LOCATING REAL EIGENVALUES OF A SPECTRAL PROBLEM IN FLUID-SOLID TYPE STRUCTURES

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Exploiting minmax characterizations for nonlinear and nonoverdamped eigenvalue problems, we prove the existence of a countable set of eigenvalues converging to ∞ and inclusion theorems for a rational spectral problem governing mechanical vibrations of a tube bundle immersed in an incompressible viscous fluid. The paper demonstrates that the variational characterization of eigenvalues is a powerful tool for studying nonoverdamped eigenproblems, and that the appropriate enumeration of the eigenvalues is of predominant importance, whereas the natural ordering of the eigenvalues may yield false conclusions.

1. Introduction

Characterizations of eigenvalues as minmax or maxmin values of the Rayleigh quotient are known to be very powerful tools when studying selfadjoint linear operators on a Hilbert space H. To name just a few applications, bounds for eigenvalues, comparison theorems, interlacing results, and monotonicity of eigenvalues can be proved easily. Generalizations to families of operators depending nonlinearly on an eigenparameter were derived for overdamped problems, that is, if a generalized Rayleigh quotient called Rayleigh functional is defined on the entire space H [6, 7, 8, 9, 10, 15], and counting the eigenvalue in an appropriate way it was shown in [11, 13] that variational characterizations of eigenvalues hold in the nonoverdamped case, too. In this paper, we apply this variational characterization to a model governing small vibrations of a tube bundle immersed in an incompressible viscous fluid.

The governing nonclassical eigenvalue problem involving the Stokes system of equations with nonlocal and rational boundary conditions was studied by Conca et al. [3] (see also the recent monograph [5]). Transforming this problem to one of determining the characteristic values of a compact (nonselfadjoint) operator they proved that there exists a countable set of eigenvalues which converge to infinity. Moreover, it was shown that the number of eigenvalues with nonvanishing imaginary part is finite, and that they are all lying in a semicircle about the origin in the left half-plane. In [2] an upper bound

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of the number of nonreal eigenvalues was provided, and upper and lower bounds of the real eigenvalues were derived. The proof of these bounds however is false.

Taking advantage of variational characterizations of real eigenvalues for selfadjoint nonlinear eigenproblems, we obtain the existence of countably many real eigenvalues in a more transparent and less technical way than in [3], and comparing the Rayleigh functional of the rational problem to the Rayleigh quotients of suitable linear eigenvalue problems, we prove upper and lower a priori bounds for the real eigenvalues outside the semicircle mentioned above.

A crucial point when applying minmax or maxmin characterizations of eigenvalues for nonoverdamped problems is to enumerate the eigenvalues correctly. The natural ordering which was used in [2] is inappropriate, and a finite element discretization of the rational eigenproblem (to which the techniques from [2, 3] and the ones used here apply as well), demonstrates that the bounds developed in [2] actually do not hold.

The aim of this paper is twofold. Firstly, it demonstrates by a fluid-structure interaction problem the direct study of which requires a mathematical analysis which is far from being trivial (cf. [2, 3, 5]), that the variational characterization of eigenvalues is a powerful tool for studying nonoverdamped nonlinear eigenvalue problems. Secondly, enumerating the eigenvalues appropriately is of predominant importance, whereas the naive enumeration of the eigenvalues may yield false conclusions.

Our paper is organized as follows. Section 2 summarizes the minmax characterization of eigenvalues of nonoverdamped eigenproblems where the eigenparameter appears nonlinearly. Section 3 outlines the rational eigenvalue problem governing small vibrations of a tube bundle immersed in an incompressible viscous fluid and collects the results in [2, 3] on the number and location of the eigenvalues. In Section 4, we derive the existence of a countable set of real eigenvalues, converging to infinity in a transparent way, from the variational characterizations in Section 2, and we obtain lower and upper bounds of these real eigenvalues. The paper closes with a numerical example revealing that the bounds given in [2] do not hold.

2. Variational characterization of eigenvalues of nonlinear eigenproblems

We consider the nonlinear eigenvalue problem

$$T(\lambda)x = 0, \tag{2.1}$$

where $T(\lambda)$ for every λ in an open real interval J is a selfadjoint and bounded operator on a Hilbert space H. As in the linear case, $\lambda \in J$ is called an eigenvalue of problem (2.1) if (2.1) has a nontrivial solution $x \neq 0$. Such an x is called an eigenelement or eigenvector corresponding to λ.

We assume that

$$f : \begin{cases} J \times H \longrightarrow \mathbb{R}, \\ (\lambda, x) \longmapsto \langle T(\lambda)x, x\rangle \end{cases} \tag{2.2}$$

is continuously differentiable, and that for every fixed $x \in H^0$, $H^0 := H \setminus \{0\}$, the real equation

$$f(\lambda, x) = 0 \tag{2.3}$$

has at most one solution in J. Then (2.3) implicitly defines a functional p on some subset D of H^0 which we call the Rayleigh functional.

We assume that

$$\frac{\partial}{\partial\lambda} f(\lambda, x)\,|_{\lambda=p(x)} > 0 \quad \text{for every } x \in D. \tag{2.4}$$

Then it follows from the implicit function theorem that D is an open set and that p is continuously differentiable on D.

For the linear eigenvalue value problem $T(\lambda) := \lambda I - A$ where $A : H \to H$ is selfadjoint and continuous the assumptions above are fulfilled, and p is the Rayleigh quotient and $D = H^0$. If A additionally is completely continuous, then A has a countable set of eigenvalues which can be characterized as minmax and maxmin values of the Rayleigh quotient by the principles of Poincaré and of Courant, Fischer, and Weyl (cf. [14]).

The nonlinear eigenproblems variational properties using the Rayleigh functional were proved for overdamped systems (i.e., if the Rayleigh functional is defined on H^0) by Duffin [6] and Rogers [9] for the finite-dimensional case and by Hadeler [7, 8], Rogers [10], and Werner [15] for the infinite-dimensional case. For nonoverdamped systems Werner and the author [13] proved a minmax characterization of Poincaré type; a maxmin characterization generalizing the principle of Courant, Fischer, and Weyl is contained in [11].

In this section, we assemble the results in [11, 13] for the nonlinear nonoverdamped eigenvalue problem (2.1).

We denote by H_j the set of all j-dimensional subspaces of H and by $V_1 := \{v \in V : \|v\| = 1\}$ the unit sphere of the subspace V of H.

We already stressed the fact that the eigenvalues of problem (2.1) have to be enumerated appropriately to derive variational characterizations for nonoverdamped problems. To this end we assume that for every fixed $\lambda \in J$ there exists $\nu(\lambda) > 0$ such that the linear operator $T(\lambda) + \nu(\lambda)I$ is completely continuous. Then the essential spectrum of $T(\lambda)$ contains only the point $-\nu(\lambda)$, and every eigenvalue $\mu > -\nu(\lambda)$ of $T(\lambda)$ can be characterized as maxmin value of the Rayleigh quotient of $T(\lambda)$. In particular, if λ is an eigenvalue of the nonlinear problem (2.1), then $\mu = 0$ is an eigenvalue of the linear problem $T(\lambda)y = \mu y$, and therefore there exists $n \in \mathbb{N}$ such that

$$\mu_n(\lambda) := \max_{V \in H_n} \min_{v \in V_1} \langle T(\lambda)v, v \rangle = 0. \tag{2.5}$$

In this case, we call λ an nth eigenvalue of the nonlinear eigenvalue problem (2.1).

With this enumeration the following minmax characterization of the eigenvalues of problem (2.1) holds which was proved in [13].

Theorem 2.1. *Under the conditions given above the following assertions hold.*

(i) *For every $n \in \mathbb{N}$ there is at most one nth eigenvalue of problem (2.1) which can be characterized by*

$$\lambda_n = \min_{\substack{V \in H_n \\ V \cap D \neq \emptyset}} \sup_{v \in V \cap D} p(v). \tag{2.6}$$

The minimum is attained by the invariant subspace W of $T(\lambda_n)$ corresponding to the n largest eigenvalues of $T(\lambda_n)$, and $\sup_{v \in W \cap D} p(v)$ is attained by all eigenvectors of (2.1) corresponding to λ_n. The set of eigenvalues of (2.1) is at most countable.

(ii) *If, conversely,*

$$\lambda_n = \inf_{\substack{V \in H_n \\ V \cap D \neq \emptyset}} \sup_{v \in V \cap D} p(v) \in J \tag{2.7}$$

for some $n \in \mathbb{N}$, then λ_n is the nth eigenvalue of (2.1) and the characterization (2.6) holds.

The characterization of the eigenvalues in Theorem 2.1 is a generalization of the minmax principle of Poincaré for linear eigenvalue problems. In a similar way as in [13], the maxmin characterization of Courant, Fischer, and Weyl can be generalized to the nonlinear case (cf. [11]).

Theorem 2.2. *If problem (2.1) has an nth eigenvalue $\lambda_n \in J$, then*

$$\lambda_n = \max_{\substack{V \in H_{n-1} \\ V^\perp \cap D \neq \emptyset}} \inf_{v \in V^\perp \cap D} p(v). \tag{2.8}$$

3. A rational eigenvalue problem in fluid-structure interaction

This section is devoted to the presentation of the mathematical model which describes the problem governing free vibrations of a tube bundle immersed in an incompressible viscous fluid whose velocity field and pressure satisfy the steady Stokes equations. The tubes are assumed to be rigid, assembled in parallel inside the fluid, and elastically mounted in such a way that they can vibrate transversally, but they cannot move in the direction perpendicular to their sections. The fluid is assumed to be contained in a cavity which is infinitely long, and each tube is supported by an independent system of springs (which simulates the specific elasticity of each tube). Due to these assumptions, three-dimensional effects are neglected, and so the problem can be studied in any transversal section of the cavity.

Considering small vibrations of the fluid (and the tubes) around the state of rest, and assuming that the fluid is viscous and incompressible, this is a nonclassical eigenvalue problem involving the Stokes system of equations with nonlinear conditions on the boundaries of the tubes, which model the fluid-solid interaction. On the boundary of the cavity we assume the standard nonslip conditions.

Mathematically, the problem can be described in the following way as shown in Figure 3.1 (cf. [3, 5]): let $\Omega_0 \subset \mathbb{R}^2$ (the section of the cavity) be an open bounded set with locally Lipschitz continuous boundary Γ_0. We assume that there exists a family $\Omega_j \neq \emptyset$, $j = 1,\dots,K$, (the sections of the tubes) of simply connected open sets such that $\bar{\Omega}_j \subset \Omega_0$

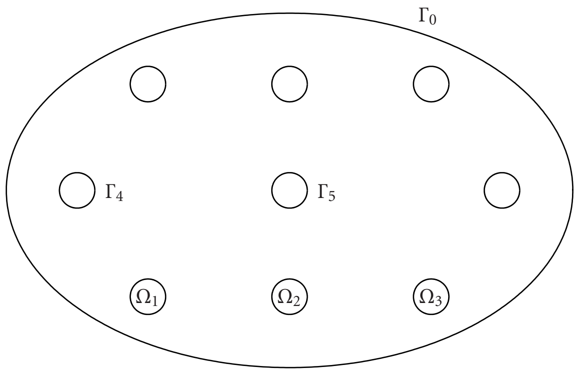

Figure 3.1. Domain Ω.

for every j, $\bar{\Omega}_j \cap \bar{\Omega}_i = \emptyset$ for $j \neq i$, and each Ω_j has a locally Lipschitz continuous boundary Γ_j. With these notations we set $\Omega := \Omega_0 \setminus \bigcup_{j=1}^{K} \bar{\Omega}_j$. Then the boundary Γ of Ω consists of $K+1$ connected components which are Γ_0 and Γ_j, $j = 1,\dots,K$.

If $\mathbf{u}(x)e^{-\omega t}$ is the velocity field of the fluid, $p(x)e^{-\omega t}$ denotes its pressure, and ν its kinematic viscosity, then the eigenvalue problem governing the free vibrations of the fluid–solid structure which was derived by Conca et al. [3] obtains the following form:

$$-2\nu \operatorname{div} e(\mathbf{u}) + \nabla p - \omega \mathbf{u} = 0 \quad \text{in } \Omega, \tag{3.1}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{3.2}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_0, \tag{3.3}$$

$$\mathbf{u} = \frac{\omega}{k_i + \omega^2 m_i} \int_{\Gamma_i} \sigma(\mathbf{u}, p)\mathbf{n}\, ds \quad \text{on } \Gamma_i. \tag{3.4}$$

Here m_i is the mass per unit length of the ith tube, and k_i represents the stiffness constant of the spring system supporting the ith tube. $e(\mathbf{u})$ is the linear strain tensor of the fluid defined by

$$2e(\mathbf{u}) = \nabla \mathbf{u} + (\nabla \mathbf{u})^T, \tag{3.5}$$

and $\sigma(\mathbf{u}, p)$ denotes its stress tensor satisfying the Stokes law

$$\sigma(\mathbf{u}, p) = -pI + 2\nu e(\mathbf{u}). \tag{3.6}$$

To rewrite problem (3.1)–(3.4) in variational form let

$$H^1(\Omega)^2 := \{\mathbf{v} \in L^2(\Omega)^2 : \nabla \mathbf{v} \in L^2(\Omega)^4\} \tag{3.7}$$

be the standard Sobolev space equipped with the usual scalar product. Then clearly

$$H := \{\mathbf{v} \in H^1(\Omega)^2 : \operatorname{div} \mathbf{v} = 0,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0,\ \mathbf{v} \text{ constant on each } \Gamma_j,\ j = 1,\dots,K\} \tag{3.8}$$

is a closed subspace of $H^1(\Omega)^2$.

It is well known from Korn's inequality that the scalar product

$$\langle \mathbf{u},\mathbf{v}\rangle := \int_\Omega e(\mathbf{u}) : e(\bar{\mathbf{v}})dx := \int_\Omega \sum_{i,j=1}^{2} e_{ij}(\mathbf{u})e_{ij}(\bar{\mathbf{v}})dx \tag{3.9}$$

defines a norm on H which is equivalent to the standard Sobolev norm. Hence, H equipped with this scalar product is a Hilbert space.

Multiplying (3.1) by $\bar{\mathbf{v}} \in H$ and integrating by parts, one gets (cf. [3]) the variational form of problem (3.1)–(3.4).

Find $\omega \in \mathbb{C}$ and $\mathbf{u} \in H$, $\mathbf{u} \neq \mathbf{0}$ such that for every $\mathbf{v} \in H$,

$$2\nu \int_\Omega e(\mathbf{u}) : e(\bar{\mathbf{v}})dx = \omega \int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}}\, dx + \sum_{j=1}^{K} \left(\omega m_j + \frac{k_j}{\omega}\right)\gamma_j(\mathbf{u}) \cdot \gamma_j(\bar{\mathbf{v}}), \tag{3.10}$$

where $\gamma_j(\mathbf{u})$ denotes the trace of $\mathbf{u}$ on Γ_j which by the definition of H is a constant vector. By standard arguments it can be shown that the eigenproblems (3.1)–(3.4) and (3.10) are equivalent in the following sense: if $(\mathbf{u}, p, \omega)$ solves the eigenproblem (3.1)–(3.4), then $(\mathbf{u}, \omega)$ is a solution of (3.10), and conversely, if $(\mathbf{u}, \omega)$ is a solution of (3.10), then there exists $p \in L^2(\Omega)$ such that $(\mathbf{u}, p, \omega)$ solves (3.1)–(3.4).

Conca et al. [3] multiplied the rational eigenproblem (3.10) by ω obtaining a quadratic problem. They proved that the eigenvalues of this problem are the characteristic values of a compact operator acting on a Hilbert space. Hence, they obtained that the set of eigenvalues of problem (3.10) is countable, and its only cluster point is ∞. Moreover, they proved the following location result.

Theorem 3.1. *Let $(\omega, \mathbf{u})$ be a solution of the rational eigenvalue problem (3.10). Then the following assertions hold:*

(i) $\mathrm{Re}(\omega) > 0$,

(ii) *if* $\mathrm{Im}(\omega) \neq 0$, *then*

$$\begin{aligned} |\omega|^2 &< \frac{k}{m} := \max\left\{\frac{k_j}{m_j} : j = 1,\dots,K\right\}, \\ \mathrm{Re}(\omega) &\geq \frac{1}{2}\mu, \qquad \sum_{j=1}^{K} k_j \left|\gamma_j(\mathbf{u})\right|^2 > 0, \end{aligned} \tag{3.11}$$

where μ denotes the smallest eigenvalue of the linear eigenproblem.

Find $\mu \in \mathbb{C}$ and $\mathbf{v} \in H$, $\mathbf{v} \neq \mathbf{0}$ such that for every $\mathbf{w} \in H$,

$$2\nu \int_\Omega e(\mathbf{v}) : e(\bar{\mathbf{w}})dx = \mu\left(\int_\Omega \mathbf{v} \cdot \bar{\mathbf{w}}\, dx + \sum_{j=1}^{K} m_j \gamma_j(\mathbf{v}) \cdot \gamma_j(\bar{\mathbf{w}})\right). \tag{3.12}$$

From Theorem 3.1(ii), it follows at once that problem (3.10) has only a finite number of nonreal eigenvalues. In [2] Conca et al. proved that the maximum number of nonreal eigenvalues is $4K$, and [1] contains a numerical example that demonstrates that this bound is attained, which is approved by our numerical example in Section 5 as well.

4. Comparison results

In this section, we prove inclusion results for the real eigenvalues $\omega_j > \sqrt{k/m}$ taking advantage of the minmax characterization for these eigenvalues and comparing the Rayleigh functional with Rayleigh quotients R_1 of the linear eigenvalue problem (3.12) and R_2 of the linear problem.

Find $\omega \in \mathbb{C}$ and $\mathbf{v} \in H$ such that for every $\mathbf{w} \in H$,

$$2\nu \int_\Omega e(\mathbf{v}) : e(\bar{\mathbf{w}})dx = \mu\left(\int_\Omega \mathbf{v} \cdot \bar{\mathbf{w}}\,dx + \sum_{j=1}^{K}\left(m_j + \frac{m}{k}k_j\right)\gamma_j(\mathbf{v}) \cdot \gamma_j(\bar{\mathbf{w}})\right). \tag{4.1}$$

Problem (3.10) fulfills the conditions of the minmax theory for the interval $J := (\sqrt{k/m}, \infty)$ since for

$$F(\omega,\mathbf{u}) := -2\nu \int_\Omega e(\mathbf{u}) : e(\bar{\mathbf{u}})dx + \omega \int_\Omega |\mathbf{u}|^2\,dx + \sum_{j=1}^{K}\left(\omega m_j + \frac{k_j}{\omega}\right)\left|\gamma_j(\mathbf{u})\right|^2, \tag{4.2}$$

we have

$$\frac{\partial}{\partial\omega}F(\omega,\mathbf{u}) = \int_\Omega |\mathbf{u}|^2\,dx + \sum_{j=1}^{K}\left(m_j - \frac{k_j}{\omega^2}\right)\left|\gamma_j(\mathbf{u})\right|^2 > 0, \tag{4.3}$$

if

$$m_j - \frac{k_j}{\omega^2} > 0 \quad \text{for every } j, \text{ i.e., } \omega^2 > \max_{j=1,\dots,K}\frac{k_j}{m_j} = \frac{k}{m}. \tag{4.4}$$

Hence, all eigenvalues $\omega_j \in J$ of problem (3.10) can be characterized by

$$\omega_j = \min_{\substack{V \in H_j \\ V \cap D \neq \emptyset}} \sup_{v \in V \cap D} p(\mathbf{v}), \tag{4.5}$$

where the Rayleigh functional p is defined by $F(\omega,\mathbf{u}) = 0$, and F is given in (4.2). By D we denote the domain of definition of p.

LEMMA 4.1. *Let R_1 be the Rayleigh quotient of the linear eigenproblem (3.12). Then it holds that*

$$p(\mathbf{u}) \le R_1(\mathbf{u}) \quad \textit{for every } \mathbf{u} \in D. \tag{4.6}$$

Proof. For every $\mathbf{u} \in H$, $\mathbf{u} \neq \mathbf{0}$ it holds that

$$\begin{aligned} F(R_1(\mathbf{u}),\mathbf{u}) &= -2\nu \int_\Omega e(\mathbf{u}) : e(\bar{\mathbf{u}})dx + R_1(\mathbf{u}) \int_\Omega |\mathbf{u}|^2\,dx + \sum_{j=1}^{K}\left(R_1(\mathbf{u})m_j + \frac{k_j}{R_1(\mathbf{u})}\right)\left|\gamma_j(\mathbf{u})\right|^2 \\ &= \frac{1}{R_1(\mathbf{u})}\sum_{j=1}^{K} k_j \left|\gamma_j(\mathbf{u})\right|^2 \ge 0. \end{aligned} \tag{4.7}$$

Hence, if $\mathbf{u} \in D$, that is, $F(\omega, \mathbf{u}) = 0$ has a solution $p(\mathbf{u}) \in J$, then it follows from (4.3) that $p(\mathbf{u}) \le R_1(\mathbf{u})$. □

Lemma 4.2. *Let R_2 denote the Rayleigh quotient of the linear eigenproblem (4.1). If $R_2(\mathbf{u}) \in J$ for some $\mathbf{u} \in H^0$, then $\mathbf{u} \in D$, and $p(\mathbf{u}) \ge R_2(\mathbf{u})$.*

Proof. For $\mathbf{u} \in H^0$ such that $R_2(\mathbf{u}) > \sqrt{k/m}$,

$$\begin{aligned} F(R_2(\mathbf{u}), \mathbf{u}) &= -2\nu \int_\Omega e(\mathbf{u}) : e(\bar{\mathbf{u}}) dx + R_2(\mathbf{u}) \int_\Omega |\mathbf{u}|^2 dx + \sum_{j=1}^K \left(R_2(\mathbf{u}) m_j + \frac{k_j}{R_2(\mathbf{u})} \right) |\gamma_j(\mathbf{u})|^2 \\ &= \sum_{j=1}^K \left(\frac{1}{R_2(\mathbf{u})} - \frac{m}{k} R_2(\mathbf{u}) \right) k_j |\gamma_j(\mathbf{u})|^2 < 0, \\ &\qquad \lim_{\omega \to \infty} F(\omega, \mathbf{u}) = \infty. \end{aligned} \tag{4.8}$$

Thus, $\mathbf{u} \in D$, and $p(\mathbf{u}) \ge R_2(\mathbf{u})$. □

We are now in the position to prove an inclusion theorem for real eigenvalues of problem (3.10).

Theorem 4.3. (i) *Assume that the jth eigenvalue*

$$\mu_j := \min_{V \in H_j} \max_{u \in V^0} R_2(\mathbf{u}) > \frac{k}{m} \tag{4.9}$$

of problem (4.1) is contained in J. Then the nonlinear eigenproblem (3.10) has a jth eigenvalue $\omega_j \in J$, and μ_j is a lower bound of ω_j:

$$\mu_j \le \omega_j. \tag{4.10}$$

Since the linear problem (4.1) has a countable set of eigenvalues the only accumulation point of which is ∞, the nonlinear problem (3.10) must have countably many real eigenvalues as well which also converge to ∞.

(ii) *If (3.10) has a jth eigenvalue $\omega_j \in J$, then*

$$\omega_j \le \eta_j := \min_{V \in H_j} \max_{u \in V^0} R_1(\mathbf{u}). \tag{4.11}$$

Proof. (i) For $V \in H_j$ let $\mathbf{u}_V \in V$ such that $R_2(\mathbf{u}_V) = \max_{v \in V^0} R_2(\mathbf{v})$. Then

$$R_2(\mathbf{u}_V) \ge \min_{W \in H_j} \max_{w \in W^0} R_2(\mathbf{w}) = \mu_j > \frac{k}{m}, \tag{4.12}$$

and Lemma 4.2 yields

$$\mathbf{u}_V \in D, \qquad p(\mathbf{u}_V) \ge R_2(\mathbf{u}_V). \tag{4.13}$$

In particular $V \cap D \neq \emptyset$ for every $V \in H_j$.

Moreover,

$$\begin{aligned}\mu_j &= \min_{V\in H_j}\max_{v\in V^0} R_2(\mathbf{v}) = \min_{V\in H_j} R_2(\mathbf{u}_V)\\ &\le \min_{V\in H_j} p(\mathbf{u}_V) \le \min_{V\in H_j}\sup_{u\in V\cap D} p(\mathbf{u}).\end{aligned} \tag{4.14}$$

By Theorem 2.1(ii), the nonlinear eigenvalue problem (3.10) has a jth eigenvalue ω_j, and $\mu_j \le \omega_j$.

(ii) Since $V\cap D \neq \emptyset$ for every $V\in H_j$, we obtain from Lemma 4.1

$$\begin{aligned}\omega_j &= \min_{\substack{V\in H_j\\ V\cap D\neq\emptyset}}\sup_{v\in V\cap D} p(\mathbf{v}) \le \min_{\substack{V\in H_j\\ V\cap D\neq\emptyset}}\sup_{v\in V\cap D} R_1(\mathbf{v})\\ &\le \min_{\substack{V\in H_j\\ V\cap D\neq\emptyset}}\max_{v\in V^0} R_1(\mathbf{v}) = \min_{V\in H_j}\max_{v\in V^0} R_1(\mathbf{v}) = \eta_j.\end{aligned} \tag{4.15}$$

□

Remark 4.4. Multiplying the nonlinear eigenproblem (3.10) by ω, we consider the resulting quadratic eigenproblem.

Find $\rho := 1/\omega \in \mathbb{C}$ and $\mathbf{v}\in H$ such that for every $\mathbf{w}\in H$,

$$\left(\int_\Omega \mathbf{v}\cdot\bar{\mathbf{w}}\,dx + \sum_{j=1}^{K} m_j\gamma_j(\mathbf{v})\cdot\gamma_j(\bar{\mathbf{w}})\right) - \rho\left(2\nu\int_\Omega e(\mathbf{v}):e(\bar{\mathbf{w}})dx\right) + \rho^2\sum_{j=1}^{K} k_j\gamma_j(\mathbf{v})\cdot\gamma_j(\bar{\mathbf{w}}) = 0 \tag{4.16}$$

as positive perturbation of finite range of the linear eigenproblem (3.12). Conca, Duran, and Planchard claimed the following bounds.

Let $0<\tilde{\omega}_1\le\tilde{\omega}_2\le\cdots$ be the real eigenvalues of the nonlinear eigenproblem (3.10) ordered by magnitude and regarding their multiplicity, and let $0<\eta_1\le\eta_2\le\cdots$ be the eigenvalues of the linear problem (3.12). Then it holds that

$$\tilde{\omega}_j \le \eta_j \quad \text{for } j=1,\dots,2K, \tag{4.17}$$

$$\eta_{j-2K} \le \tilde{\omega}_j \le \eta_j \quad \text{for } j \ge 2K+1, \tag{4.18}$$

where K denotes the number of tubes.

We already pointed out in [12] that the natural enumeration whereby we call the smallest eigenvalue the first one, the second smallest the second one, and so forth is not appropriate for the quadratic eigenvalue problem (4.16), and therefore the proof of these bounds is not correct. The numerical example in the next section demonstrates that the bounds (4.17) and (4.18) actually do not hold.

For those eigenvalues ω_j contained in J, the bounds (4.17) and (4.18) can be adjusted if we replace the natural ordering of the eigenvalues $\tilde{\omega}_j$ by the enumeration introduced in Section 2. The upper bound $\omega_j\le\eta_j$ is already contained in Theorem 4.3(ii).

The lower bound is obtained from the maxmin characterization in Theorem 2.1. Let $W=\operatorname{span}\{\mathbf{u}_1,\dots,\mathbf{u}_{n-2K-1}\}$ denote the subspace of H spanned by the eigenelements of problem (3.12) corresponding to the $n-2K-1$ smallest eigenvalues, and let

$$Z=\left\{\mathbf{u}\in H:\sum_{j=1}^{K}k_j\gamma_j(\mathbf{u})\gamma_j(\bar{\mathbf{v}})=0 \text{ for every } \mathbf{v}\in H\right\}^{\perp}. \tag{4.19}$$

Then obviously $p(\mathbf{u})=R_1(\mathbf{u})$ for every $\mathbf{u}\in D\cap Z$, and we obtain from Rayleigh's principle and the maxmin characterization in Theorem 2.1

$$\begin{aligned}\eta_{n-2K} &= \min_{u\in W^{\perp}} R_1(\mathbf{u}) \le \min_{u\in (W+Z)^{\perp}} R_1(\mathbf{u})\\ &\le \inf_{u\in (W+Z)^{\perp}\cap D} p(\mathbf{u}) \le \max_{\dim V\le n-1}\ \inf_{u\in V^{\perp}\cap D} p(\mathbf{u})=\omega_n.\end{aligned} \tag{4.20}$$

5. Numerical experiments

While the variational form (3.10) was convenient for the theoretical study of problem (3.1)–(3.4), its numerical treatment requires to deal with the incompressibility condition $\operatorname{div}\mathbf{u}=0$ implicitly, and to use a mixed variational formulation, which reads as follows (cf. [1, 4]).

Find $(\mathbf{u},p,\omega)\in H\times L^2(\Omega)\times\mathbb{C}$, $(\mathbf{u},p)\neq(\mathbf{0},0)$ such that for every $(\mathbf{v},q)\in H\times L^2(\Omega)$,

$$\begin{gathered}2\nu\int_{\Omega}e(\mathbf{u}):e(\bar{\mathbf{v}})dx+\int_{\Omega}p\operatorname{div}\bar{\mathbf{v}}\,dx=\omega\int_{\Omega}\mathbf{u}\cdot\bar{\mathbf{v}}\,dx+\sum_{j=1}^{K}\left(\omega m_j+\frac{k_j}{\omega}\right)\gamma_j(\mathbf{u})\cdot\gamma_j(\bar{\mathbf{v}}),\\ \int_{\Omega}\bar{q}\operatorname{div}\mathbf{u}\,dx=0.\end{gathered} \tag{5.1}$$

Here H denotes the space as follows:

$$H:=\left\{\mathbf{v}\in H^1(\Omega)^2:\mathbf{v}=\mathbf{0}\text{ on }\Gamma_0,\ \mathbf{v}\text{ constant on each }\Gamma_j,\ j=1,\dots,K\right\} \tag{5.2}$$

which again is a closed subspace of $H^1(\Omega)^2$.

We discretized this problem by finite elements using piecewise quadratic ansatz functions on a regular triangulation of Ω for the velocity field, and piecewise linear functions on the same triangulation for the pressure yielding a rational matrix eigenvalue problem which can be reduced to a general matrix eigenvalue problem and solved using standard numerical software. The convergence properties of this approach are studied in [4].

We consider problem (5.1) where $\Omega_0=(0,3)\times(0,3)$ is the section of the cave, and four structures are contained in it with sections $\Omega_1=(0.8,1.0)\times(0.8,1.0)$, $\Omega_2=(2.0,2.2)\times(0.8,1.0)$, $\Omega_3=(0.8,1.0)\times(2.0,2.2)$, and $\Omega_4=(2.0,2.2)\times(2.0,2.2)$. In all experiments we chose $\nu=1$ and $m:=m_j=1$, $j=1,2,3,4$, and we assumed that all $k_j=:k$ are identical.

For $k\ge 30.82$ the discrete version of (5.1) has nonreal eigenvalues, and for $k\ge 106.03$ there exist 16 nonreal eigenvalues demonstrating that the bound $4K$ on the number of nonreal eigenvalues is attained.

For $k=400$ the smallest real eigenvalue is $\tilde{\omega}_1=13.478$, whereas the smallest eigenvalue of (3.12) is $\eta_1=9.671$ demonstrating that (4.17) does not hold. Finally, $k=1$ contradicts the lower bound in (4.18), since $\tilde{\omega}_9=9.605$, whereas $\eta_1=9.672$.

Table 5.1

j	μ_j	η_j	$\omega_j(k=1)$	$\omega_j(k=400)$
1	$5.5273441e+00$	$9.6715372e+00$	$9.6051792e+00$	—
2	$6.3743399e+00$	$1.1103915e+01$	$1.1055398e+01$	—
3	$6.4656339e+00$	$1.2237010e+01$	$1.2164933e+01$	—
4	$7.1377728e+00$	$1.2907175e+01$	$1.2899003e+01$	—
5	$8.6717113e+00$	$1.3728925e+01$	$1.3663594e+01$	—
6	$9.5210984e+00$	$1.4538864e+01$	$1.4516629e+01$	—
7	$1.0069363e+01$	$1.5059600e+01$	$1.5026228e+01$	—
8	$1.0290363e+01$	$1.5874023e+01$	$1.5872178e+01$	—
9	$1.3605487e+01$	$1.7630409e+01$	$1.7588566e+01$	—
10	$1.3716557e+01$	$1.9280833e+01$	$1.9246874e+01$	—
11	$1.5190485e+01$	$1.9647977e+01$	$1.9604771e+01$	—
12	$1.5870909e+01$	$1.9893125e+01$	$1.9848079e+01$	—
13	$2.2897653e+01$	$2.3661814e+01$	$2.3655921e+01$	$2.2948292e+01$
14	$3.2789694e+01$	$3.3880279e+01$	$3.3877702e+01$	$3.3216627e+01$
15	$3.5497102e+01$	$3.6107300e+01$	$3.6106314e+01$	$3.5808040e+01$
16	$3.5548471e+01$	$3.6185088e+01$	$3.6183766e+01$	$3.5828882e+01$
17	$3.7479680e+01$	$3.7852470e+01$	$3.7851786e+01$	$3.7663430e+01$
18	$3.7929796e+01$	$3.8388335e+01$	$3.8387384e+01$	$3.8142407e+01$
19	$3.8082108e+01$	$3.8535861e+01$	$3.8535044e+01$	$3.8307897e+01$
20	$4.3651778e+01$	$4.3695701e+01$	$4.3695639e+01$	$4.3677373e+01$
21	$4.9197938e+01$	$4.9379084e+01$	$4.9378915e+01$	$4.9323111e+01$

Table 5.1 contains the smallest eigenvalues μ_j and η_j of the linear problems (4.1) and (3.12), respectively, which for $m = 1$ and identical k_j are bounds for eigenvalues greater than $\sqrt{k}$. In columns 4 and 5 we added the smallest real eigenvalues ω_j of the rational eigenproblem for $k = 1$ and $k = 400$ satisfying $\omega_j > \sqrt{k}$ where these eigenvalues are enumerated in the way introduced in Section 2.

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