpha|^{\delta-1} \tag{3.3}$$

with $\alpha \in [1,d-1]$ and $d = 2$ or 3. Consequently, one obtains from (3.3)

$$|G^*(x,y)| \le c_3|\xi_\alpha|^{-2(1-\delta)+3-d} \tag{3.4}$$

with an arbitrary constant c_3 and $d = 2$ or 3. This shows that $G^*(x,y)$ is a weakly singular kernel of type $|x-y|^{-2(1-\delta)}$ and hence it is integrable.

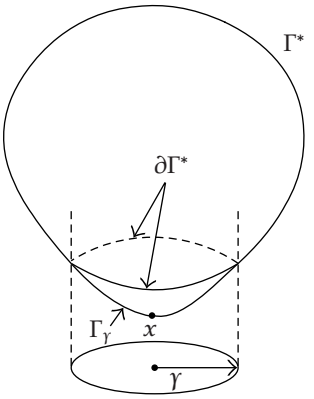

Figure 3.1. Convex case.

In order to calculate $\int_\Gamma G^*(x,y)d\Gamma_y$, we use Stoke's theorem [6]. For the following, we consider a closed surface Γ and an arbitrary point $y=(y_1,y_2,y_3)\in\Gamma$. At this point, the normal to the area A is constructed. Let the functions $P_1(y)$, $P_2(y)$, and $P_3(y)$ be any twice differentiable functions of y_1, y_2, and y_3 and n is the normal. Stoke's theorem in three dimensions provides the following relation:

$$\begin{aligned}&\int_{\partial A}(P_1dy_1+P_2dy_2+P_3dy_3)\\&\quad=\int_A\left[\left(\frac{\partial P_3}{\partial y_2}-\frac{\partial P_2}{\partial y_3}\right)n_1(y)+\left(\frac{\partial P_1}{\partial y_3}-\frac{\partial P_3}{\partial y_1}\right)n_2(y)+\left(\frac{\partial P_2}{\partial y_1}-\frac{\partial P_1}{\partial y_2}\right)n_3(y)\right]dA.\end{aligned}\tag{3.5}$$

Hence this relation can now be applied to express area integrals in view factor computations in terms of boundary integrals. To this end, we consider the surface Γ as shown in Figure 3.1, let $\Gamma_\gamma=Z(x,\gamma)\cap\Gamma$ be a small neighbourhood of the point x, and define Γ^* as $\Gamma^*=\Gamma\setminus\Gamma_\gamma$.

Here $Z(x,\gamma)$ is a cylinder which is defined by the relation $x_1^2+x_2^2\le\gamma^2$. Since Γ^* is not independent of x, the integral $\int_\Gamma G^*(x,y)d\Gamma_y$ can be expressed as

$$F_\gamma(x)=\int_\Gamma G^*(x,y)d\Gamma_y=\int_{\Gamma_\gamma}G^*(x,y)d\Gamma_y+\int_{\Gamma^*}G^*(x,y)d\Gamma_y,\tag{3.6}$$

where the first integral tends to zero for $\gamma\to0$ because of the weakly singular kernel $G^*(x,y)$. Hence (3.6) is reduced to

$$F_\gamma(x)=\lim_{\gamma\to0}\int_{\Gamma^*}G^*(x,y)d\Gamma_y.\tag{3.7}$$

Since the view factor $G^*(x,y)$ is smooth in Γ^*, the application of Stoke's theorem leads to

$$\begin{aligned} F_\gamma(x) &= \lim_{\gamma\to 0}\int_{\Gamma^*} G^*(x,y)d\Gamma_y = \lim_{\gamma\to 0}\int_{\partial\Gamma^*} \nabla\times\vec{P}(y)\cdot n(y)dy \\ &= \lim_{\gamma\to 0}\oint_{\partial\Gamma^*}(P_1dy_1+P_2dy_2+P_3dy_3), \end{aligned} \tag{3.8}$$

where $P_1(y)$, $P_2(y)$, and $P_3(y)$ are given in [6], respectively, by

$$\begin{aligned} P_1(y) &= \frac{-n_2(x)(x_3-y_3)+n_3(x)(x_2-y_2)}{2\pi|x-y|^2}, \\ P_2(y) &= \frac{n_1(x)(x_3-y_3)-n_3(x)(x_1-y_1)}{2\pi|x-y|^2}, \\ P_3(y) &= \frac{-n_1(x)(x_2-y_2)+n_2(x)(x_1-y_1)}{2\pi|x-y|^2}. \end{aligned} \tag{3.9}$$

The normal to the area element is perpendicular to both the x_1- and x_2-axes and parallel to the x_3-axis. Hence (3.8) becomes

$$\begin{aligned} F_\gamma(x) &= \frac{1}{2\pi}\lim_{\gamma\to 0}\oint_{\partial\Gamma^*}\frac{(x_2-y_2)dy_1-(x_1-y_1)dy_2}{|x-y|^2} \\ &= \frac{1}{2\pi}\lim_{\gamma\to 0}\oint_{\partial\Gamma^*}\frac{-y_2dy_1+y_1dy_2}{y_1^2+y_2^2+y_3^2}, \end{aligned} \tag{3.10}$$

using the fact that the area element is located at the origin of the coordinate system. With the help of the relation $y_1^2+y_2^2=\gamma^2$, we get

$$\begin{aligned} F_\gamma(x) = &\underbrace{\frac{1}{2\pi}\lim_{\gamma\to 0}\oint_{\partial\Gamma^*}\frac{1}{\gamma^2}(-y_2dy_1+y_1dy_2)}_{:=I_1} \\ &+\underbrace{\frac{1}{2\pi}\lim_{\gamma\to 0}\oint_{\partial\Gamma^*}\frac{-y_3^2(-y_2dy_1+y_1dy_2)}{(\gamma^2+y_3^2)\gamma^2}}_{:=I_2}. \end{aligned} \tag{3.11}$$

Let the boundary of the domain Γ^* be described by the triple $(y_1,y_2,f(y,y_2))$; then the first integral I_1 will be integrated over the circle $y_1^2+y_2^2=\gamma^2$. Using the polar coordinates $y_1=\gamma\cos\theta$ and $y_2=\gamma\sin\theta$, one obtains directly

$$I_1 = \frac{1}{2\pi}\cdot\frac{1}{\gamma^2}\int_0^{2\pi}\gamma^2 d\theta = 1. \tag{3.12}$$

For the second integral, we have $y_3=f(y_1,y_2)$. Applying Taylor's expansion, it can easily be shown that $I_2=0$. Hence, we have the desired result for convex enclosure geometries (3.2). Next we have to show that this result holds also for the nonconvex case; see Figure 3.2. Therefore, we consider the set $\Gamma\backslash\Gamma_y$, where $\Gamma_y=\{x\in\Gamma \mid \beta(x,y)=1\}$.

This set consists in general of many disjoint components. For the sake of simplicity, we take one of these components and denote it by D_i, where D_i is the boundary of Γ_i. Clearly,

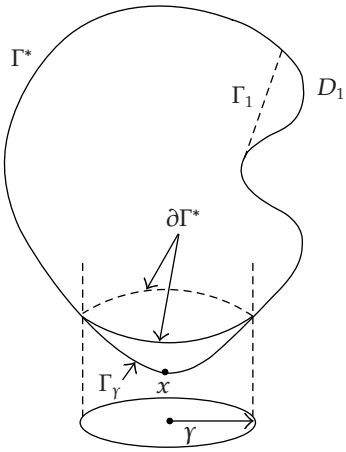

Figure 3.2. Nonconvex case.

all Γ_i are dependent on the choice of D_i. Due to the discontinuity of the visibility function $\beta(x,y)$, the Stoke theorem cannot be applied directly for $G(x,y)$, but we write first

$$\int_{\Gamma^*} G(x,y)d\Gamma_y = \int_{\Gamma^*} G^*(x,y)d\Gamma_y - \sum_i \int_{D_i} \nabla \times \vec{P}(y) \cdot n(y)dy. \tag{3.13}$$

Since the second integral vanishes over the closed surface D_i, the assertion follows directly. □

Lemma 3.2. *Let Γ be a closed surface of the class C^2. Then $G^*(x,y)$ in (2.17) is a bounded kernel, that is,*

$$\left| G^*(x,y) \right| \le \widetilde{C} \tag{3.14}$$

with a suitable chosen constant $\widetilde{C}$.

Proof. Under the assumption that $\Gamma \subset C^2$, the following requirements are fulfilled.

(1) In every point of the surface exists a tangential plane.

(2) If θ is the angle between the normals at the points x and y and $r_{1,2}$ denotes the distance between these two points, the inequality

$$|\theta| < Ar_{1,2}, \quad \theta \in (0, 2\pi), \tag{3.15}$$

holds, where A is a positive number independent from the choice of the points x and y.

(3) For all points x_0 of the surface, there exists a fixed number d with the property that the point of the surface which is located within the sphere of radius d around x_0 is intersected by a parallel to the normal in x_0 at most in one point.

Let the ζ-axis be the normal at the surface point x_0 and take the two ξ- and η-axes to be the tangential plane containing the point x_0 such that the three axes form an orthonormal system. The corresponding unit vectors are denoted by e_1, e_2, and e_3. As a consequence of the third condition above, a part of the surface which lies inside the Ljapunow sphere takes the form $\zeta = \Psi(\xi,\eta)$. The existence of the tangential plane and its continuous change imply the existence of the first partial derivatives Ψ_ξ and Ψ_η which are continuous due to requirement (2). Assume that d is sufficiently small, that is,

$$Ad \le 1, \tag{3.16}$$

so that the angle between the normal at x_0 and the normal at any arbitrary point of the surface which lies inside the sphere does not exceed the value $\pi/2$. Denoting with r_0 the distance $|x_0 - y_0|$, one obtains

$$\cos\theta_0 \ge 1 - \frac{1}{2}\theta_0^2 \ge 1 - \frac{1}{2}A^2 r_0^2 > \frac{1}{2}. \tag{3.17}$$

On the other hand, we have

$$\frac{1}{\cos\theta_0} = \sqrt{1+\Psi_\xi^2+\Psi_\eta^2} \le 1 + A^2 r_0^2 \le 2 \tag{3.18}$$

and therefore,

$$\Psi_\xi^2 + \Psi_\eta^2 \le 2A^2 r_0^2 + A^4 r_0^2. \tag{3.19}$$

The introduction of the polar coordinates $\xi = \varrho_0\cos\theta$, $\eta = \varrho_0\sin\theta$ leads to

$$\Psi_{\varrho_0}^2 = \left(\Psi_\xi\cos\theta + \Psi_\eta\sin\theta\right)^2 \le \Psi_\xi^2 + \Psi_\eta^2. \tag{3.20}$$

Using (3.19) together with the estimate $|\Psi| \le \sqrt{3}\varrho_0$ and therefore $r_0 \le 2\varrho_0$, we get

$$\left|\Psi_{\varrho_0}\right| \le 2\sqrt{3}A\varrho_0. \tag{3.21}$$

Finally, it follows from (3.17) that

$$1 - \cos\theta_0 \le 2A^2\varrho_0^2. \tag{3.22}$$

As a consequence of (3.19), the estimate

$$\left|\cos(n,e_1)\right| = \frac{\sqrt{\Psi_\xi^2+\Psi_\eta^2}}{\sqrt{1+\Psi_\xi^2+\Psi_\eta^2}} \le \left|\Psi_\xi\right| \le \sqrt{3}Ar_0 \tag{3.23}$$

holds, where n is the unit vector of the outward normal of Γ at an arbitrary point. Analogously, we get

$$\left|\cos(n,e_2)\right| \le \sqrt{3}Ar_0, \qquad \left|\cos(n,e_3)\right| = \cos\theta_0. \tag{3.24}$$

Summarizing the estimates above, we get

$$|\Psi| \le c\varrho_0^2, \qquad |\cos(n,e_1)| < c\varrho_0,$$
$$|\cos(n,e_2)| \le c\varrho_0, \qquad |\cos(n,e_3)| \ge \frac{1}{2}. \tag{3.25}$$

From (3.23), it follows that

$$|\cos((x-y),n(x))| = \left|\frac{n(x)\cdot(x-y)}{r_{1,2}}\right| \le \Psi_\xi \le D_1 r_{1,2}, \tag{3.26}$$

and similarly the estimate

$$|\cos((y-x),n(y))| = \left|\frac{n(y)\cdot(y-x)}{r_{1,2}}\right| \le D_1 r_{1,2} \tag{3.27}$$

with $D_1 = \sqrt{3}A$. Therefore, we get, for the kernel,

$$|G^*(x,y)| = \left|\frac{\cos((x-y),n(x))\cdot\cos((y-x),n(y))}{r_{1,2}^2}\right| \le \tilde{c}, \tag{3.28}$$

where $\tilde{c} = 3A^2/\pi$ with $A = \sup_{x,y\in\Gamma}(\theta/r_{1,2})$. □

We remark that in the two-dimensional case for $G^*(x,y)$ in (2.17), the estimate

$$|G^*(x,y)| \le \tilde{c} r_{1,2} \tag{3.29}$$

holds with some constant $\tilde{c}$.

Lemma 3.3. *For the integral kernel $G(x,y)$, it holds that $G(x,y) \ge 0$. The mapping $\tilde{K}: L^p(\Gamma) \to L^p(\Gamma)$ is compact for $1 \le p \le \infty$. Furthermore,*

(a) *$\tilde{K}1 = 1$ and $\|\tilde{K}\| = 1$ in L^p for $1 \le p \le \infty$,*
(b) *the spectral radius $\varrho(\tilde{K}) = 1$.*

Proof. For the convex case, $G^*(x,y)$ is obviously not negative. For the nonconvex case, the visibility factor $\beta(x,y) \equiv 0$ whenever $G^*(x,y) < 0$, hence $G(x,y) \ge 0$ and, consequently, the integral operator $\tilde{K}$ is not negative.

From Lemma 3.1, it follows that the kernel $G(x,y)$ is integrable and $\tilde{K}$ is a weakly singular integral operator. Hence the mapping $\tilde{K}: L^p(\Gamma) \to L^p(\Gamma)$ is compact. We now estimate the norm of this integral operator $\tilde{K}$. For $1 < p < \infty$ and $q_0 \in L^p(\Gamma)$, we have with $1/p + 1/q = 1$,

$$\begin{aligned} |\tilde{K}q_0(x)| &= \left|\int_{\Gamma_y} G(x,y)^{1/p+1/q} q_0(y)\,d\Gamma_y\right| \\ &\le \left(\int_{\Gamma_y} G(x,y)\,d\Gamma_y\right)^{1/q} \left(\int_{\Gamma_y} G(x,y)\,|q_0(y)|^p\,d\Gamma_y\right)^{1/p}. \end{aligned} \tag{3.30}$$

Since $\int_{\Gamma_y} G(x,y)d\Gamma_y = 1$ (see Lemma 3.1), it follows that

$$|\widetilde{K}q_0(x)| \le \left(\int_{\Gamma_y} G(x,y)\,|q_0(y)|^p d\Gamma_y\right)^{1/p}. \tag{3.31}$$

Furthermore, we get

$$\begin{aligned} ||\widetilde{K}q_0(x)||_{L^p}^p &= \int_{\Gamma_x} |\widetilde{K}q_0(x)|^p d\Gamma_x \\ &\le \int_{\Gamma_y} |q_0(y)|^p \int_{\Gamma_x} G(x,y)d\Gamma_x d\Gamma_y = ||q_0(y)||_{L^p}^p. \end{aligned} \tag{3.32}$$

Hence we obtain $\|\widetilde{K}\| \le 1$ in all spaces L^p, $1 \le p \le \infty$. Equality can be achieved by choosing $q = 1$ which is clearly an eigenvector of $\widetilde{K}$ with eigenvalue 1.

Finally, it follows from the fact $\widetilde{K}1 = 1$ and the Hilbert theorem that the integral operator $\widetilde{K}$ has an eigenvalue λ_0 with $|\lambda_0| = \|\widetilde{K}\| = 1$. □

Lemma 3.4. *The integral operator $\widetilde{K}$ is for the convex case, that is, $\beta(x,y) \equiv 1$, a classical pseudodifferential operator of order $\alpha = -2$. The kernel of this integral operator possesses a pseudohomogeneous expansion of the form*

$$G^*(x,y) \sim |u-v|^{-\alpha-2} \sum_{\nu\ge 0} \Psi_\nu(x,\theta)|u-v|^\nu \sim r^{-\alpha-2} \sum_{\nu\ge 0} \Psi_\nu(x,\theta) r^\nu. \tag{3.33}$$

In the two-dimensional case (either convex or nonconvex), the kernel possesses a pseudohomogeneous expansion of the form

$$G^*(x,y) \sim (s-s_0) \sum_{\nu\ge 0} C_\nu(x)(s-s_0)^\nu. \tag{3.34}$$

In the two-dimensional convex case, the integral operator $\widetilde{K}$ is even a pseudodifferential operator of order $-\infty$.

Proof. One can write the kernel of the integral operator $\widetilde{K}$ as a convolution kernel in a pseudohomogeneous expansion form. In the case when Γ has a quadratic parameter representation and $u = \Phi^{-1}(x)$, one obtains [9]

$$y - x = \Phi(v) = bv_1 + cv_2 + dv_1^2 + 2ev_1v_2 + fv_2^2 \tag{3.35}$$

with vectors $b,c,d,e,f \in \mathbb{R}^3$. For the normal, one has

$$n(v) = \frac{\Phi_1 \times \Phi_2}{|\Phi_1 \times \Phi_2|}, \tag{3.36}$$

where Φ_1 and Φ_2 are given by the parameter representation of Γ as

$$\Phi_1 = \frac{\partial \Phi}{\partial v_1} = b + 2(v_1 d + v_2 e), \qquad \Phi_2 = \frac{\partial \Phi}{\partial v_2} = e + 2(v_1 e + v_2 f),$$
$$\Phi_1 \times \Phi_2 = b \times c + 2Q_1(v) + 4Q_2(v) \tag{3.37}$$

with

$$\begin{aligned} Q_1(v) &= v_1(b \times e + d \times c) + v_2(b \times f + e \times c), \\ Q_2(v) &= v_1^2(d \times e) + v_1 v_2(d \times f) + v_2^2(e \times f). \end{aligned} \tag{3.38}$$

Consequently,

$$(\Phi_1 \times \Phi_2)(y - x) = v_1^2 b(d \times c) + 2v_1 v_2 c(b \times e) + v_2^2 c(b \times f). \tag{3.39}$$

Using the polar coordinates in the parameter plane $v - u = r(\cos\theta, \sin\theta)^T$, we obtain

$$n(y) \cdot (y - x) = \frac{r^2}{|\Phi_1 \times \Phi_2|}\left[b(d \times c)\cos^2\theta + 2c(b \times e)\cos\theta\sin\theta + c(b \times f)\sin^2\theta\right]. \tag{3.40}$$

Analogously, $n(x)(x - y)$ has in $u = \Phi^{-1}(x) = 0$ an expansion of the form

$$n(x) \cdot (x - y) = \frac{-(b \times c)}{|b \times c|} r^2 \left[d\cos^2\theta + 2e\cos\theta\sin\theta + f\sin^2\theta\right]. \tag{3.41}$$

From [9], it holds also that

$$\varrho^{-4} = |x - y|^{-4} = r^{-4} \sum_{\nu=0}^{\infty} (\ell_2^{-2-\nu}(\theta) P_{3\nu}(\cos\theta, \sin\theta)) r^\nu, \tag{3.42}$$

where $P_{3\nu}$ is a homogeneous polynomial of degree 3ν and $\ell_2(\theta)$ is given by

$$\ell_2(\theta) = |b|^2\cos^2\theta + bc\sin\theta\cos\theta + |c|^2\sin^2\theta. \tag{3.43}$$

Finally, one obtains for $G^*(x, y)$ in (2.17) the expansion

$$G^*(x, y) = \frac{|b \times c|}{4\pi|\Phi_1 \times \Phi_2|}\left\{\left[L\cos^2\theta + 2M\cos\theta\sin\theta + N\sin^2\theta\right]^2 \sum_{\nu=0}^{\infty} (\ell_2^{-2-\nu} P_{3\nu})(\theta) r^\nu\right\}, \tag{3.44}$$

where L, M, and N are the coefficients of the second fundamental form defined by

$$\begin{aligned} d(b \times c) &= -(d \times c)b = -\frac{1}{2}|b \times c|L, \\ e(b \times c) &= -(b \times e)c = -\frac{1}{2}|b \times c|M, \\ f(b \times c) &= -(d \times f)c = -\frac{1}{2}|b \times c|N. \end{aligned} \tag{3.45}$$

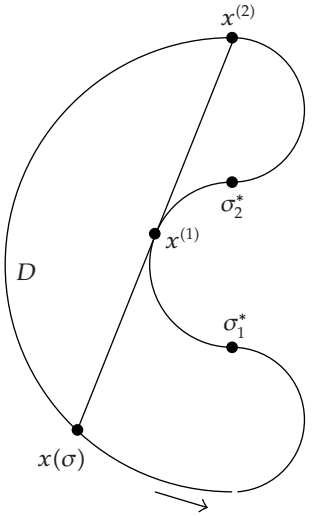

Figure 3.3. Parametric representation.

From (3.44), it follows that the integral operator $\widetilde{K}$ is for $\beta(x,y) \equiv 1$, that is, for convex Γ, a classical pseudodifferential operator of the order $\alpha = -2$. The kernel possesses a pseudohomogeneous expansion of the form

$$G^*(x,y) \sim |u-v|^{-\alpha-2} \sum_{\nu \geq 0} \Psi_\nu(x,\theta)|u-v|^\nu \sim r^{-\alpha-2} \sum_{\nu \geq 0} \Psi_\nu(x,\theta) r^\nu. \tag{3.46}$$

□

Lemma 3.5. *Let Γ be any closed curve of the class C^2. Then in the two-dimensional case, $\widetilde{K}$ defines a continuous mapping $\widetilde{K} : L^2(\Gamma) \to H^1(\Gamma)$ if $G(x,y)$ is the kernel of the radiosity equation as defined in (2.17) and (2.18).*

Proof. First let $G^*(x,y)$ be defined as in (2.17) and

$$\Phi(x) = \int_\Gamma G^*(x,y)\beta(x,y)q_0(y)d\Gamma_y. \tag{3.47}$$

Consider the simple case similar to the situation in Figure 3.3.

We use the following abbreviations: $y = y(s)$, $x^{(i)} = x(\sigma^{(i)})$ with $\sigma^{(i)} = \sigma^{(i)}(\sigma)$ for $i = 1,2$. Γ^+ and Γ^- are open parts with $x(\sigma_1^*), x(\sigma_2^*) \notin \Gamma^+, \Gamma^-$ and $x(\sigma_1^*), x(\sigma_2^*) \in \overline{\Gamma}^+, \overline{\Gamma}^-$.

Choose $\sigma^{(1)}$ in such a way that $x(\sigma_1^*) - x(\sigma)$ is for all $\sigma_1^* \in (\sigma, \sigma^{(1)})$ no longer parallel to $x^{(2)} - x(\sigma)$. Then with the help of these abbreviations, (3.47) can be expressed as

$$\Phi\big(x(\sigma)\big) = \int_{x(\sigma^{(2)})}^{x(\sigma^{(1)})} G^*\big(x(\sigma), y(s)\big) q_0\big(y(s)\big) d\Gamma_{y(s)}. \tag{3.48}$$

Applying Leibniz rule of differentiation, one obtains

$$\begin{aligned}\frac{d\Phi(\sigma)}{d\sigma} &= \int_{x(\sigma^{(2)})}^{x(\sigma^{(1)})} \frac{dG^*(x(\sigma),y(s))}{d\sigma}\cdot q_0(y(s))\,d\Gamma_{y(s)} \\ &\quad + \frac{dx(\sigma^{(1)})}{d\sigma}\cdot G^*(x(\sigma),x(\sigma^{(1)}))\cdot q_0(x(\sigma^{(1)})) \\ &\quad - \frac{dx(\sigma^{(2)})}{d\sigma}\cdot G^*(x(\sigma),x(\sigma^{(2)}))\cdot q_0(x(\sigma^{(2)})).\end{aligned} \tag{3.49}$$

Since the normal at the point $x^{(1)}$ is perpendicular to the straight line between $x(\sigma)$ and $x^{(2)}$, the kernel $G^*(x(\sigma),x(\sigma^{(1)}))=0$ and therefore (3.49) is reduced to

$$\begin{aligned}\frac{d\Phi(\sigma)}{d\sigma} &= \int_{x(\sigma^{(2)})}^{x(\sigma^{(1)})} \frac{dG^*(x(\sigma),y(s))}{d\sigma}\cdot q_0(y(s))\,d\Gamma_{y(s)} \\ &\quad - \frac{dx(\sigma^{(2)})}{d\sigma}\cdot G^*(x(\sigma),x(\sigma^{(2)}))\cdot q_0(x(\sigma^{(2)})).\end{aligned} \tag{3.50}$$

For $\Gamma\in C^2$, it follows that $G^*(x(\sigma),y(s))$, and $(dG^*/d\sigma)(x(\sigma),y(s))$ are continuous kernels and therefore the integral

$$I=\int_{x(\sigma^{(2)})}^{x(\sigma^{(1)})} \frac{dG^*(x(\sigma),y(s))}{d\sigma}\cdot q_0(y(s))\,d\Gamma_{y(s)} \quad \text{for } q_0(y(s))\in L^2(\Gamma) \tag{3.51}$$

is bounded in $L^2(\Gamma)$. From the definition of $x^{(2)}$, we obtain

$$\frac{d\sigma^{(2)}\cdot\cos((x^{(2)}-x),n(\sigma^{(2)}))}{(|x^{(2)}-x(\sigma)|-|x^{(1)}-x(\sigma)|)} = \frac{d\sigma\cdot\cos((x-x^{(2)}),n(\sigma))}{|x^{(1)}-x(\sigma)|} \tag{3.52}$$

and since $(x-x^{(2)})$ and $(x-x^{(1)})$ are parallel, this leads to

$$\begin{aligned}&\frac{d\sigma^{(2)}}{d\sigma}G^*(x(\sigma),x(\sigma^{(2)})) \\ &\quad = \frac{(|x^{(2)}-x(\sigma)|-|x^{(1)}-x(\sigma)|)}{|x^{(1)}-x(\sigma)|\,|x^{(2)}-x|}\cdot\cos^2((x-x^{(1)}),n(\sigma)).\end{aligned} \tag{3.53}$$

A continuous curve with nonvanishing curvature is also a C-curve [4], that is, there exist constants $c_0>0$, $c_1>0$ such that for all points on the curve, we have

$$\begin{aligned}|x(\sigma^{(1)})-x(\sigma)| &\le |\sigma^{(1)}-\sigma| \le c_0|x^{(1)}-x(\sigma)|, \\ |\cos((x-x^{(1)}),n(\sigma))| &\le c_1|\sigma^{(1)}-\sigma|.\end{aligned} \tag{3.54}$$

Altogether, we obtain the estimate

$$\left|\frac{d\sigma^{(2)}}{d\sigma}G^*(x(\sigma),x^{(2)})\right| \le 1\cdot c_0\cdot c_1^2|\sigma^{(1)}-\sigma| \le M_1 \tag{3.55}$$

and from Lemma 3.2 with a constant M_0, we know $|G^*(x(\sigma),x^{(2)})| \le M_0$. This leads immediately to the following estimate:

$$\begin{aligned}
&\left\| \frac{dx(\sigma^{(2)})}{d\sigma} \cdot G^*(x(\sigma),x(\sigma^{(2)})) \cdot q_0(x(\sigma^{(2)})) \right\|^2_{L^2(\Gamma)} \\
&\quad = \int_\Gamma \left| \frac{d\sigma^{(2)}}{d\sigma} \cdot G^*(x(\sigma),x(\sigma^{(2)})) q_0(x(\sigma^{(2)})) \right|^2 d\sigma \\
&\quad \le M_1 \cdot M_0 \int_\Gamma |q_0(\sigma^{(2)})|^2 \left| \frac{d\sigma^{(2)}}{d\sigma} \right| d\sigma \le M_1 \cdot M_0 \|q_0\|^2_{L^2(\Gamma)},
\end{aligned} \tag{3.56}$$

which shows the assertion. □

Lemma 3.6. *The integral operator $A = (I - K)$ is L^2-elliptic. Furthermore, A is a positive definite operator which satisfies the Gårding inequality on Γ.*

Proof. Let the integral operator K be defined as $K = (1-\varepsilon)\widetilde{K}$, where $\widetilde{K}$ is given by (3.1). From Lemma 3.3, it follows that

$$\|Kq_0\|_{L^2(\Gamma)} \le (1-\varepsilon)\|q_0\|_{L^2(\Gamma)}. \tag{3.57}$$

Furthermore, K satisfies the inequality

$$\langle Kq_0, q_0\rangle_{L^2(\Gamma)} \le (1-\varepsilon)\langle q_0, q_0\rangle_{L^2(\Gamma)}. \tag{3.58}$$

Inequality (3.58) with $A = (I - K)$ leads to

$$\varepsilon\langle q_0, q_0\rangle_{L^2(\Gamma)} \le \langle Aq_0, q_0\rangle_{L^2(\Gamma)} \le (2-\varepsilon)\langle q_0, q_0\rangle_{L^2(\Gamma)}. \tag{3.59}$$

Furthermore, A satisfies the Gårding inequality, that is, for all $q_0 \in L^2(\Gamma)$ and $\varepsilon \ge 0$, the following holds:

$$\operatorname{Re}(Aq_0, q_0) = \operatorname{Re}\int_\Gamma q_0 A q_0 \, d\Gamma_x \ge \varepsilon\|q_0\|^2_{L^2(\Gamma)}. \tag{3.60}$$

□

4. Existence theorem for the radiosity integral equation

A simple method to prove the existence of the solution of the integral equation (2.16) is the application of Banach's fixed point theorem. The successive approximation method can be used and the convergence of the Neumann series can be proved. We want to show first that the integral operator

$$K = (1-\varepsilon)\widetilde{K} : L^p(\Gamma) \longrightarrow L^p(\Gamma) \quad \text{for } 1 < p < \infty \tag{4.1}$$

defines a contraction mapping, that is, there exists a constant $0 \le c < 1$ such that

$$\|Kq_0 - K\widetilde{q_0}\|_{L^p(\Gamma)} \le c\|q_0 - \widetilde{q_0}\|_{L^p(\Gamma)} \tag{4.2}$$

holds. From the definition

$$Kq_0 - K\widetilde{q_0} = (1-\varepsilon)\int_\Gamma G(x,y)\cdot(q_0(y) - \widetilde{q_0}(y))d\Gamma_y \tag{4.3}$$

and the application of Hölder's inequality follows

$$|Kq_0 - K\widetilde{q_0}| \le |(1-\varepsilon)|\left(\int_\Gamma G(x,y)d\Gamma_y\right)^{1/q}\cdot\left(\int_\Gamma G(x,y)|q_0 - \widetilde{q_0}|^p d\Gamma_y\right)^{1/p} \tag{4.4}$$

with $1/p + 1/q = 1$. Since $\int_\Gamma G(x,y)d\Gamma_y = 1$ (see Lemma 3.1), we get

$$|Kq_0 - K\widetilde{q_0}| \le |(1-\varepsilon)|\left(\int_\Gamma G(x,y)|q_0(y) - \widetilde{q_0}(y)|^p d\Gamma_y\right)^{1/p}. \tag{4.5}$$

Then one obtains

$$\|Kq_0 - K\widetilde{q_0}\|^p_{L^p(\Gamma)} \le |(1-\varepsilon)|^p\cdot\int_{\Gamma_y}|q_0(y) - \widetilde{q_0}(y)|^p\int_{\Gamma_x}G(x,y)d\Gamma_x d\Gamma_y \tag{4.6}$$

so that we finally have

$$\|Kq_0 - K\widetilde{q_0}\|^p_{L^p(\Gamma)} \le |1-\varepsilon|^p\cdot\|q_0(y) - \widetilde{q_0}(y)\|_{L^p(\Gamma)}. \tag{4.7}$$

Due to the inequality $0 < \varepsilon < 1$, for the constant c, we get $c := |1-\varepsilon|^p < 1$. Hence the integral operator K is contractive on $L^p(\Gamma)$ and the iteration scheme $q_{0,n+1} = Kq_{0,n}$ for $n = 1,2,\ldots$ is convergent. $\{q_{0,n}\}$ converges to some q_0 in the space $L^p(\Gamma)$, which solves the equation $Kq_0 = q_0$ in $L^p(\Gamma)$. The uniqueness of $q_0 \in L^p(\Gamma)$ follows directly from the contraction of K due to

$$0 < \|q_0 - \widetilde{q_0}\|_{L^p(\Gamma)} = \|Kq_0 - K\widetilde{q_0}\|_{L^p(\Gamma)} \le c\cdot\|q_0 - \widetilde{q_0}\|_{L^p(\Gamma)}, \quad c < 1. \tag{4.8}$$

Consequently, we have

$$(1-c)\cdot\|q_0 - \widetilde{q_0}\|_{L^p(\Gamma)} \le 0. \tag{4.9}$$

Since q_0 and $\widetilde{q_0}$ are two fixed points of K with $(1-c) > 0$ and $\|q_0 - \widetilde{q_0}\| > 0$, then (4.9) implies $q_0 = \widetilde{q_0}$ and one gains the assertion.

5. The numerical realization in three dimensions

For the numerical simulation of the radiosity equation, we use the boundary element method. The weak formulation of (2.16) in $L^2(\Gamma)$ reads as follows: find $q_0 \in L^2(\Gamma)$ such

that for all $v \in L^2(\Gamma)$, there holds

$$\int_\Gamma q_0(x)v(x)d\Gamma_x = \sigma \int_\Gamma \varepsilon T^4(x)v(x)d\Gamma_x + \int_\Gamma (1-\varepsilon(x)) \int_\Gamma G(x,y)q_0(y)d\Gamma_y v(x)d\Gamma_x. \tag{5.1}$$

We consider a Bubnov-Galerkin formulation and choose bilinear trial and basis functions $\phi_{k(k=1,\dots,N)}$ with local support $\Gamma_k \subset \Gamma$. Then the Galerkin equations read as follows: find $q_{0,h}(x) = \sum_{i=1}^N q_0^{(i)}\phi_i(x) \in V_h$ such that

$$\sum_{i=1}^N q_0^{(i)} \underbrace{\int_{\Gamma_j} \phi_i(x)\phi_j(x)d\Gamma_x}_{=:M_{ij}} - \sum_{i=1}^N q_0^{(i)} \underbrace{\int_{\Gamma_j} (1-\varepsilon(x)) \int_{\Gamma_i} G(x,y)\phi_j(x)\phi_i(y)d\Gamma_y d\Gamma_x}_{=:S_{ij}} = \underbrace{\sigma \int_{\Gamma_j} \varepsilon(x)T^4(x)\phi_j(x)d\Gamma_x}_{=:f_j} \tag{5.2}$$

holds for all $j = 1,\dots,N$. We can write (5.2) in the following short form:

$$Cq_0 := (M-S)q_0 = f, \tag{5.3}$$

using the abbreviations $M := (M_{ij})_{i,j=1,\dots,N}$ for the mass matrix, $S := (S_{ij})_{i,j=1,\dots,N}$ for the view factor matrix, and $f = (f_j)_{j=1,\dots,N}$ for the right-hand side of the discretized equation.

Either the mass matrix M and the right-hand side f can be calculated analytically exact for special geometries or numerical integration is applied. To keep the numerical integration error small, we handle the weak singularity of the integral kernel in the case of a nonsmooth boundary by employing double partial integration; see [7, 8].

The main problem is the efficient detection of the shadow zones to calculate the visibility function $\beta(x,y)$ appearing as part of the visibility matrix S for nonconvex enclosures. To reduce the computational effort, in [5] a geometrical algorithm was developed to determine the shadow function in the two-dimensional case for polygonal domains. This algorithm was transformed to the three-dimensional case for an enclosure with polyhedral boundary and consists of the following steps. First, we decide whether the geometry is convex or nonconvex using an angle criterion. Then, an element-orientated preview factor matrix is calculated to reduce the number of elements we have to deal with in the last step, the nodewise calculation of the view factors. With this algorithm, we obtain reasonable results since less than 5 percent of all view factors have to be calculated numerically. For more details, see [3, 8, 11].

Some solution methods for the discrete heat equation (5.3), for example, cg-method with or without preconditioning, direct solvers, multigrid methods, have been compared in [6]. In the three-dimensional case from our experience, the conjugate gradient algorithm with preconditioning has turned out to be the most efficient method and will be applied in the following example.

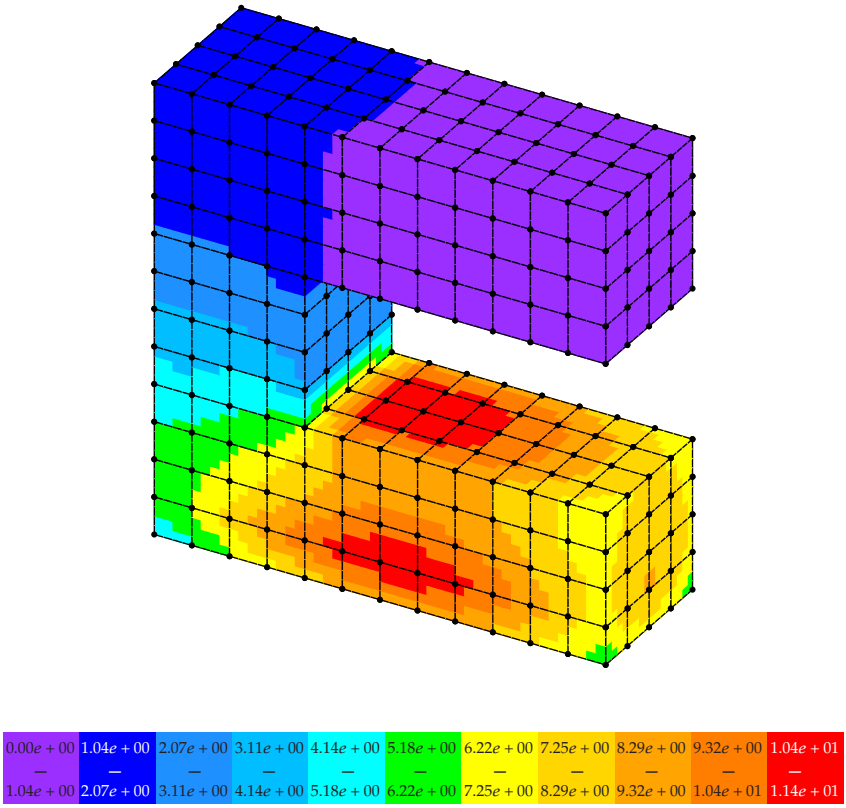

Figure 5.1. Outgoing radiative flux q_0.

As a nonconvex geometry, we take an aperture as depicted and use a quadrangular discretization of the surface Γ into 480 elements. The emissivity coefficient is chosen as $\varepsilon = 0.2$, the Stefan-Boltzmann constant has the value $\sigma = 5.67 \cdot 10^{-8}\ \mathrm{W}/(\mathrm{m}^2 K^4)$, and the temperature source on the bottom will be given by the function $T = 500\sqrt[4]{x(1.5-x)y(0.5-y)}K$. The error is controlled a posteriori by the residual.

Then the outgoing radiative flux looks as in Figure 5.1.

References

[1] R. Bialecki, *Boundary element calculation of the radiative heat sources*, Advanced Computational Methods in Heat Transfer II. Vol. 1 (L. C. Wrobel, C. A. Brebbia, and A. J. Nowak, eds.), Elsevier, London, 1992, pp. 205–217.

[2] ———, *Solving Heat Radiation Problems Using the Boundary Element Method*, Topics in Engineering, vol. 15, Computational Mechanics Publications, Southampton, 1993.

[3] F. Durand, *A multidisciplinary survey of visibility*, ACM SIGGRAPH Course Notes on Visibility, Problems, Techniques, and Applications, ACM Press, New York, 2000, pp. 1–146.

[4] D. Gaier, *Integralgleichungen erster Art und konforme Abbildung*, Math. Z. **147** (1976), no. 2, 113–129.

[5] M. Maischak, *Tätigkeitsbericht SFB 259, Teilprojekt B7*, Mathematisches Institut A, Universität Stuttgart, Germany, 1997.

[6] N. Qatanani, *Lösungsverfahren und Analysis der Integralgleichung für das Hohlraum-Strahlungs-Problem* [*Solution method and analysis of the integral equation for the problem of cavity radiation*], Ph.D. thesis, Universität Stuttgart, Stuttgart, 1996.

[7] M. Schulz and H. Schulz, *A new method for the computation of the view factor matrix for three dimensional heat radiation problems*, in preparation.

[8] M. Schulz and W. L. Wendland, *Mathematical analysis and numerical realization of heat radiation problems*, in preparation.

[9] C. Schwab and W. L. Wendland, *Kernel properties and representations of boundary integral operators*, Math. Nachr. **156** (1992), 187–218.

[10] R. Siegel and J. R. Howell, *Thermal Radiation Heat Transfer*, Hemisphere Publishing, New York, 1981.

[11] S. Teller and P. Hanrahan, *Global visibility algorithms for illumination computations*, SIGGRAPH 1993, ACM Press, New York, 1993, pp. 239–246.

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