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Research Article

Travelling Wave Solutions for the KdV-Burgers-Kuramoto and Nonlinear Schrödinger Equations Which Describe Pseudospherical Surfaces

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We use the geometric notion of a differential system describing surfaces of a constant negative curvature and describe a family of pseudospherical surfaces for the KdV-Burgers-Kuramoto and nonlinear Schrödinger equations with constant Gaussian curvature −1. Travelling wave solutions for the above equations are obtained by using a sech-tanh method and Wu's elimination method.

1. Introduction

The theory of integrable systems has been an active area of mathematics for the past thirty years. Different aspects of the subject have fundamental relations with mechanics and dynamics, applied mathematics, algebraic structures, theoretical physics, analysis including spectral theory and geometry [1–3]. Most differential geometers have some knowledge and experience with finite dimensional integrable systems as they appear in sympectic geometry (mechanics) or ordinary differential equations (ODEs), although the reformulation of part of this theory as algebraic geometry is not commonly known [4].

There are two quite separate methods of extension of these ideas to partial differential equations (PDEs): one based on algebraic constructions and one based on spectral theory and analysis. These are less familiar still to geometers.

Many geometric equations are known to have integrable aspects, especially if one takes into account that most experts do not have a good definition of "integrable" as applied

to PDEs, particularly elliptic examples. In addition to those we mention in our historical discussion, the equations for harmonic maps (sigma models) from surfaces into groups [5–7], harmonic tori in symmetric spaces [8], constant mean curvature surfaces in space forms [9], isometric immersions of space forms in other space forms [10, 11], and the theory of affine spheres [12] and affine minimal surfaces [13] are all examples of "elliptic" integrable systems.

The ideas surrounding string theory resulted in a series of deep and not completely understood connections between representation theory of certain algebras and many of the more classical theories of integrable systems in mathematics [14–16]. Most recently, supersymmetric quantum field theories produce in a natural way moduli spaces of vacua or ground states which have new geometry generated by the supersymmetry. Since the supersymmetry generalizes the classical symmetries which produce integrals for the Euler-Lagrange equations via Noether's theorem, the connection with integrability is perhaps not surprising [17–19]. However, this does not explain entirely the use of integrable systems in hyper-Kähler geometry [20], Seiberg-Witten theory [21], special Kähler geometry, and quantum cohomology [22].

The 19th century geometers were mainly interested in the local theory of surfaces in R^3, which we might regard as the prehistory of these modern constructions. The sine-Gordon equation arose first through the theory of surfaces of constant Gauss curvature -1, and the reduced 3-wave equation can be found in Darboux's work on triply orthogonal systems of R^3 [23].

It is well-known that a differential equation (DE) for a real-valued function $u(x,t)$, or a differential system for a 2-vector-valued function $u(x,t)$, is said to describe pseudospherical surfaces (pss) if it is the necessary and sufficient condition for the existence of smooth real functions f_{ij}, $1 \le i \le 3$, $1 \le j \le 2$, depending only on u and a finite number of derivatives, such that the one-forms

$$\omega_i = f_{i1}dx + f_{i2}dt, \quad 1 \le i \le 3, \tag{1.1}$$

satisfy the structure equations of a surface of constant Gaussian curvature -1

$$d\omega_1 = \omega_3 \wedge \omega_2, \qquad d\omega_2 = \omega_1 \wedge \omega_3, \qquad d\omega_3 = \omega_1 \wedge \omega_2. \tag{1.2}$$

A DE for a real valued function $u(x,t)$ is kinematically integrable if it is the integrability condition of a one-parameter family of linear problems [24–30]

$$v_x = P(\eta)v, \qquad v_t = Q(\eta)v \tag{1.3}$$

in which $P(\eta)$ and $Q(\eta)$ are $SL(2,R)$-valued functions of x, t and (u and its derivatives) up to a finite order. Thus, an equation is kinematically integrable if it is equivalent to the zero curvature condition

$$\frac{\partial P(\eta)}{\partial t} - \frac{\partial Q(\eta)}{\partial x} + \left[P(\eta), Q(\eta)\right] = 0, \tag{1.4}$$

where $\operatorname{tr} P(\eta) = \operatorname{tr} Q(\eta) = 0$, for each η (spectral parameter or eigenvalue). In addition, a DE will be said to be strictly kinematically integrable if it is kinematically integrable and diagonal entries of the matrix $P(\eta)$ introduced above are η and $-\eta$.

The main aim of this paper is to explain the relationships between local differential geometry of surfaces and integrability of evolutionary nonlinear evolution equations (NLEEs). New travelling wave solutions for the KdV-Burgers-Kuramoto (KBK) and nonlinear Schrödinger (NLS) equations which describe pss are obtained.

The paper is organized as follows. The correspondence between KBK, NLS equations and their families of pss is established in Section 2. In Section 3, a new exact soliton solution is obtained for the KBK equation by using a sech-tanh method and Wu's elimination method. In Section 4, we construct a new travelling wave solution for the NLS equation by using the same above way. Finally, we give some conclusions in Section 5.

2. The KBK and NLS equations that describe pss

The inverse scattering method (ISM) was introduced first for the Korteweg-de Vries equation (KdVE) [18]. Later it was extended by Zakharov and Shabat [31] to a 2×2 scattering problem for the NLS equation and that was subsequently generalized by Ablowitz, Kaup, Newell, and Segur (AKNS) [32] to include a variety of NLEEs. Khater et al. [28] generalized the results of Konno and Wadati [33] by considering v as a three-component vector and Ω as a traceless 3×3 matrix one-form. The above definition of a DE is equivalent to saying that the DE for u is the integrability condition for the problem

$$dv = \Omega v, \quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \tag{2.1}$$

where v is a vector and the 3×3 matrix Ω (Ω_{ij}, $i, j = 1, 2, 3$) is traceless

$$\operatorname{tr}\Omega = 0, \tag{2.2}$$

and consists of a one-paramter (η), family of one-forms in the independent variables (x, t), the dependent variable u, and its derivatives. Khater et al. [28] introduced the inverse scattering problem (ISP):

$$\begin{aligned} v_{1x} &= f_{31}v_2 - f_{11}v_3, & v_{2x} &= -f_{31}v_1 - \eta v_3, & v_{3x} &= -f_{11}v_1 - \eta v_2, \\ v_{1t} &= f_{32}v_2 - f_{12}v_3, & v_{2t} &= -f_{32}v_1 - f_{22}v_3, & v_{3t} &= -f_{12}v_1 - f_{22}v_2. \end{aligned} \tag{2.3}$$

The associated integrability conditions for (1.3) or (2.1), which are obtained by cross-differentiation, then take the matrix form

$$d\Omega - \Omega\wedge\Omega = 0, \tag{2.4}$$

or the component form

$$\begin{aligned} f_{12,x} - f_{11,t} &= f_{31}f_{22} - \eta f_{32}, \\ f_{22,x} &= f_{11}f_{32} - f_{12}f_{31}, \\ f_{32,x} - f_{31,t} &= f_{11}f_{22} - \eta f_{12}. \end{aligned} \tag{2.5}$$

We will restrict ourselves to the case where $f_{21} = \eta$. More precisely, we say that a DE for $u(x,t)$ describes a pss if it is a necessary and sufficient condition for the existence of functions f_{ij}, $1 \le i \le 3$, $1 \le j \le 2$, depending on $u(x,t)$ and its derivatives, $f_{21} = \eta$, such that the one-forms in (1.1) satisfy the structure equations (1.2) of a pss. It follows from this definition that for each nontrivial solution u of the DE, one gets a metric defined on M^2, whose Gaussian curvature is -1.

It has been known, for a long time, that the sine-Gordon (SG) equation describes a pss. In this paper, we extend the same analysis to include the KBK and NLS equations.

Examples: let M^2 be a differentiable surface, parametrized by coordinates x, t.

(a) *The KBK equation*

Consider

$$\begin{aligned} \omega_1 &= \left(\frac{-1}{2}u + g(x,t)\right)dx + \left(\frac{1}{2}u_x + \frac{1}{4}u^2 + f(x,t)\right)dt, \\ \omega_2 &= \eta dx + \left(\frac{1}{2}\eta u - \eta g(x,t)\right)dt, \\ \omega_3 &= -\eta dx + \left(\frac{-1}{2}\eta u + \eta g(x,t)\right)dt, \end{aligned} \tag{2.6}$$

in which the functions $g(x,t)$ and $f(x,t)$ satisfy the equations

$$g_x + g^2 + f = 0, \qquad f_x - g_t = \frac{1}{2}\left(\alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx}\right). \tag{2.7}$$

Then M^2 is a pss if and only if u satisfies the KBK equation

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0, \tag{2.8}$$

where α, β, and γ are constants.

(b) *The NLS equation*

Consider

$$\begin{aligned} \omega_1 &= 2w dx + \left(-4\eta w + 2v_x\right)dt, \\ \omega_2 &= 2\eta dx + \left(2\left(v^2 + w^2\right) - 4\eta^2\right)dt, \\ \omega_3 &= -2v dx + \left(4\eta v + 2w_x\right)dt. \end{aligned} \tag{2.9}$$

Then M^2 is a pss if and only if u satisfies the NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = 0, \quad \text{where } u = v + iw. \tag{2.10}$$

3. Travelling wave solutions for the KBK equation

Now we will find travelling wave solutions $u(x,t)$ for the KBK equation (2.8). The solutions of KBK equation possess their actual physical application; this is the reason why so many methods, such as Wiss-Tabor-Carnevale transformation method [34], tanh-function method

[35], truncated expansion method [36], and so on, have been applied to obtain the solution of KBK equation. In this section, we obtain a new travelling wave solution class for KBK equations by using a sech-tanh method [37, 38] and Wu's elimination method [39]. The main idea of the algorithm is as follows. Suppose there is a PDE of the form

$$f\left(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots\right) = 0, \tag{3.1}$$

where f is a polynomial. By assuming travelling wave solutions of the form

$$u(x,t) = \phi(\rho), \quad \rho = \lambda(x - kt + c), \tag{3.2}$$

where k, λ are constant parameters to be determined, and c is an arbitrary constant, from the two equations (3.1) and (3.2) we obtain an ODE

$$f\left(\phi', \phi'', \phi''', \ldots\right) = 0, \tag{3.3}$$

where $\phi' = d\phi/d\rho$. According to the sech-tanh method [37–41], we suppose that (3.3) has the following formal travelling wave solution:

$$\phi(\rho) = \sum_{i=1}^{n} \operatorname{sech}^{i-1}\rho\left(B_i \operatorname{sech}\rho + A_i \tanh\rho\right) + A_0, \tag{3.4}$$

where $A_0, \ldots, A_n$ and $B_1, \ldots, B_n$ are constants to be determined. Then we proceed as follows.

(i) Equating the highest-order nonlinear term and highest-order linear partial derivative in (3.3) yields the value of n.

(ii) Substituting (3.4) into (3.3), we obtain a polynomial equation involving $\tanh\rho \operatorname{sech}^i\rho$, $\operatorname{sech}^i\rho$ for $i = 0, 1, 2, \ldots, n$ (with n being positive integer).

(iii) Setting the constant term and coefficients of $\operatorname{sech}\rho, \tanh\rho, \operatorname{sech}\rho\tanh\rho, \operatorname{sech}^2\rho, \ldots$, in the equation obtained in (ii) to zero, we obtain a system of algebraic equations about the unknown numbers k, λ, A_0, A_i, B_i for $i = 1, 2, \ldots, n$.

(iv) Using the Mathematica and Wu's elimination methods, the algebraic equations in (iii) can be solved.

These yield the solitary wave solutions for the system (3.3). We remark that the above method yields solutions that include terms $\operatorname{sech}\rho$ or $\tanh\rho$, as well as their combinations. There are different forms of those obtained by other methods, such as the homogenous balance method [42, 43]. We assume formal solutions of the form

$$u(x,t) = \phi(\rho), \quad \rho = \lambda(x - kt + c), \tag{3.5}$$

where k, λ are constant parameters to be determined later, and c is an arbitrary constant. Substituting from (3.5) and (2.8), we obtain an ODE

$$-k\phi' + \phi\phi' + \lambda\alpha\phi'' + \beta\lambda^2\phi''' + \gamma\lambda^3\phi'''' = 0. \tag{3.6}$$

(i) We suppose that (3.6) has the following formal solution:

$$\begin{aligned}\phi(\rho) = {} & A_0 + A_1 \operatorname{sech}\rho + B_1 \tanh\rho + A_2 \operatorname{sech}^2\rho + B_2 \operatorname{sech}\rho \tanh\rho \\ & + A_3 \operatorname{sech}^3\rho + B_3 \operatorname{sech}^2\rho \tanh\rho.\end{aligned} \tag{3.7}$$

(ii) From (3.6) and (3.7), we get

$$\begin{aligned}
& -k\phi' + \phi\phi' + \lambda\alpha\phi'' + \beta\lambda^2\phi''' + \gamma\lambda^3\phi'''' \\
& \quad = \left(-A_1B_1 - A_0B_2 + kB_2 + \lambda\alpha A_1 - \beta\lambda^2 B_2 + \gamma\lambda^3 A_1\right)\operatorname{sech}\rho \\
& \qquad + \left(-A_0A_1 - B_1B_2 + kA_1 + \alpha\lambda B_2 - \beta\lambda^2 A_1 + \gamma\lambda^3 B_2\right)\operatorname{sech}\rho\tanh\rho \\
& \qquad + \left(A_0B_1 - 2A_2B_1 - 2A_1B_2 - 2A_0B_3 - kB_1 + 2kB_3 + 4\alpha\lambda A_2\right. \\
& \qquad\qquad \left. + 4\beta\lambda^2 B_1 - 8\beta\lambda^2 B_3 + 16\gamma\lambda^3 A_2\right)\operatorname{sech}^2\rho \\
& \qquad + \left(-A_1^2 - 2A_0A_2 + B_1^2 - B_2^2 - 2B_1B_3 + 2kA_2 - 2\alpha\lambda B_1 + 4\alpha\lambda B_3 - 8\beta\lambda^2 A_2\right. \\
& \qquad\qquad \left. - 8\gamma\lambda^3 B_1 + 16\gamma\lambda^3 B_3\right)\operatorname{sech}^2\rho\tanh\rho \\
& \qquad + \left(2A_1B_1 - 3A_3B_1 + 2A_0B_2 - 3A_2B_2 - 3A_1B_3 - 2kB_2 - 2\alpha\lambda A_1 + 9\alpha\lambda A_3\right. \\
& \qquad\qquad \left. + 20\beta\lambda^2 B_2 - 20\gamma\lambda^3 A_1 + 81\gamma\lambda^3 A_3\right)\operatorname{sech}^3\rho \\
& \qquad + \left(-3A_1A_2 - 3A_0A_3 + 3B_1B_2 - 3B_2B_3 + 3kA_3 - 6\alpha\lambda B_2\right. \\
& \qquad\qquad \left. + 6\beta\lambda^2 A_1 - 27\beta\lambda^2 A_3 - 60\gamma\lambda^3 B_2\right)\operatorname{sech}^3\rho\tanh\rho \\
& \qquad + \left(3A_2B_1 + 3A_1B_2 - 4A_3B_2 + 3A_0B_3 - 4A_2B_3 - 3kB_3 - 6\alpha\lambda A_2\right. \\
& \qquad\qquad \left. - 6\beta\lambda^2 B_1 + 60\beta\lambda^2 B_3 - 120\gamma\lambda^3 A_2\right)\operatorname{sech}^4\rho \\
& \qquad + \left(-2A_2^2 - 4A_1A_3 + 2B_2^2 + 4B_1B_3 - 2B_3^2 - 12\alpha\lambda B_3 + 24\beta\lambda^2 A_2\right. \\
& \qquad\qquad \left. + 24\gamma\lambda^3 B_1 - 240\gamma\lambda^3 B_3\right)\operatorname{sech}^4\rho\tanh\rho \\
& \qquad + \left(4A_3B_1 + 4A_2B_2 + 4A_1B_3 - 5A_3B_3 - 12\alpha\lambda A_3 - 24\beta\lambda^2 B_2\right. \\
& \qquad\qquad \left. + 24\gamma\lambda^3 A_1 - 408\gamma\lambda^3 A_3\right)\operatorname{sech}^5\rho \\
& \qquad + \left(-5A_2A_3 + 5B_2B_3 + 60\beta\lambda^2 A_3 + 120\gamma\lambda^3 B_2\right)\operatorname{sech}^5\rho\tanh\rho \\
& \qquad + \left(5A_3B_2 + 5A_2B_3 - 60\beta\lambda^2 B_3 + 120\gamma\lambda^3 A_2\right)\operatorname{sech}^6\rho \\
& \qquad + \left(-3A_3^2 + 3B_3^2 + 360\gamma\lambda^3 B_3\right)\operatorname{sech}^6\rho\tanh\rho + \left(6A_3B_3 + 360\gamma\lambda^3 A_3\right)\operatorname{sech}^7\rho \\
& \quad = 0.
\end{aligned} \tag{3.8}$$

(iii) Setting the coefficients of $\operatorname{sech}^j\rho\tanh^i\rho$ for $i = 0,1$ and $j = 1,2,\ldots,7$ to zero, we have the following set of overdetermined equations in the unknowns A_0, A_1, A_2, A_3, B_1, B_2, B_3, λ, and k.

(iv) We now solve the above set of equations (3.8) by using Mathematica and Wu's elimination method, and obtain the following solutions:

$$\begin{gathered}\lambda^2 = \frac{(5 \pm 19)\alpha}{64\gamma}, \qquad \beta^2 = 16\alpha\gamma, \qquad k = 5\beta\lambda^2 + \frac{\alpha\beta}{4\gamma}, \\ A_0 = A_1 = B_1 = B_2 = A_3 = 0, \qquad A_2 = 15\beta\lambda^2, \qquad B_3 = -120\gamma\lambda^3.\end{gathered} \tag{3.9}$$

Substituting (3.9) into (3.7), we obtain

$$u(x,t) = 15\lambda^2 \operatorname{sech}^2 \rho(\beta - 8\gamma\lambda \tanh \rho), \quad \text{where } \rho = \lambda(x - kt + c). \tag{3.10}$$

4. Travelling wave solutions for the NLS equation

Now we will find travelling wave solutions $u(x,t)$ for the NLS equation (2.10). Equation (2.10) can be written in the real form $u = v + iw$ as follows:

$$\begin{gathered}v_t + w_{xx} + 2(v^2 + w^2)w = 0, \\ -w_t + v_{xx} + 2(v^2 + w^2)v = 0.\end{gathered} \tag{4.1}$$

We assume formal solutions of the form

$$v(x,t) = \phi(\rho), \quad w(x,t) = \theta(\rho), \quad \rho = \lambda(x - kt + c), \tag{4.2}$$

where k, λ are constant parameters to be determined later, and c is an arbitrary constant. Substituting from (4.2) into (4.1), we obtain two ODEs:

$$\begin{gathered}-k\lambda\phi' + \lambda^2\theta'' + 2(\phi^2 + \theta^2)\theta = 0, \\ k\lambda\theta' + \lambda^2\phi'' + 2(\phi^2 + \theta^2)\phi = 0.\end{gathered} \tag{4.3}$$

(i) Equating the highest-order nonlinear term and highest-order linear partial derivative in (4.3) yields $n = 1$. Then (4.3) has the following formal solutions:

$$\begin{gathered}\phi(\rho) = A_0 + A_1 \operatorname{sech} \rho + B_1 \tanh \rho, \\ \theta(\rho) = a_0 + a_1 \operatorname{sech} \rho + b_1 \tanh \rho.\end{gathered} \tag{4.4}$$

(ii) With the aid of Mathematica, substituting (4.4) into (4.3), then we obtain a polynomial equation involving $\tanh^i \rho \operatorname{sech}^j \rho$ for $i = 0, 1$, $j = 0, 1, 2, 3$.

$$\begin{aligned}
&-k\lambda\phi' + \lambda^2\theta'' + 2(\phi^2 + \theta^2)\theta \\
&\quad = (2a_0^3 + 2a_0A_0^2 + 4b_1A_0B_1 + 6a_0b_1^2 + 2a_0B_1^2) \\
&\qquad + (\lambda^2 a_1 + 6a_0^2 a_1 + 2a_1A_0^2 + 4a_0A_0A_1 + 6a_1b_1^2 + 4b_1A_1B_1 + 2a_1B_1^2)\operatorname{sech}\rho \\
&\qquad + (2b_1^3 + 6a_0^2 b_1 + 4a_0A_0B_1 + 2b_1A_0^2 + 2b_1B_1^2)\tanh\rho \\
&\qquad + (k\lambda A_1 + 12a_0a_1b_1 + 4b_1A_0A_1 + 4a_1A_0B_1 + 4a_0A_1B_1)\operatorname{sech}\rho\tanh\rho \\
&\qquad + (6a_0a_1^2 + 4a_1A_0A_1 + 2a_0A_1^2 - 6a_0b_1^2 - k\lambda B_1 - 4b_1A_0B_1 - 2a_0B_1^2)\operatorname{sech}^2\rho \\
&\qquad + (-2\lambda^2 b_1 + 6a_1^2 b_1 - 2b_1^3 + 2b_1A_1^2 + 4a_1A_1B_1 - 2b_1B_1^2)\operatorname{sech}^2\rho\tanh\rho \\
&\qquad + (-2\lambda^2 a_1 + 2a_1^3 + 2a_1A_1^2 - 6a_1b_1^2 - 4b_1A_1B_1 - 2a_1B_1^2)\operatorname{sech}^3\rho \\
&\quad = 0, \\
&k\lambda\theta' + \lambda^2\phi'' + 2(\phi^2 + \theta^2)\phi \\
&\quad = (2a_0^2A_0 + 6A_0B_1^2 + 2A_0B_1^2 + 4a_0b_1B_1 + 2A_0^3) \\
&\qquad + (\lambda^2 A_1 + 6A_0^2A_1 + 2A_1b_1^2 + 4a_0a_1A_0 + 6A_1B_1^2 + 4a_1b_1B_1 + 2A_1a_0^2)\operatorname{sech}\rho \\
&\qquad + (2B_1^3 + 6A_0^2B_1 + 4a_0A_0b_1 + 2B_1a_0^2 + 2b_1^2B_1)\tanh\rho \\
&\qquad + (-k\lambda a_1 + 12A_0A_1B_1 + 4a_1A_0b_1 + 4a_0a_1B_1 + 4a_0A_1b_1)\operatorname{sech}\rho\tanh\rho \\
&\qquad + (6A_0A_1^2 + 4a_0a_1A_1 + 2A_0a_1^2 - 6A_0B_1^2 + k\lambda b_1 - 4b_1a_0B_1 - 2A_0b_1^2)\operatorname{sech}^2\rho \\
&\qquad + (-2\lambda^2 B_1 + 6A_1^2B_1 - 2B_1^3 + 2B_1a_1^2 + 4a_1A_1b_1 - 2B_1b_1^2)\operatorname{sech}^2\rho\tanh\rho \\
&\qquad + (-2\lambda^2 A_1 + 2A_1^3 + 2a_1^2A_1 - 6A_1B_1^2 - 4b_1a_1B_1 - 2A_1b_1^2)\operatorname{sech}^3\rho \\
&\quad = 0.
\end{aligned} \tag{4.5}$$

(iii) Setting the constant term and coefficients of $\tanh^i \rho \operatorname{sech}^j \rho$ for $i = 0, 1$, $j = 0, 1, 2, 3$, in the equation obtained in (ii) to zero, we obtain a system of algebraic equations about the unknown numbers A_0, A_1, B_1, a_0, a_1, b_1, and k.

(iv) Now we solve the above set of (4.5) by using Mathematica and Wu's elimination method, and we obtain the following solution:

$$A_0 = A_1 == a_1 = b_1 = 0, \qquad a_0 = \pm\lambda\, B_1 = \pm i\lambda, \qquad k = \mp i\lambda. \tag{4.6}$$

Substituting (4.6) into (4.4), we obtain

$$v(x,t) = \phi(\rho) = \pm i\lambda\tanh\rho, \qquad w(x,t) = \theta(\rho) = \pm\lambda, \tag{4.7}$$

then the solution of NLS equation (2.10) takes the form

$$u(x,t) = \pm i\lambda(1 + \tanh\rho), \quad \rho = \lambda(x \pm i\lambda t + c). \tag{4.8}$$

5. Conclusions

We find the relationship between KBK, NLS equations and their families of pss. With the help of Mathematica, many travelling solutions for the KBK and NLS equations of the pseudospherical class are obtained by using a sech-tanh method and Wu's elimination method. We obtained some new solitary wave solutions and periodic solutions.

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