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Research Article

Minimizing Banking Risk in a Lévy Process Setting

F. Gideon, J. Mukuddem-Petersen, and M. A. Petersen

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The primary functions of a bank are to obtain funds through deposits from external sources and to use the said funds to issue loans. Moreover, risk management practices related to the withdrawal of these bank deposits have always been of considerable interest. In this spirit, we construct Lévy process-driven models of banking reserves in order to address the problem of hedging deposit withdrawals from such institutions by means of reserves. Here reserves are related to outstanding debt and act as a proxy for the assets held by the bank. The aforementioned modeling enables us to formulate a stochastic optimal control problem related to the minimization of reserve, depository, and intrinsic risk that are associated with the reserve process, the net cash flows from depository activity, and cumulative costs of the bank's provisioning strategy, respectively. A discussion of the main risk management issues arising from the optimization problem mentioned earlier forms an integral part of our paper. This includes the presentation of a numerical example involving a simulation of the provisions made for deposit withdrawals via Treasuries and reserves.

1. Introduction

We apply the quadratic hedging approach developed in [1] to a situation related to bank deposit withdrawals. In incomplete markets, this problem arises due to the fact that random obligations cannot be replicated with probability one by trading in available assets. For any hedging strategy, there is some residual risk. More specifically, in the quadratic hedging approach, the variance of the hedging error is minimized. With regard to this,

our contribution addresses the problem of determining risk minimizing hedging strategies that may be employed when a bank faces deposit withdrawals with fixed maturities resulting from lump sum deposits.

In the recent past, more attention has been given to modeling procedures that deviate from those that rely on the seminal Black-Scholes financial model (see, e.g., [2, 3]). Some of the most popular and tractable of these procedures are related to Lévy process-based models. In this regard, our paper investigates the dynamics of banking items such as loans, reserves, capital, and regulatory ratios that are driven by such processes. An advantage of Lévy-processes is that they are very flexible since for any time increment Δt, any infinitely divisible distribution can be chosen as the increment distribution of periods of time Δt. In addition, they have a simple structure when compared with general semimartingales and are able to take different important stylized features of financial time series into account. A specific motivation for modeling banking items in terms of Lévy processes is that they have an advantage over the more traditional modeling tools such as Brownian motion (see, e.g., [4–7]), since they describe the noncontinuous evolution of the value of economic and financial indicators more accurately. Our contention is that these models lead to analytically and numerically tractable formulas for banking items that are characterized by jumps.

Some banking activities that we wish to model dynamically are constituents of the assets and liabilities held by the bank. With regard to the former, it is important to be able to measure the volume of Treasuries and reserves that a bank holds. Treasuries are bonds issued by a national treasury and may be modeled as a risk-free asset (bond) in the usual way. In the modern banking industry, it is appropriate to assign a price to reserves and to model it by means of a Lévy process because of the discontinuity associated with its evolution and because it provides a good fit to real-life data. Banks are interested in establishing the level of Treasuries and reserves on demand deposits that the bank must hold. By setting a bank's individual level of reserves, roleplayers assist in mitigating the costs of financial distress. For instance, if the minimum level of required reserves exceeds a bank's optimally determined level of reserves, this may lead to deadweight losses. While the academic literature on pricing bank assets is vast and well developed, little attention is given to pricing bank liabilities. Most bank deposits contain an embedded option which permits the depositor to withdraw funds at will. Demand deposits generally allow costless withdrawal, while time deposits often require payment of an early withdrawal penalty. Managing the risk that depositors will exercise their withdrawal option is an important aspect of our contribution. The main thrust of our paper is the hedging of bank deposit withdrawals. In this spirit, we discuss an optimal risk management problem for commercial banks which use the Treasuries and reserves to cater for such withdrawals. In this regard, the main risks that can be identified are reserve, depository, and intrinsic risk that are associated with the reserve process, the net cash flows from depository activity, and cumulative costs of the bank's provisioning strategy, respectively.

In the sequel, we use the notational convention "subscript t or s" to represent (possibly) random processes, while "bracket t or s" is used to denote deterministic processes. In the ensuing discussion, for the sake of completeness, we firstly provide a general description of a Lévy process and an associated measure and then describe the Lévy

decomposition that is appropriate for our analysis. In this regard, we assume that $\phi(\xi)$ is the characteristic function of a distribution. If for every positive integer n, $\phi(\xi)$ is also the nth power of a characteristic function, we say that the distribution is *infinitely divisible*. For each infinitely divisible distribution, a stochastic process $L = (L_t)_{0\le t}$ called a *Lévy process* exists. This process initiates at zero, has independent and stationary increments and has $(\phi(u))^t$ as a characteristic function for the distribution of an increment over $[s, s+t]$, $0 \le s, t$, such that $L_{t+s} - L_s$. Every Lévy process is a semimartingale and has a cádlág version (right continuous with left-hand limits) which is itself a Lévy process. We will assume that the type of such processes that we work with is always cádlág. As a result, sample paths of L are continuous a.e. from the right and have limits from the left. The *jump of* L_t at $t \ge 0$ is defined by $\Delta L_t = L_t - L_{t^-}$. Since L has stationary independent increments, its *characteristic function* must have the form

$$\mathbf{E}[\exp\{-i\xi L_t\}] = \exp\{-t\Psi(\xi)\} \tag{1.1}$$

for some function Ψ called the *Lévy* or *characteristic exponent* of L. The *Lévy-Khintchine formula* is given by

$$\begin{aligned}\Psi(\xi) = i\gamma\xi + \frac{c^2}{2}\xi^2 &+ \int_{|x|<1}[1-\exp\{-i\xi x\} - i\xi x]\nu(dx)\\ &+ \int_{|x|\ge 1}[1-\exp\{-i\xi x\}]\nu(dx), \quad \gamma, c \in \mathbb{R}\end{aligned} \tag{1.2}$$

and for some σ-finite measure ν on $\mathbb{R}\setminus\{0\}$ with

$$\int \inf\{1, x^2\}\nu(dx) = \int \inf(1\wedge x^2)\nu(dx) < \infty. \tag{1.3}$$

An infinitely divisible distribution has a *Lévy triplet* of the form

$$[\gamma, c^2, \nu(dx)], \tag{1.4}$$

where the measure ν is called the *Lévy measure*.

The Lévy-Khintchine formula given by (1.2) is closely related to the Lévy process, L. This is particularly true for the *Lévy decomposition* of L which we specify in the rest of this paragraph. From (1.2), it is clear that L must be a linear combination of a Brownian motion and a quadratic jump process X which is independent of the Brownian motion. We recall that a process is classified as *quadratic pure jump* if the continuous part of its quadratic variation $\langle X\rangle^c \equiv 0$, so that its quadratic variation becomes

$$\langle X\rangle_t = \sum_{0<s\le t}(\Delta X_s)^2, \tag{1.5}$$

where $\Delta X_s = X_s - X_{s^-}$ is the jump size at time s. If we separate the Brownian component, Z, from the quadratic pure jump component X, we obtain

$$L_t = X_t + cZ_t, \tag{1.6}$$

where X is quadratic pure jump and Z is standard Brownian motion on $\mathbb{R}$. Next, we describe the Lévy decomposition of Z. Let $Q(dt,dx)$ be the Poisson measure on $\mathbb{R}^+ \times \mathbb{R} \setminus \{0\}$ with expectation (or intensity) measure $dt \times \nu$. Here dt is the Lebesque measure and ν is the Lévy measure as before. The measure $dt \times \nu$ (or sometimes just ν) is called the compensator of Q. The Lévy decomposition of X specifies that

$$\begin{aligned} X_t &= \int_{|x|<1} x\left[Q((0,t],dx) - t\nu(dx)\right] + \int_{|x|\geq 1} xQ((0,t],dx) + t\mathbf{E}\left[X_1 - \int_{|x|\geq 1} x\nu(dx)\right] \\ &= \int_{|x|<1} x\left[Q((0,t],dx) - t\nu(dx)\right] + \int_{|x|\geq 1} xQ((0,t],dx) + \gamma t, \end{aligned} \tag{1.7}$$

where

$$\gamma = \mathbf{E}\left[X_1 - \int_{|x|\geq 1} x\nu(dx)\right]. \tag{1.8}$$

The parameter γ is called the *drift of* X. In addition, in order to describe the Lévy decomposition of L, we specify more conditions that L must satisfy. The most important supposition that we make about L is that

$$\mathbf{E}\left[\exp\{-hL_1\}\right] < \infty, \quad \forall h \in (-h_1, h_2), \tag{1.9}$$

where $0 < h_1$, $h_2 \leq \infty$. This implies that L_t has finite moments of all orders and in particular, $\mathbf{E}[X_1] < \infty$. In terms of the Lévy measure ν of X, we have, for all $h \in (-h_1, h_2)$, that

$$\begin{gathered} \int_{|x|\geq 1} \exp\{-hx\}\nu(dx) < \infty, \\ \int_{|x|\geq 1} x^\alpha \exp\{-hx\}\nu(dx) < \infty, \quad \forall \alpha > 0, \\ \int_{|x|\geq 1} x\nu(dx) < \infty. \end{gathered} \tag{1.10}$$

The above assumptions lead to the fact that (1.7) can be rewritten as

$$X_t = \int_{\mathbb{R}} x\left[Q((0,t],dx) - t\nu(dx)\right] + t\mathbf{E}[X_1] = M_t + at, \tag{1.11}$$

where

$$M_t = \int_{\mathbb{R}} x\left[Q((0,t],dx) - t\nu(dx)\right] \tag{1.12}$$

is a martingale and $a = \mathbf{E}[X_1]$.

In the specification of our model, we assume that the Lévy measure $\nu(dx)$ of L satisfies

$$\int_{|x|>1} |x|^3 \nu(dx) < \infty. \tag{1.13}$$

As in the general discussion above, this allows a decomposition of L of the form

$$L_t = cZ_t + M_t + at, \quad 0 \le t \le T, \tag{1.14}$$

where $(cZ_t)_{0\le t\le \tau}$ is a Brownian motion with standard deviation $c > 0$, $a = \mathbf{E}(L_1)$ and the martingale

$$M_t = \int_0^t \int_{\mathbb{R}} xM(ds, dx), \quad 0 \le t \le T, \tag{1.15}$$

is a square-integrable. Here, we denote the *compensated Poisson random measure on* $[0, \infty) \times \mathbb{R} \setminus \{0\}$ related to L by $M(dt, dx)$. Subsequently, if $\nu = 0$, then we will have that $L_t = Z_t$, where Z_t is appropriately defined Brownian motion.

Our work generalizes several aspects of the contribution [5] (see, also, [8–10]) by extending the description of bank behavior in a continuous-time Brownian motion framework to one in which the dynamics of bank items may have jumps and be driven by Lévy processes. As far as information on these processes is concerned, Protter [11, Chapter I, Section 4] and Jacod and Shiryaev [12, Chapter II] are standard texts (see, also, [13, 14]). Also, the connections between Lévy processes and finance are embellished upon in [15] (see, also, [16, 17]). If there is a deviation from the Black-Scholes paradigm, one typically enters into the realm of incomplete market models. Most theoretical financial market models are incomplete, with academics and practitioners alike agreeing that "real-world" markets are also not complete. The issue of completeness goes hand-in-hand with the uniqueness of the martingale measure (see, e.g., [18]). In incomplete markets, we have to choose an equivalent martingale measure that may emanate from the market. For the purposes of our investigation, for bank Treasuries and reserves, we choose a risk-neutral martingale measure, $\mathbf{Q}_g$, that is related to the classical Kunita-Watanabe measure (see [19]). We observe that, in practice, it is quite acceptable to estimate the risk-neutral measure directly from market data via, for instance, the volatility surface. It is well known that if the (discounted) underlying asset is a martingale under the original probability measure, $\mathbf{P}$, the optimal hedging strategy is given by the Galtchouck-Kunita-Watanabe decomposition as observed in [1]. In the general case, the underlying asset has some drift under $\mathbf{P}$, and the solution to the minimization problem is much more technical as it possesses a feedback component.

A vast literature exists on the properties of Treasuries and reserves and their interplay with deposit withdrawals. For instance, [20] (and the references contained therein) provides a neat discussion about Treasuries and loans and the interplay between them. Reserves are discussed in such contributions as [21–26]. Firstly, [21] investigates the role of a central bank in preventing and avoiding financial contagion. Such a bank, by imposing reserve requirements on the banking industry, trades off the cost of reducing the resources available for long-term investment with the benefit of raising liquidity to face an adverse shock that could cause contagious crises. We have that [22] presents a computational model for optimal reserve management policy in the banking industry. Also, [23] asserts that the standard view of the monetary transmission mechanism depends on the central bank ability to manipulate the overnight interest rate by controlling reserve

supply. They note that in the 90's, there was a marked decline in the level of reserve balances in the US accompanied at first by an increase in federal funds rate volatility. The article [24] examines how a bank run may affect the investment decisions made by a competitive bank. The basic premise is that when the probability of a run is small, the bank will offer a contract that admits a bank-run equilibrium. They show that in this case, the bank will hold an amount of liquid reserves exactly equal to what the withdrawal demand will be if a run does not occur; precautionary or excess liquidity will not be held. The paper [25] asserts that the payment of interest on bank reserves by the government assists in the implementation of monetary policy. In particular, it is demonstrated that paying interest on reserves financed by labor tax reduces welfare. Finally, [26] asserts that reserve requirements allow period-average smoothing of interest rates but are subject to reserve avoidance activities. A system of voluntary, period-average reserve commitments could offer equivalent rate-smoothing advantages. A common theme in the aforementioned contributions about reserves is the fact that they can be viewed as a proxy for general banking assets and that reserve dynamics are closely related to the dynamics of the deposits.

In the current section, we provide preliminary information about Lévy processes and distinguish our paper from the preexisting literature. Under the conditions highlighted above, the main problems addressed in the rest of our contribution is subsequently identified.

In Section 2, we extend some of the modeling and optimization issues highlighted in [9] (see, also, [5, 8, 10]) by presenting jump diffusion models for various bank items. Here, we introduce a probability space that is the product of two spaces that models the uncertainty associated with the bank reserve portfolio and deposit withdrawals. As a consequence of this approach, the intrinsic risk of the bank arises now not only from the reserve portfolio but also from the deposit withdrawals. Throughout we consider a depository contract that stipulates payment to the depositor on the contract's maturity date. We concentrate on the fact that deposit withdrawals are catered for by the Treasuries and reserves held by the bank. The stochastic dynamics of the latter mentioned items and their sum are presented in Sections 2.1.1 and 2.1.2, respectively. In Section 2.2, our main focus is on depository contracts that permit a cohort of depositors to withdraw funds at will, with the stipulation that the payment of an early withdrawal is only settled at maturity. This issue is outlined in more detail in Section 2.2.1. Furthermore, in Section 2.2.2, we suggest a way of counting deposit withdrawals by cohort depositors from which the bank has taken a single deposit at the initial time, $t = 0$.

Section 3 explores the relationship between the risk management of deposit withdrawals and reserves and the dynamic models for Treasuries and reserves. Moreover, Section 3.1 briefly explains basic risk concepts and Section 3.2 provides some risk minimization results that directly pertain to our studies. In Theorem 3.1, we derive a generalized GKW decomposition of the arbitrage-free value of the sum of cohort deposits depending on the reserve price. Theorem 3.2 provides a hedging strategy for bank reserve-dependent depository contracts in an incomplete reserve market setting. Intrinsic risk and the said strategies are derived with the (local) risk minimization theory contained in [1], assuming that bank deposits held accumulate interest on a risk-free basis. In order

to derive a hedging strategy for a bank reserve-dependent depository contract we require the generalized GKW decomposition for both its intrinsic value and the product of the inverse of Treasuries and the arbitrage free value of the sum of the cohort deposits. We accomplish this by assuming that the bank takes deposits (from a certain cohort of depositors with prespecified characteristics) as a single lump sum at the beginning of a specified time interval and holds it until withdrawal some time later. More specifically, under these conditions, we show that the reserve risk (risk of losses from earning opportunity costs through bank and Federal government operations) is not diversifiable by raising the number of depository contracts within the portfolio. This is however the case with depository risk originating from the amount and timing of net cash flows from deposits and deposit withdrawals emanating from a cession of the depository contract. We conclude Section 3 by considering the risk management of reserve, depository and intrinsic risk in our Lévy process setting (see Section 3.3 for more details).

In Section 4, we analyze the main risk management issues arising from the Lévy process-driven banking model that we constructed in the aforegoing sections. Some of the highlights of this section are mentioned below. A description of the role that bank assets play is presented in Section 4.1. Furthermore, we provide more information about depository contracts and the stochastic counting process for deposit withdrawals in Section 4.2. Moreover, Section 4.3 provides a numerical simulation of provisioning via the sum of Treasuries and reserves. Risk minimization and the hedging of withdrawals is discussed in Section 4.4. In addition to the solutions to the problems outlined above, Section 5 offers a few concluding remarks and possible topics for future research.

2. Lévy process-driven banking model

Our main objective is to construct a Lévy process-driven stochastic dynamic model that consists of assets, A, (uses of funds) and liabilities, Γ, (sources of funds). In our contribution, these items can specifically be identified as

$$A_t = \Lambda_t + \mathtt{T}(t) + R_t, \qquad \Gamma_t = \Delta_t, \tag{2.1}$$

where Λ, T, R, and Δ are loans, Treasuries, reserves, and outstanding debt, respectively.

2.1. Assets. In this subsection, the bank assets that we discuss are loans, provisions, Treasuries, reserves, and unweighted and risk-weighted assets. In order to model the uncertainty associated with these items, we consider the filtered probability space $(\Omega_1, \mathbf{G}, (\mathcal{G}_t)_{0\le t\le T}, \mathbf{P}_1)$.

2.1.1. Treasuries and reserves. *Treasuries* are the debt financing instruments of the federal government. There are four types of Treasuries, namely, treasury bills, treasury notes, treasury bonds, and savings bonds. All of the Treasuries besides savings bonds are very liquid and are heavily traded on the secondary market. We denote the *interest rate on Treasuries* or *treasury rate* by $r^{\mathtt{T}}(t)$. In the sequel, the dynamics of the Treasuries will be described by

$$d\mathtt{T}(t) = r^{\mathtt{T}}(t)\mathtt{T}(t)dt, \quad \mathtt{T}(0) = \mathtt{t} > 0. \tag{2.2}$$

Bank reserves are the deposits held in accounts with a national agency (e.g., the Federal Reserve for banks) plus money that is physically held by banks (vault cash). Such reserves are constituted by money that is not lent out but is earmarked to cater for withdrawals by depositors. Since it is uncommon for depositors to withdraw all of their funds simultaneously, only a portion of total deposits will be needed as reserves. As a result of this description, we may introduce a reserve-deposit ratio, η, for which

$$R_t = \eta \Delta_t, \quad \Delta_t = \frac{1}{\eta} R_t, \quad 0 < \eta \leq 1. \tag{2.3}$$

The bank uses the remaining deposits to earn profit, either by issuing loans or by investing in assets such as Treasuries and stocks. The individual rationality constraint implies that reserves may implicitly earn at least their opportunity cost through certain bank operations and Federal government subsidies. For instance, members of the Federal Reserve in the United States may earn a return on required reserves through government debt trading, foreign exchange trading, other Federal Reserve payment systems, and affinity relationships (outsourcing) between large and small banks. We note that vault cash in the automated teller machines (ATMs) network also qualifies as required reserves. The conclusion is that banks may earn a positive return on reserves. In the sequel, we take the above discussion into account when assuming that the dynamics of the reserves are described by

$$dR_t = R_{t^-} \{ [r^R(t) - f^R(t) + a^R \sigma^R(t)] dt + \sigma^R(t) (c^R dZ_t^R + dM_t^R) \}, \quad R_0 = r > 0, \tag{2.4}$$

where r^R is the deterministic rate of (positive) return on reserves earned by the bank, f^R is the fraction of the reserves consumed by deposit withdrawals, and σ^R is the volatility in the level of reserves. In order to have $R_t > 0$, we assume that $\sigma^R \Delta R_t > -1$ for all t a.s. Here, in a manner analogous to (1.14), we assume that L^R admits the decomposition

$$L_t^R = c^R Z_t^R + M_t^R + a^R t, \quad 0 \leq t \leq T, \tag{2.5}$$

where $(c^R Z_t^R)_{0 \leq t \leq \tau}$ is a Brownian motion with standard deviation $c^R > 0$, $a^R = \mathbf{E}(L_1^R)$ and

$$M_t^R = \int_0^t \int_{\mathbb{R}} x M^R(ds, dx), \quad 0 \leq t \leq \tau, \tag{2.6}$$

is a square-integrable martingale. We know that the SDE (2.4) has the explicit solution

$$\begin{aligned} R_t = R_0 \exp \left\{ \int_0^t c^R \sigma^R(s) dZ_s^R + \int_0^t \sigma^R(s) dM_s^R + \int_0^t \left[a^R \sigma^R(s) + r^R(s) - f^R(s) - \frac{c^{R2} \sigma^{R2}(s)}{2} \right] ds \right\} \\ \times \prod_{0 \leq s \leq t} (1 + \sigma^R(s) \Delta M_s^R) \exp(-\sigma^R(s) \Delta M_s^R). \end{aligned} \tag{2.7}$$

We can use the notation $\widehat{R}_t = \mathrm{T}^{-1}(t) R_t$ to denote the value of the discounted reserves. It is clear that $\widehat{R}_t$ has a nonzero drift term so that it is only a semimartingale rather than a martingale. In order for $\widehat{R}$ to be a martingale, under the approach of risk neutral valuation, a $\mathbf{P}_1$-equivalent martingale measure is required. There are infinitely many such

measures in incomplete markets (see [27] for the incomplete information case). However, an equivalent martingale measure that fits the bill is the generalized Kunita-Watanabe (GKW) measure, $\mathbf{Q}_g$, (see [19]) whose Girsanov parameter may be represented by

$$G_t = \frac{r^{\mathrm{T}}(t) - r^R(t) + f^R(t) - a^R\sigma^R(t)}{\sigma^R(t)\left(c^{R2} + \nu\right)},$$
$$\nu = \int_{\mathbb{R}} x^2 \nu(dx), \quad G_t \Delta R_t > -1 \quad \forall t \in [0,T]. \tag{2.8}$$

In the sequel, the *compensated jump measure of* L^R *under* $\mathbf{Q}_g$ is denoted by $M^{\mathbf{Q}}(dt,dx)$ and the Lévy measure $\nu(dx)$ under $\mathbf{Q}_g$ has the form

$$\nu_t^{\mathbf{Q}}(dx) = \left(1 + G_t x\right)\nu(dx). \tag{2.9}$$

In addition, $\widehat{R}$ is a square-integrable martingale under $\mathbf{Q}_g$ (cf. (1.13)) that satisfies

$$d\widehat{R}_t = \sigma_t^R \widehat{R}_{t^-}\left(c^R dZ_t^{\mathbf{Q}} + dM_t^{\mathbf{Q}}\right). \tag{2.10}$$

Here $Z^{\mathbf{Q}}$ is standard Brownian motion and

$$M_t^{\mathbf{Q}} = M_t - \int_0^t \int_{\mathbb{R}} G_s x^2 \nu(dx)ds = \int_0^t \int_{\mathbb{R}} x M^{\mathbf{Q}}(ds,dx) \tag{2.11}$$

is a square-integrable $\mathbf{Q}_g$-martingale. Under the above martingale, L^R may not be a Lévy process since it may violate the fact that a semimartingale has stationary increments if and only if its characteristics are linear in time (cf. Jacod and Shiryaev [12, Chapter II, Corollory 4.19]).

2.1.2. Provisions for deposit withdrawals. In the main, provisioning for deposit withdrawals involve decisions about the volume of Treasuries and reserves held by the bank. Without loss of generality, in the sequel, we suppose that the provisions for deposit withdrawals correspond with the sum of Treasuries and reserves as defined by (2.2) and (2.4), respectively.

For withdrawal provisioning, we assume that the stochastic dynamics of the sum of Treasuries and reserves, W, is given by

$$dW_t = W_{t^-}\left[\left(r^{\mathrm{T}}(t) + \pi_t\left(r^R(t) - f^R(t) - r^{\mathrm{T}}(t) + a^R\sigma^R(t)\right)\right)dt + \pi_t\sigma^R(t)\left(c^R dZ_t^R + dM_t^R\right)\right]$$
$$- k(t)dt; \quad W_{0^-} = \mathtt{t} + r = w \geq 0,\ W_t = W_t^u = \mathrm{T}^u(t) + R_t^u \geq 0,\ \forall t \geq 0, \tag{2.12}$$

where $\pi_t = R_t/W_t$ and the *depository value*, k, is the rate at which Treasuries are consumed by deposit withdrawals.

2.2. Liabilities. In the sequel, we assume that the bank deposit withdrawals are represented by the filtered probability space $(\Omega_2, \mathbf{H}, \mathbb{H}, \mathbf{P}_2)$. Here, $\mathbb{H}$ is the natural filtration generated by $\mathbf{I}(T_i \leq t)$, $i = 1,\ldots,n^x$, $\mathcal{H}_0$ is trivial and $\mathcal{H}_T = \mathbf{H}$. We suppose risk neutrality of the bank towards deposit withdrawals, which means that $\mathbf{P}_2$ is the risk neutral measure.

2.2.1. Depository contracts. A *depository contract* is an agreement that stipulates the conditions for deposit taking and holding by the bank and withdrawal by the depositor. Depository contracts typically specify the payment of some maturity amount that could be fixed or a function of some specified traded bank asset. Furthermore, we define the *deposit holding time* as the time between bank deposit taking and its withdrawal by the depositor. Our main supposition is that such times are mutually independent and identically distributed (i.i.d.). This assumption implies that depository contracts may be picked to form a cohort of individual contracts that have been held for an equal amount of time, x, with n^x denoting the number of such contracts. Ultimately, this situation leads to the description of the remaining deposit holding time by the i.i.d. nonnegative random variables $T_1,\dots,T_{n^x}$. Under the assumption that the distribution of T_i is absolutely continuous, the *deposit survival conditional probability* may be represented by

$$\mathbf{P}_2(T_i > t+x \mid T_i > x) = \exp\left\{-\int_0^t \omega_{x+\tau}\,d\tau\right\}, \tag{2.13}$$

where the *withdrawal rate function* is denoted by ω_{x+t}. Roughly speaking, for a deposit withdrawal at time instant $T, \mathbf{P}_2(T > t+x \mid T > x)$ provides information about the probability that a deposit will still be held by a bank at $x+t$ conditional on a single deposit being taken by the bank at x.

In the sequel, reserves are related to outstanding debt (see, e.g., (2.3)) and acts as a proxy for the assets held by the bank. This suggests that the *sum of cohort deposits*, D^c, may be dependent on the bank reserves, R_t, and as a consequence may be denoted by $D_t^c(R_t)$. For $\mathtt{T}$ and R from (2.2) and (2.4), respectively, suppose that $D_t^c(R_t)$ is a $\mathcal{G}_t$-measurable function with

$$\sup_{u\in[0,T]} \mathbf{E}^{\mathbf{Q}}\left[\left(\mathtt{T}^{-1}(u)D_u^c(R_u)\right)^2\right] < \infty. \tag{2.14}$$

We suppose that deposit withdrawals may take place at any time, $u \in [0,T]$, but that payment is deferred to the term of the contract. As a consequence, the contingent claim $D_u^c(R_u)$ must be time-dependent. From risk-neutral valuation, the *arbitrage-free value function*, $F_t(R_t,u)$, *of the sum of cohort deposits*, $D_t^c(R_t)$, is

$$F_t(R_t,u) = \begin{cases} \mathbf{E}^{\mathbf{Q}}\left[\mathtt{T}(t)\mathtt{T}^{-1}(u)D_u^c(R_u) \mid \mathcal{G}_t\right], & 0 \le t < u \le T, \\ \mathtt{T}(t)\mathtt{T}^{-1}(u)D_u^c(R_u), & 0 \le u \le t \le T. \end{cases} \tag{2.15}$$

From [11, Chapter I, Theorem 32], for $0 \le t < u \le T$ and $x \ge 0$, we have

$$F_t(x,u) = \mathbf{E}_{t,x}^{\mathbf{Q}}\left[\mathtt{T}(t)\mathtt{T}^{-1}(u)D_u^c(R_u)\right] = \mathbf{E}^{\mathbf{Q}}\left[\mathtt{T}(t)\mathtt{T}^{-1}(u)D_u^c(R_u) \mid R_t = x\right], \tag{2.16}$$

with $F.(\cdot,u) \in C^{1,2}([0,T]\times[0,\infty))$ and $D_xF_t(x,u)$ bounded. Furthermore, we consider

$$j_t(x,u) = \mathtt{T}^{-1}(t)\left\{F_t\left(R_{t^-}\left(1+\sigma^R(t)x\right),u\right) - F_t(R_{t^-},u)\right\} \tag{2.17}$$

to be the *value of the jump in the reserve process induced by a jump of the underlying Lévy process*, L^R.

In the case where (2.14) holds, the depository contract terminated at t receives the payout,

$$D^c_{T_i}(R_{T_i})\mathtt{T}(T)\mathtt{T}^{-1}(T_i) \tag{2.18}$$

at time T. By way of consistency with our framework, the present value of the bank's depository obligation generated by the entire portfolio of depository contracts is considered to be $\mathbf{Q}$-a.s. of the form

$$\begin{aligned} D &= \mathtt{T}^{-1}(T)\sum_{i=1}^{n^x} D^c_{T_i}(R_{T_i})\mathtt{T}^{-1}(T_i)\mathtt{T}(T)\mathbf{I}(T_i \le T) \\ &= \sum_{i=1}^{n^x}\int_0^T D^c_u(R_u)\mathtt{T}^{-1}(u)d\mathbf{I}(T_i \le u) = \int_0^T D^c_u(R_u)\mathtt{T}^{-1}(u)dN^I_u. \end{aligned} \tag{2.19}$$

2.2.2. Stochastic counting process for deposit withdrawals. We assume that the bank takes a single deposit from each of n^x cohort depositors at $t = 0$. Furthermore, we model the *number of deposit withdrawals*, N^I, by

$$N^I_t = \sum_{i=1}^{n^x}\mathbf{I}(T_i \le t), \qquad l^x_t = n^x - N^I_t = \sum_{i=1}^{n^x}\mathbf{I}(T_i > t). \tag{2.20}$$

The *compensated counting process*, $M^I = (M^I_t)_{0\le t\le T}$, expressible as

$$M^I_t = N^I_t - \int_0^t \iota_u du, \quad \text{where } \iota_t dt \equiv l^x_{t^-}\omega_{x+t}dt = \mathbf{E}[dN^I_t \mid \mathcal{H}_{t^-}], \tag{2.21}$$

defines an $\mathbf{H}$-martingale with

$$\langle M^I\rangle_t = \int_0^t \iota_u du, \quad 0 \le t \le T, \tag{2.22}$$

where ι is the *(stochastic) intensity of* N^I (cf. with [12, Chapter II, Proposition 3.32]). In other words, ι is more or less the product of the withdrawal rate function, ω_{x+t}, and the remaining number of cohort depositors just before time instant t.

2.2.3. Cost of deposit withdrawals. Another modeling issue relates to the possibility that *unanticipated deposit withdrawals*, w, will occur. By way of making provision for these withdrawals, the bank is inclined to hold reserves, R, and Treasuries, $\mathtt{T}$, that are very liquid. In our contribution, we propose that w may be associated with the *probability density function*, $f(w)$, that is independent of time. In this regard, we may suppose that the unanticipated deposit withdrawals have a uniform distribution with support $[\underline{\Delta}, \overline{\Delta}]$ so that the *cost of liquidation*, c^l, or additional external funding is a quadratic function of the sum of Treasuries and reserves, $W = \mathtt{T} + R$. In addition, for any t, if

$$w > W_t, \tag{2.23}$$

then bank assets are liquidated at some *penalty rate*, r_t^p. In this case, the *cost of deposit withdrawals* is

$$c^w(W_t) = r_t^p \int_{W_t}^{\infty} [w - W_t] f(w) dw = \frac{r_t^p}{2\Delta} [\overline{\Delta} - W_t]^2. \tag{2.24}$$

3. Risk and the banking model

Our model has far-reaching implications for risk management in the banking industry. For instance, we can apply the quadratic hedging theory developed in [1, 28] to derive a risk minimizing strategy for deposit withdrawals. An approach that we adopt in this case involves the introduction of a probability space that is the product of two spaces modeling the uncertainty associated with the bank's provision for deposit withdrawals via Treasuries and reserves and the withdrawals themselves given by

$$(\Omega_1, \mathbf{G}, \mathbb{G}, \mathbf{P}_1), \qquad (\Omega_2, \mathbf{H}, \mathbb{H}, \mathbf{P}_2), \tag{3.1}$$

respectively. In the sequel, we represent the product probability space by $(\Omega, \mathbf{F}, \mathbb{F}, \mathbf{P})$, where the filtration, $\mathbb{F}$, is characterized by

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t. \tag{3.2}$$

Here, $\mathcal{G}_t$ and $\mathcal{H}_t$ are stochastically independent. As an equivalent martingale measure, $\mathbf{Q}$, we use the product measure of the generalized GKW measure $\mathbf{Q}_g$ and of the risk-neutral deposit withdrawal law $\mathbf{P}_2$. As a consequence of this approach, the intrinsic risk of the bank arises now not only from the Treasuries/reserves provisioning portfolio but also from the deposit withdrawals.

3.1. Basic risk concepts. We assume that the actual provisions for deposit withdrawals are constituted by Treasuries and reserves with price processes $\mathtt{T} = (\mathtt{T}(t))_{0 \le t \le T}$ and $R = (R_t)_{0 \le t \le T}$, respectively. Suppose that $n_t^{\mathtt{T}}$ and n_t^R are the number of Treasuries and reserves held in the withdrawal provisioning portfolio, respectively. Let $L^2(\mathbf{Q}_{\hat{R}})$ be the space of square-integrable predictable processes $n^R = (n_t^R)_{0 \le t \le T}$ satisfying

$$\mathbf{E}^{\mathbf{Q}} \left\{ \int_0^T (n_s^R)^2 d\langle \hat{R} \rangle_s \right\} < \infty, \tag{3.3}$$

where $\hat{R}_t = \mathtt{T}^{-1}(t) R_t$. For the discounted reserve price, $\hat{R}_t$, we call $\Theta_t = (n_t^R, n_t^{\mathtt{T}})$, $0 \le t \le T$, a *provisioning strategy* if

(1) $n^R \in L^2(\mathbf{Q}_{\hat{R}})$;

(2) $n^{\mathtt{T}}$ is adapted;

(3) the *discounted provisioning portfolio value process*

$$\hat{V}_t(\Theta) = V_t(\Theta) \mathtt{T}^{-1}(t); \quad V_t(\Theta) = n_t^R R_t + n_t^{\mathtt{T}} \mathtt{T}(t) \in L^2(\mathbf{Q}), \quad 0 \le t \le T; \tag{3.4}$$

(4) $\hat{V}_t(\Theta)$ is cádlág.

The *(cumulative) cost process* $c(\Theta)$ associated with a provisioning strategy, Θ, is

$$c_t(\Theta) = \widehat{V}_t(\Theta) - \int_0^t n_s^R d\widehat{R}_s, \quad 0 \le t \le T. \tag{3.5}$$

The *intrinsic* or *remaining risk process*, $\mathtt{R}(\Theta)$, associated with a strategy is

$$\mathtt{R}_t(\Theta) = \mathbf{E}^{\mathbf{Q}}\left[\left(c_T(\Theta) - c_t(\Theta)\right)^2 \mid \mathcal{F}_t\right], \quad 0 \le t \le T. \tag{3.6}$$

It is clear that this concept is related to the conditioned expected square value of future costs. The strategy $\Theta = (n_t^R, n_t^{\mathtt{T}})$, $0 \le t \le T$ is *mean self-financing* if its corresponding cost process $c = (c_t)_{0\le t\le T}$ is a martingale. Furthermore, the strategy Θ is *self-financing* if and only if

$$\widehat{V}_t(\Theta) = \widehat{V}_0(\Theta) + \int_0^t n_u^R d\widehat{R}_u, \quad 0 \le t \le T. \tag{3.7}$$

A strategy $\widetilde{\Theta}$ is called an *admissible time t continuation* of Θ if $\widetilde{\Theta}$ coincides with Θ at all times before t and $V_T(\Theta) = D$**Q**-a.s. Moreover, a provisioning strategy is called *risk minimizing* if for any $t \in [0, T)$, Θ minimizes the remaining risk. In other words, for any admissible continuation $\widetilde{\Theta}$ of Θ at t we have

$$\mathtt{R}_t(\Theta) \le \mathtt{R}_t(\widetilde{\Theta}), \quad \mathbf{P}\text{-a.s.} \tag{3.8}$$

The contribution [1] shows that a unique risk minimizing provisioning strategy Θ^D can be found using the generalized GKW decomposition of the *intrinsic value process*, $V^* = (V_t^*)_{0\le t\le T}$, *of a contingent withdrawal*, D, given by

$$V_t^* = \mathbf{E}^{\mathbf{Q}}[D \mid \mathcal{F}_t] = \mathbf{E}^{\mathbf{Q}}[D] = \int_0^t n_s^{RD} d\widehat{R}_s + K_t^D, \quad 0 \le t \le T, \tag{3.9}$$

where $K^D = (K_t^D)_{0\le t\le T}$ is a zero-mean square-integrable martingale, orthogonal to the square-integrable martingale $\widehat{R}$ and $n^{RD} \in L^2(\mathbf{Q}_{\widehat{R}})$. Furthermore, Θ_t^D is mean self-financing and given by

$$\Theta_t^D = \left(n_t^{RD}, V_t^* - n_t^{RD}\widehat{R}_t\right), \quad 0 \le t \le T. \tag{3.10}$$

In this case, we have the intrinsic or remaining risk process

$$\mathtt{R}_t(\Theta^D) = \mathbf{E}^{\mathbf{Q}}\left[\left(K_T^D - K_t^D\right)^2 \mid \mathcal{F}_t\right], \quad 0 \le t \le T. \tag{3.11}$$

3.2. Generalized GKW decomposition of $\mathtt{T}^{-1}(t)F_t(R_t, u)$. Suppose that $n_t^{\mathtt{T}}$ and n_t^R are the number of Treasuries and reserves in the provisioning strategy, $\Theta = (n^R, n^{\mathtt{T}})$, respectively. Next, we produce the generalized GKW decomposition of $\mathtt{T}^{-1}(t)F_t(R_t, u)$, in order to eventually derive a hedging strategy for a reserve-dependent deposit withdrawal.

Theorem 3.1 (generalized GKW decomposition of $\mathrm{T}^{-1}(t)F_t(R_t,u)$). *Let $F_t(R_t,u)$ and j be defined by (2.15) and (2.17), respectively and assume that*

$$\nu_t^{\mathbf{Q}} = \int_{\mathbf{R}} x^2 \nu_t^{\mathbf{Q}}(dx), \quad \kappa_t = \tilde{s}^2 + \nu_t^{\mathbf{Q}}, \quad t \in [0,T]. \tag{3.12}$$

For $0 \le t < u \le T$, the predictable process

$$n_t^R(u) = \frac{\tilde{s}^2}{\kappa_t} D_x F_t(R_{t^-},u) + \frac{1}{\sigma(t)\hat{R}_{t^-}\kappa_t} \int_{\mathbf{R}} x j_t(x,u)\nu_t^{\mathbf{Q}}(dx) \tag{3.13}$$

and continuous and discontinuous terms are defined by

$$\begin{aligned} \vartheta_t^{(1)}(u) &= \tilde{s}\sigma(t)\hat{R}_{t^-}\{D_x F_t(\hat{R}_{t^-},u) - n_t^R(u)\}, \\ \vartheta_t^{(2)}(y,u) &= j_t(y,u) - y\sigma(t)\hat{R}_{t^-} n_t^R(u), \end{aligned} \tag{3.14}$$

respectively. In this situation, the generalized GKW decomposition of $\mathrm{T}^{-1}(t)F_t(R_t,u)$ is given by

$$\mathrm{T}^{-1}(t)F_t(R_t,u) = F_0(R_0,u) + \int_0^t n_s^R(u)\,d\hat{R}_s + K_t(u), \tag{3.15}$$

where

$$K_t(u) = \int_0^t \vartheta_t^{(1)}(u)\,dZ_s^{\mathbf{Q}} + \int_0^t \int_{\mathbf{R}} \vartheta_t^{(2)}(y,u) N^{\mathbf{Q}}(ds,dy) \tag{3.16}$$

is orthogonal to $\hat{R}$.

Proof. We base our proof on the additivity of the projection in the GKW decomposition. From (1.13), (2.11), and the fact that $D_x F_t(x,u)$ is bounded, the integrals driven by $\hat{R}, N^{\mathbf{Q}}(\cdot,\cdot)$, and $Z^{\mathbf{Q}}$ are well-defined and square-integrable martingales. Furthermore, we note that [29, Proposition 10.5] determines n_t^R for the generalized GKW decomposition in the Lévy process case. Under the equivalent measure, $\mathbf{Q}$, this result extends quite naturally to the case of the additive process L. We are able to deduce from Ito's formula in [11, Chapter II, Theorem 33], that the discounted arbitrage-free value, $\mathrm{T}^{-1}(t)F_t(R_t,u)$, admits the decomposition

$$\mathrm{T}^{-1}(t)F_t(R_t,u) = F_0(R_0,u) + \int_0^t D_x F_s(R_{s^-},u)\,d\hat{R}_s + \tilde{K}_t(u), \quad 0 \le t \le T, \tag{3.17}$$

where

$$\tilde{K}_t(u) = \int_0^t \int_{\mathbf{R}} \{J_s(y,u) - D_x F_s(R_{s^-},u)\sigma_s \hat{R}_{s^-} y\} N^{\mathbf{Q}}(ds,dy). \tag{3.18}$$

This formula, along with the differential (2.10), allows the orthogonal part (3.16) in the hypothesis of Theorem 3.1 to be computed via

$$K_t(u) = \int_0^t \{D_x F_s(R_{s^-},u) - n_s^R(u)\}\,d\hat{R}_s + \tilde{K}_t(u). \tag{3.19}$$

□

3.3. Risk-minimizing strategy for depository contracts. In the discussion thus far, bank obligations generated by depository contracts unfortunately do not correspond to a T-claim so that special assumptions are required. A way of transforming the aforementioned obligations into a T-claim is to suppose that deposit withdrawals are deferred to the term of the contract and are accumulated with a risk-free interest rate of r. In the wake of this specification, the depository contract terminated at time t would receive the payout $D^c_{T_i}(R_{T_i})\mathtt{T}(T)\mathtt{T}^{-1}(T_i)$ at time T, in the case where (2.14) holds. These contracts by deferment usually have short time horizons. By way of consistency with our framework, the present value of the bank obligation generated by the entire portfolio of depository contracts is considered to be $\mathbf{Q}$-a.s. of the form

$$\begin{aligned} D &= \mathtt{T}^{-1}(T)\sum_{i=1}^{n^x} D^c_{T_i}(R_{T_i})\mathtt{T}^{-1}(T_i)\mathtt{T}(T)\mathbf{I}(T_i \le T) \\ &= \sum_{i=1}^{n^x}\int_0^T D^c_u(R_u)\mathtt{T}^{-1}(u)d\mathbf{I}(T_i \le u) \\ &= \int_0^T D^c_u(R_u)\mathtt{T}^{-1}(u)dN^I_u. \end{aligned} \tag{3.20}$$

Also, we recall that the intrinsic risk process associated with D may be given by

$$\mathtt{R}_t(\Theta^*) = \mathbf{E}^{\mathbf{Q}}\left[\left(K^D_T - K^D_t\right)^2 \mid \mathcal{F}_t\right]. \tag{3.21}$$

The independence of the reserve market and deposit withdrawals enables us to represent the *intrinsic value of the entire depository contract portfolio*, V^*_t, as

$$V^*_t = \int_0^t \mathtt{T}^{-1}(u)D^c_u(R_u)dN^I_u + \int_t^T \mathtt{T}^{-1}(t)F_t(R_t,u)l^x_t\mathbf{P}_2(T_i > u-t+x \mid T_i > x+t)\omega_{x+u}du, \tag{3.22}$$

with initial value

$$V^*_0 = \int_0^T F_0(R_0)n^x\mathbf{P}_2(T_i > u+x \mid T_i > x)\omega_{x+u}du. \tag{3.23}$$

Under certain conditions, V_0 may be the single deposit taken by the bank at $t = 0$. The risk minimization approach adopted in the ensuing main result is dependent on the fact that the value of the optimal provisioning strategy, Θ^*, is exactly equal to the sum of cohort deposits that have already been withdrawn and expected possible future withdrawals as in

$$\widehat{V}_t(\Theta^*) = V^*_t = \int_0^t \mathtt{T}^{-1}(u)D^c_u(R_u)dN^I_u + \mathbf{E}^{\mathbf{Q}}\left[\int_t^T \mathtt{T}^{-1}(t)F_u(R_u)dN^I_u \mid \mathcal{F}_t\right]. \tag{3.24}$$

Theorem 3.2 (GKW decomposition of V^*_t and risk-minimizing strategy). *Suppose that n^R, $\vartheta^{(1)}_t$, $\vartheta^{(2)}_t$, and V^*_t are given by (3.13), (3.14) in Theorem 3.1 and (3.22), respectively.*

(1) *For (3.20), with $0 \le t \le T$, the intrinsic value process, V^*, has the GKW decomposition*

$$V_t^* = V_0^* + \int_0^t l_{s^-}^x n_s^R d\widehat{R}_s + K_t^H, \tag{3.25}$$

where

$$n_t^R = \int_t^T \mathbf{P}_2(T_i > u - t + x \mid T_i > x + t)\omega_{x+u} n_t^R(u)du, \tag{3.26}$$

$$K_t^D = \int_0^t l_{s^-}^x \vartheta_s^{(1)} dZ_s^{\mathbf{Q}} + \int_0^t \int_{\mathbb{R}} l_{s^-}^x \vartheta_t^{(2)}(y) N^{\mathbf{Q}}(ds, dy) + \int_0^t v_s dM_s^I, \tag{3.27}$$

are orthogonal to $\widehat{R}$. Here,

$$\begin{aligned} \vartheta_t^{(1)} &= \int_t^T \mathbf{P}_2(T_i > u - t + x \mid T_i > x + t)\omega_{x+u}\vartheta_t^{(1)}(u)du, \\ \vartheta_t^{(2)}(y) &= \int_t^T \mathbf{P}_2(T_i > u - t + x \mid T_i > x + t)\omega_{x+u}\vartheta_t^{(2)}(y, u)du, \\ v_t &= \mathrm{T}^{-1}(T)D_t^c(R_t) - \int_t^T \mathrm{T}^{-1}(T)F_t(R_{t^-}, u)\mathbf{P}_2(T_i > u - t + x \mid T_i > x + t)\omega_{x+u}du. \end{aligned} \tag{3.28}$$

(2) *For $0 \le t \le T$, under $\mathbf{Q}$, the unique admissible risk-minimizing hedging strategy, $\Theta^* = (n^{R*}, n^{\mathrm{T}*})$, for the bank obligation in (3.20) is given by*

$$\begin{aligned} n_t^{R*} &= l_{t^-}^x \int_t^T \mathbf{P}_2(T_i > u - t + x \mid T_i > x + t) n_t^R(u)du, \\ n_t^{\mathrm{T}*} &= \int_0^t \mathrm{T}^{-1}(u)D_u^c(R_u)dN_u^I \\ &\quad + l_t^x \int_t^T \mathrm{T}^{-1}(t)F_t(R_t, u)\mathbf{P}_2(T_i > u - t + x \mid T_i > x + t)\omega_{x+u}du - n_t^{R*}\widehat{R}_t. \end{aligned} \tag{3.29}$$

For $0 \le t \le T$, the intrinsic risk process is

$$\begin{aligned} \mathrm{R}_t(\Theta^*) &= \int_t^T \left\{ \mathbf{E}^{\mathbf{Q}}[(l_t^x)^2 \mid \mathcal{F}_t]\mathbf{E}^{\mathbf{Q}}\left[\vartheta_s^{(1)2} + \int_{\mathbb{R}} (\vartheta_s^{(2)}(y))^2 \nu_s^{\mathbf{Q}}(dy) \mid \mathcal{F}_t\right]\right\} ds \\ &\quad + l_t^x \int_t^T \mathbf{E}^{\mathbf{Q}}[v_s^2(R_s) \mid \mathcal{F}_t]\mathbf{P}_2(T_i > s - t + x \mid T_i > x + t)\omega_{x+s}ds. \end{aligned} \tag{3.30}$$

Proof. (1) The proof of (3.25) in the first part of the theorem relies on the stochastic Fubini theorem (see, e.g., [11, Chapter II]).

(2) In order to complete the proof, we make use of a combination of isometry results, Tonelli's theorem, and orthogonality. In this regard, we bear in mind that

$$\begin{aligned} d\iota_s &= l_s^x \omega_{x+s} ds, \\ \mathbf{E}^{\mathbf{Q}}[l_s^x \mid \mathcal{F}_t] &= l_{s^-}^x \mathbf{P}_2(T_i > s - t + x \mid T_i > x + t). \end{aligned} \tag{3.31}$$

□

4. Analysis of the main risk management issues

The dynamic models of bank items constructed in this paper are compliant with the dictates of the Basel II capital accord. For instance, the properties of our models are positively correlated with the methods currently being used to assess the riskiness of bank provisioning portfolios and their minimum capital requirement (see [30, 31]).

4.1. Assets. In this subsection, we analyze aspects of the bank assets such as provisions for deposit withdrawals, Treasuries, and reserves.

4.1.1. Treasuries and reserves. As was mentioned in Section 2.1.1, Treasuries are bonds issued by a national treasury and may be modeled as a risk-free asset (bond) in the usual way. In modern times, it is possible to assign a price to reserves and to model them by means of Lévy processes. This is due to the discontinuity associated with their evolution and because they provide a good fit to real-life data. In this regard, several interesting contributions have led to the choice of representation (2.12) for the dynamics of the sum of the Treasuries and reserves. Amongst these is a paper by Chan (see [27]) that treats the case where the Lévy decomposition of general assets corresponds to our decomposition. The size of the depository value, k, from (2.12), can vary greatly.

Two economic aspects of required reserves on bank deposits are noteworthy. Firstly, their impact on bank-intermediated investment versus direct investment and, secondly, their opportunity cost. The main function of bank reserves is to serve as a buffer to mitigate inefficient liquidation of a bank's assets in order to meet the demand for liquidity by investors. Due to some transaction costs or information costs, investors may prefer bank-intermediated investment to direct investment. Banks offer investors competitive deposit returns compared to the liquidation value of investment to attract funds from investors. If the Federal Reserve allows banks to set their individual optimal level of reserves, this might mitigate costs associated with required reserves. If banks implement the social optimum, this may introduce additional fragility into the banking system. We argue that required reserves might lead to deadweight loss if they are set above a bank's optimally determined reserves.

4.1.2. Provisions for deposit withdrawals. As has been noted in Section 2.1.2, deposits are subject to the risk of early withdrawal. This phenomenon can be associated with some interesting trends. For instance, as interest rates fall, depositors become less inclined to withdraw their deposits and the volume of provisions for deposit withdrawals may decrease. On the other hand, at higher rates of interest, depositors have a greater propensity towards withdrawal and provisions for deposit withdrawals may increase. Furthermore, an increase in interest rate volatility diminishes the adverse impact of the correlation between interest rates and withdrawals. As a consequence, the optimal deposit rate may decrease which results in the widening of the optimal intermediation margin. An increase in the volatility of withdrawals exacerbates the impact of the correlation between interest rates and the propensity to withdraw. A negative correlation between interest rates and propensity to withdraw would be to the advantage of bank management because an increase in this correlation increases optimal deposit rates.

4.2. Liabilities. This subsection provides an analysis of aspects of the discussion about liabilities in Section 2.2.

4.2.1. Depository contracts. In Section 2.2.1, we noted that (2.3) suggests an association between reserves and deposits that allows the sum of cohort deposits, D^c, to be dependent on the bank reserves, R_t, and denote this by $D_t^c(R_t)$. In reality, D^c will not only be dependent on R but on all the unweighted assets and several other banking items. The building of a model to incorporate this dependence is the subject of much debate at present.

A simplifying modification of the depository contract can be considered where $T_i = T$ for all i. In this situation, (2.18) becomes

$$D_T^c(R_T) = D_T^c(R_T)\mathtt{T}(T)\mathtt{T}^{-1}(T). \tag{4.1}$$

4.2.2. Stochastic counting process for deposit withdrawals. In standard banking models, the volume of deposits and their withdrawals usually equates the input prices Lerner index with the inverse of the elasticity of supply to determine the optimal rate paid for deposits. These models have no explicit time dimension so that deposits have no meaningful term to maturity. Their implicit time to maturity is homogeneous across accounts and, without modeling the intertemporal behavior of interest rates, deposits must be assumed to be held until maturity. Clearly, the models above ignore critical aspects of bank input pricing, namely, depository contracts have different times to maturity and the fact that depositors can withdraw funds before maturity. Both these issues are addressed in our contribution.

4.2.3. Cost of deposit withdrawals. In Section 2.2.3, the rate term for auxiliary profits, $\mu^a(s)$, may be generated from activities such as special screening, monitoring, liquidity provision, and access to the payment system. Also, this additional profit may arise from imperfect competition, barriers to entry, exclusive access to cheap deposits, or tax benefits.

4.3. Simulations and numerical examples. In the sequel, further insight is gained by considering a simulation of a trajectory for the stochastic dynamics of the sum of the Treasuries and reserves, W (denoted by SDSTR), as given by (2.12).

4.3.1. Parameters and values. We consider the SDSTR problem with the constant rates and variance functions with parameters set out in Table 4.1.

Below is the trajectory for the simulated SDSTR given by (2.12) and is numerically simulated by using the parameter choices in Table 4.1.

4.3.2. Properties of the trajectory. Figure 4.1 shows the simulated trajectory for the *CIR* process of the SDSTR problem with W being given by (2.12). Here, different values for the banking parameters are collected in Table 4.1. The number of jumps of the trajectory was limited to 1000, with the initial values for $\mathtt{T}$ and R fixed at 1 and 20, respectively.

In the main, provisioning for the deposit withdrawals involve decisions about the volume of Treasuries and reserves held by the banks. For withdrawal provisioning, we assume that the stochastic dynamics of W are given by (2.12). This stochastic dynamic

Table 4.1. Parameters use in SDSTR.

Parameters	Values
r^T	0.4
π	2
r^R	0.8
f^R	5
a^R	1.25
σ^R	0.4
c^R	0.0006
κ	1.5
γ	1.9992
λ	2

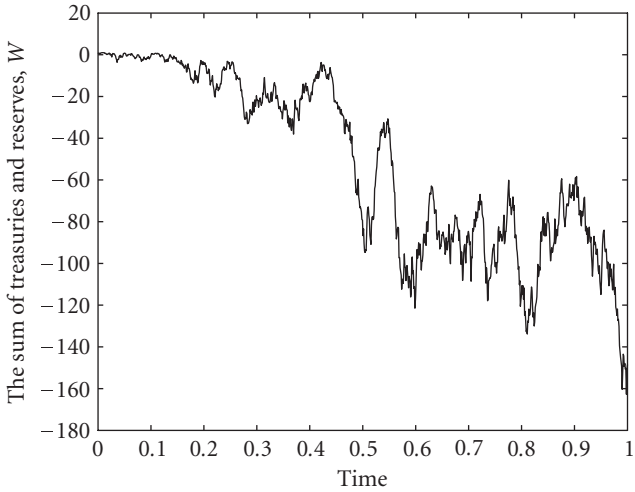

Figure 4.1. Trajectory of simulated SDSTR of W.

model enables us to analyze the interplay between deposit withdrawals and the provisioning for these withdrawals via Treasuries and reserves. In this spirit, we consider the stochastic dynamics of W that is driven by a CIR process with its trajectory given by Figure 4.1. As has been noted in Section 2.1.2, deposits are subjected to the risk of early withdrawals. According to the trajectory in Figure 4.1, we associate the latter assertion with the following trends: the trajectory initial depicted a steady path that shows the correspondence between the provisions of deposit withdrawals and W. Subsequently, a decrease in value occurs which may be due to the influence of an interest rate change. A model with such financial behavior forces depositors to become less inclined to withdraw their deposits. In this case, the volume of provisions for deposits withdrawals may decrease. Finally, we observe that the trajectory does not make an allowance for deposit

withdrawals with $W < 0$. This is an indication that the bank may have inadequate securities and assets. With the higher interest rate volatility of 0.4, the adverse impact of the correlation between interest rate and withdrawals is diminished.

4.4. Risk and the banking model. This subsection provides some comments about the connection between our dynamic banking models and risk management.

4.4.1. Basic risk concepts. The cost process, $c(\Theta)$, presented in Section 3.1, corresponds to the value of the provisioning portfolio, $\widehat{V}(\Theta)$ less the accumulated return from reserves, R. Usually, the aggregate costs, $c_t(\Theta)$, incurred on the time interval $[0,t]$ decompose easily into the cost incurred on $(0,t]$ and an initial cost $c_0(\Theta) = V_0(\Theta)$.

4.4.2. Generalized GKW decomposition of $\mathrm{T}^{-1}(t)F_t(R_t,u)$. From (3.13) and (3.14) in Theorem 3.1, we have two distinct parts associated with the continuous and discontinuous (jump) components of n^R given by $D_xF_t(x,R_{t^-},u)$ and $j_t(x,u)$, respectively. In the Brownian framework of the Black-Scholes asset market, a hedging strategy for reserves (assets) results from

$$n_t^R = D_xF_t(R_{t^-},u). \tag{4.2}$$

By contrast, Theorem 3.1 suggests that the hedging strategy for reserves with jumps is (3.13). This means that if L corresponds with Z, then $\nu_t^{\mathbf{Q}}(dx) = 0$ and $\kappa_t = \tilde{s}^2$.

4.4.3. Risk minimizing strategy for depository contracts. K^D from (3.27) in Theorem 3.2 has interesting ramifications for risk (minimization) management of the bank provisioning portfolio. For instance, K^D allows for the possibility that the reserve risk can be reduced to intrinsic risk. In this regard, small changes in K^D can be represented as

$$dK_t^D = l_{s^-}^x \vartheta_s^{(1)} dZ_s^{\mathbf{Q}} + \int_{\mathbb{R}} l_{s^-}^x \vartheta_t^{(2)}(y) N^{\mathbf{Q}}(ds,dy) + v_s dM_s^I, \tag{4.3}$$

and interpreted as the losses incurred by the bank. If we analyze the first two terms of K^D and dK^D in (3.27) and (4.3), respectively, we may conclude that the integrals with respect to $Z^{\mathbf{Q}}$ and $N^{\mathbf{Q}}$ are the drivers of reserve risk. In the context of incomplete reserve markets in a Lévy-process setting, this demonstrates the influence of bank reserves on deposit risk. Here, the reliance of the provisioning strategy components, $\vartheta_t^{(1)}$ and $\vartheta_t^{(2)}$, on D_x and $j_t(x)$, results in a risk increase that originates from the continuous part of the bank reserve process. Note that the reserve risk driver is a function of the expected number of depository contracts surviving on the time interval $[t^-,T]$. Here, a deposit withdrawal results in a decrease in dK_t^D. Also, the last term of dK^D in (4.3), that contains

$$dM_t^I = dN_t^I - \iota_t dt, \tag{4.4}$$

can be interpreted as the source of risk for the entire bank provisioning portfolio.

From the first formula in (3.29) of Theorem 3.2, it is clear that the optimal investment in reserves, n_t^{R*}, is heavily dependent on the number of cohort deposits that are withdrawn during the time interval $[t^-,T]$. In particular, as the number of withdrawals, N^I, increases ($l_{t^-}^x$ decreases), it is likely that n_t^{R*} will decrease. This trend is also possible for

the optimal investment in the bank Treasuries, $n_t^{\mathrm{T}*}$, from the second formula in (3.29). The second part of Theorem 3.2 leads to an expression for the initial risk of the provisioning strategy, $\mathtt{R}_0$. In this regard, for $s > t$, and $l_{s^-}^x$ defined by (2.20), the filtration $\mathcal{F}_t$ follows a binomial distribution with the survival probability being $\mathbf{P}_2(T_i > u - t + x \mid T_i > x + t)$. As a consequence, for $0 \le t \le T$, we have that

$$\begin{aligned}\mathbf{E}^{\mathbf{Q}}\left[\left(l_s^x\right)^2 \mid \mathcal{F}_t\right] &= l_s^x \mathbf{P}_2\left(T_i > s - t + x \mid T_i > x + t\right)\left(1 - \mathbf{P}_2\left(T_i > s - t + x \mid T_i > x + t\right)\right) \\ &\quad + \left(l_t^x\right)^2 \left(\mathbf{P}_2\left(T_i > s - t + x \mid T_i > x + t\right)\right)^2,\end{aligned} \tag{4.5}$$

$$\begin{aligned}\mathtt{R}_0(\Theta^*) = n^x \int_0^T \Big\{ &\mathbf{P}_2\left(T_i > s + x \mid T_i > x\right)\left[1 - \mathbf{P}_2\left(T_i > s + x \mid T_i > x\right) + n^x \mathbf{P}_2\left(T_i > s + x \mid T_i > x\right)\right] \\ &\times \mathbf{E}^{\mathbf{Q}}\left[\vartheta_s^{(1)2} + \int_{\mathbb{R}} \left(\vartheta_s^{(2)}(y)\right)^2 \nu_s^{\mathbf{Q}}(dy)\right] \\ &+ \mathbf{E}^{\mathbf{Q}}\left[v_s^2(R_s)\right] \mathbf{P}_2\left(T_i > s + x \mid T_i > x\right) \omega_{x+s} \Big\} ds.\end{aligned} \tag{4.6}$$

A first observation is that the reserve risk component of $\mathtt{R}_0$ in (4.6) has the form

$$\begin{aligned}n^x \int_0^T \Big\{ &\mathbf{P}_2\left(T_i > s + x \mid T_i > x\right)\left[1 - \mathbf{P}_2\left(T_i > s + x \mid T_i > x\right) + n^x \mathbf{P}_2\left(T_i > s + x \mid T_i > x\right)\right] \\ &\times \mathbf{E}^{\mathbf{Q}}\left[\vartheta_s^{(1)2} + \int_{\mathbb{R}} \left(\vartheta_s^{(2)}(y)\right)^2 \nu_s^{\mathbf{Q}}(dy)\right] \Big\} ds.\end{aligned} \tag{4.7}$$

It is clear that for the reserve risk component (4.7), if n^x increases, then division of the risk component by $(n^x)^2$ does not result in the value of the said component tending to 0. This means that in our incomplete information setting, by contrast to the findings in the Brownian motion framework, a portion of risk resulting from the holding of reserves cannot be hedged against by merely increasing the number of depository contracts, n^x. As far as bank deposit risk is concerned, $\mathtt{R}_0$ and the *relative initial risk ratio* given by

$$\rho_0 = \frac{\sqrt{\mathtt{R}_0}}{n^x} \tag{4.8}$$

may be used to measure the risk associated with the nonhedgeable part of the sum of cohort deposits. Also, the deposit risk component of $\mathtt{R}_0$ given by (4.6) behaves as in

$$n^x \longrightarrow \infty \Longrightarrow 1/n^{x2} \left\{ n^x \int_0^T \mathbf{E}^{\mathbf{Q}}\left[v_s^2(R_s)\right] \mathbf{P}_2\left(T_i > s + x \mid T_i > x\right) \omega_{x+s} ds \right\} \longrightarrow 0. \tag{4.9}$$

It is not clear how a general risk analysis can be done for the *relative risk ratio* of the form

$$\rho_t = \frac{\sqrt{\mathtt{R}_t}}{l_t^x}. \tag{4.10}$$

The simplifying modification of the depository contract in (4.1) also has important ramifications for risk management. This scenario considers the cohort claim at terminal

time, $D^c(R_T)$, rather than the more general $D_t^c(R_t, u)$ (see, e.g., (2.15) and (4.1)), so that it becomes redundant to consider the variable "u." Henceforth, the bank obligation generated by the entire portfolio of depository contracts is considered to be the $\mathcal{F}_T$-measurable discounted deposit withdrawal

$$D^s = \mathtt{T}^{-1}(T)D^c(R_T)\sum_{i=1}^{n^x}\mathbf{I}(T_i > T) = \mathtt{T}^{-1}(T)D^c(R_T)l_T^x. \tag{4.11}$$

In the simplified case, the independence of the reserves and the bank withdrawals enables us to represent the intrinsic value of the entire depository contract portfolio, V_t^{s*}, as

$$V_t^{s*} = l_t^x\mathbf{P}_2(T_1 > T - t + x \mid T_1 > x + t)\mathtt{T}_t^{-1}F_t(R_t), \quad 0 \le t \le T, \tag{4.12}$$

with initial value

$$V_0^{s*} = n^x\mathbf{P}_2(T_1 > T + x \mid T_1 > x)F_0(R_0). \tag{4.13}$$

Under certain conditions, V_0^s may be considered to be the single deposit taken by the bank at $t = 0$.

In order to hedge deposit withdrawals, it is appropriate to adopt a (local) risk minimization approach, since

$$V_t^s(\Theta^{s*}) = V_t^{s*}, \quad \forall t \in (0, T). \tag{4.14}$$

Suppose that the intrinsic value, V_t^{s*}, is given by (4.12) and the variables $\vartheta_t^{(1)s}$ and $\vartheta_t^{(2)s}$ are analogous to those given by (3.14). For the depository contract in (4.11), with $0 \le t \le T$, the intrinsic value process, V^*, has the generalized GKW decomposition

$$V_t^{s*} = V_0^{s*} + \int_0^t l_{s^-}^x\mathbf{P}_2(T_1 > T - s + x \mid T_1 > x + s)n_s^R d\widehat{R}_s + K_t^{D^s}, \tag{4.15}$$

where

$$\begin{aligned} K_t^{D^s} = &\int_0^t l_{s^-}^x\mathbf{P}_2(T_1 > T - s + x \mid T_1 > x + s)\vartheta_t^{(1)s}dZ_s^{\mathbf{Q}} \\ &+ \int_0^t\int_{\mathbb{R}} l_{s^-}^x\mathbf{P}_2(T_1 > T - s + x \mid T_1 > x + s)\vartheta_t^{(2)s}(y)N^{\mathbf{Q}}(ds, dy) \\ &+ \int_0^t -\mathtt{T}_s^{-1}F_s(R_{s^-})\mathbf{P}_2(T_1 > T - s + x \mid T_1 > x + s)dM_s^I, \end{aligned} \tag{4.16}$$

is orthogonal to $\widehat{R}$. For $0 \le t \le T$, under the equivalent probability measure $\mathbf{Q}$, the unique admissible risk-minimizing provisioning strategy, $\Theta^{s*} = (n^{Rs*}, n^{\mathtt{T}s*})$, for the deposit withdrawal in (4.11) has the form

$$\begin{aligned} n_t^{Rs*} &= l_{t^-}^x\mathbf{P}_2(T_1 > T - t + x \mid T_1 > x + t)n_t^R, \\ n_t^{\mathtt{T}s*} &= l_t^x\mathbf{P}_2(T_1 > T - t + x \mid T_1 > x + t)\mathtt{T}^{-1}(t)F_t(R_t) - n_t^{Rs*}\widehat{R}_t. \end{aligned} \tag{4.17}$$

For $0 \le t \le T$, the intrinsic or remaining risk process may be expressed as

$$\begin{aligned}
\mathrm{R}_t^s(\Theta^{s*}) = \int_t^T & \Big\{ \mathbf{P}_2^2(T_1 > T - s + x \mid T_1 > x + s)\mathbf{E}^{\mathbf{Q}}\big[(l_t^x)^2 \mid \mathcal{F}_t\big] \\
& \times \mathbf{E}^{\mathbf{Q}}\Big[\vartheta_s^{(1)2} + \int_{\mathbb{R}} \big(\vartheta_s^{(2)s}(y)\big)^2 \nu_s^{\mathbf{Q}}(dy) \mid \mathcal{F}_t\Big]\Big\} ds \\
& + l_t^x \mathbf{P}_2(T_1 > T - t + x \mid T_1 > x + t) \\
& \times \int_t^T \mathbf{P}_2(T_1 > T - s + x \mid T_1 > x + s)\omega_{x+s}\mathrm{T}^{-2}(s)\mathbf{E}^{\mathbf{Q}}\big[F_s^2(R_s) \mid \mathcal{F}_t\big] ds.
\end{aligned} \tag{4.18}$$

5. Concluding remarks

For incomplete bank reserves, we derived a (locally) risk-minimizing hedging strategy for deposit withdrawals. This context provides a fertile environment for the derivation of general Lévy-driven models for reserves. Furthermore, we investigate the generalized GKW decomposition of the intrinsic value of the sum of cohort deposits contingent on a reserve process. This leads naturally to a solution of a risk minimization problem for banks that provides a hedging strategy for deposit withdrawals. The specific risk types related to our study are intrinsic, reserve and depository risk that are associated with the cumulative cost of the bank provisioning strategy, reserve processes and the amount and timing of net cash flows from deposits, and deposit withdrawals emanating from a cession of the depository contract, respectively. In addition, we provide a discussion of the main risk management issues mentioned above.

An open problem is to directly determine the hedging strategies for loan processes that are semimartingales and are not subjected to transformation by an equivalent measure into a martingale. Further issues that have not been resolved yet relate to the generalization from 1 to n (multidimensional) reserve types (cf., e.g., (2.4)) and deterministic to stochastic interest rates (cf., e.g., (2.2)).

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F. Gideon: Department of Mathematics and Applied Mathematics, Faculty of Science, North-West University (Potchefstroom Campus), Private Bag X 6001, Potchefstroom 2520, South Africa
Email address: frednard.gideon@nwu.ac.za

J. Mukuddem-Petersen: Department of Mathematics and Applied Mathematics, Faculty of Science, North-West University (Potchefstroom Campus), Private Bag X 6001, Potchefstroom 2520, South Africa
Email address: janine.mukuddempetersen@nwu.ac.za

M. A. Petersen: Department of Mathematics and Applied Mathematics, Faculty of Science, North-West University (Potchefstroom Campus), Private Bag X 6001, Potchefstroom 2520, South Africa
Email address: mark.petersen@nwu.ac.za