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*Research Article*

# Optimal Control Systems by Time-Dependent Coefficients Using CAS Wavelets

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This paper considers the problem of controlling the solution of an initial boundary-value problem for a wave equation with time-dependent sound speed. The control problem is to determine the optimal sound speed function which damps the vibration of the system by minimizing a given energy performance measure. The minimization of the energy performance measure over sound speed is subjected to the equation of motion of the system with imposed initial and boundary conditions. Using the modal space technique, the optimal control of distributed parameter systems is simplified into the optimal control of bilinear time-invariant lumped-parameter systems. A wavelet-based method for evaluating the modal optimal control and trajectory of the bilinear system is proposed. The method employs finite CAS wavelets to approximate modal control and state variables. Numerical examples are presented to demonstrate the effectiveness of the method in reducing the energy of the system.



## 1. Introduction

Dynamic stability related to parametric resonance is a very important factor in structural dynamics. For example, instability caused by parametric resonance is believed to be the reason for the famous Tacoma bridge collapse in 1940 [1]. A suitable control of the coefficients may provide an effective protection against this phenomenon.

Control in the coefficients is known to be a very effective method in structures governed by elliptic equations [1]. However, not much information is known about the effect produced by control in coefficient for hyperbolic equations [2, 3]. In this paper, we study a control problem for a structure dynamic system governed by a hyperbolic equation where the control is a time dependent coefficient.

The model considered in this work is motivated by recent developments in the area of smart materials [4]. The properties of these materials can be changed by applying external fields, such as electrical, magnetic, or temperature; this is referred to as a phase transformation.

A structure made with such a material is considered where control consists of eliminating a transient disturbance in the structure by varying the material properties in the response to the deformation. The modal dynamic of a structure is governed by a scalar wave equation, where the control variable is the sound speed in the medium. The basic bilinear optimal control problem becomes the minimization of the energy function of the system in a given period of time with a minimum sound speed. Using modal expansion, the optimal control of the distributed parameter system is reduced to the optimal control of a bilinear time-varying lumped parameter system. The parameterization approach is used to approximate the state-variable and each component of the control variable using finite-term wavelets with unknown coefficients. Therefore, the quadratic problem is transformed into a mathematical programming problem with the objective of minimizing the unknown coefficients to give suboptimal solution of the problem. A necessary condition for the optimality of the unknown coefficients is derived as a system of linear algebraic equations for which the solution is used to obtain the optimal control sound speed and optimal state function.

The bilinear system is a kind of nonlinear system where some related problems such as optimal control are much more difficult to solve than those of linear systems. In literature, many authors [5–9] have tried various methods to overcome the difficulties of solving bilinear systems. In this paper, the focus will be on obtaining the optimal state solution of a wave equation governed by a bilinear system using CAS wavelets taking advantage of some needed properties of this type of wavelets [10, 11]. Compared to conventual method such as Fourier series or finite elements, CAS wavelets with their local properties enable arbitrary functions (even with discontinuity) to be approximated more efficient. To demonstrate the effectiveness of the proposed approach, numerical results will show confirm that the proposed method significantly minimizes the energy of the system.

## 2. Optimal Control Problem Setting

Let $\Omega_x$ be an open, bounded, and simply connected subset of $n$-dimensional Euclidean space $\mathbb{R}^n$. Let $\Omega_t$ denote a given time interval $(0, t_f)$ with finite terminal time $t_f$. Consider the wave equation, defined on $Q = \Omega_x \times \Omega_t$:

$$u_{tt} = a(t)\Delta u, \tag{2.1}$$

where $\Delta$ is the Laplacian operator, and $u(x,t)$ is the disturbance of position $x$ and time $t$. The wave speed $\sqrt{a(t)}$ is assumed to be a function of time. For simplicity, let $u$ satisfy the boundary and initial conditions:

$$\begin{gathered} u(x,t) = 0, \quad \Delta u = 0, \quad \forall x \in \partial\Omega_x, \\ u(x,0) = w_0(x), \quad u_t(x,0) = w_1(x), \quad x \in \Omega_x, \end{gathered} \tag{2.2}$$

where

$$w_0(x) \in H^2(\Omega_x) = \left\{ h(x) : \frac{\partial^i h}{\partial x^i} \in L^2(\Omega_x),\ i = 1,2 \right\}, \tag{2.3}$$

and $w_1(x) \in L^2(\Omega_x)$.

Let the admissible control set be

$$A_{\text{ad}} = \left\{ a(t) : a(t) \in L^2(\Omega_t) \right\}. \tag{2.4}$$

Associated with the wave equation (2.1) is the modified energy $J[a(t)]$ at terminal time $t_f$:

$$J[a(t)] = \mu_1 \int_{\Omega_x} u^2(x, t_f) dx + \mu_2 \int_{\Omega_x} u_t^2(x, t_f) dx + \mu_3 \int_{\Omega_t} a^2(t) dt, \tag{2.5}$$

where $\mu_1$, $\mu_2$, and $\mu_3$ are weighing constants satisfying the condition $\mu_1 + \mu_2 > 0$ and $\mu_3 > 0$. The last term on the right-hand side of (2.5) is a penalty term on control energy.

The optimal control problem is stated as follows: *determine the optimal control function* $a^*(t) \in A_{\text{ad}}$ *such that*

$$J[a^*(t)] = \min_{a(t) \in A_{\text{ad}}} J[a(t)] \tag{2.6}$$

*subject to* (2.1) *and* (2.2).

## 3. Control Problem in Modal Space

We pose the problem at hand as a control problem for an finite system of ordinary differential equations by using modal space expansion. Let

$$u(x, t) = \sum_{n=1}^{N} z_n(t) \varphi_n(x), \tag{3.1}$$

where $\varphi_n(x)$ are normalized eigenfunctions associated with eigenvalues $w_n^2$. This implies that $\varphi_n(x)$ satisfies the eignvalue-problem

$$\begin{gathered} \Delta \varphi_n(x) + w_n^2 \varphi_n(x) = 0, \quad x \in \Omega_x, \\ \varphi_n(x) = 0, \quad x \in \partial\Omega_x, \\ \Delta \varphi_n(x) = 0, \quad x \in \partial\Omega_x. \end{gathered} \tag{3.2}$$

It can be shown that the set $\varphi_n(x)$ forms an orthonormal set, and hence $z_n(t)$ satisfies

$$\frac{d^2}{dt^2} z_n(t) + a(t) w_n^2 z_n(t) = 0, \quad n = 1, 2, \ldots, N \tag{3.3}$$

with initial conditions

$$z_n(0) = z_{0n}, \quad \frac{d}{dt} z_n(0) = z_{1n}, \quad n = 1, \ldots, N. \tag{3.4}$$

In view of the expansion (3.1), the performance index becomes

$$J_N[a(t)] = \sum_{n=1}^{N}\left[\mu_1 z_n^2(t_f) + \mu_2\left(\frac{d}{dt}z_n(t_f)\right)^2\right] + \mu_3\int_0^{t_f} a^2(t)dt. \tag{3.5}$$

The optimal control problem (2.6) is now modified as follows: *determine* $a^*(t) \in A_{\text{ad}}$ *such that*

$$J_N[a^*(t)] = \min_{a(t)\in A_{\text{ad}}} J_N[a(t)] \tag{3.6}$$

*subject to* (3.3) *and* (3.4).

## 4. Properties of the CAS Wavelets

### *4.1. CAS Wavelets*

Wavelets have been used by many scientists and engineers to solve several problems in areas such as signal and image processing, control problems, and stochastic problems. Wavelets are mathematical functions that are constructed using dilation and translation of a single function called the mother wavelet denoted by $\psi(t)$ and must satisfy certain requirements. If the dilation parameter is $a$ and translation parameter is $b$, then we have the following family of wavelets:

$$\psi_{a,b}(t) = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right), \quad \text{with } a, b \in R,\ a \neq 0. \tag{4.1}$$

Restricting $a$ and $b$ to discrete values, such as $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1$, $b_0 > 0$ and $n$ and $k$ are positive integers, gives

$$\psi_{k,n}(t) = |a_0|^{k/2}\psi\left(a_0^k t - nb_0\right), \tag{4.2}$$

where $\psi_{k,n}(t)$ form a basis for $L^2(R)$. If $a_0 = 2$ and $b_0 = 1$, then it is clear that the set $\{\psi_{k,n}(t)\}$ forms an orthonormal basis for $L^2(R)$.

The CAS wavelets employed in this paper are defined as

$$\psi_{n,m}(t) = \begin{cases} 2^{k/2}\text{CAS}_m\left(2^k t - n\right), & \text{if } \dfrac{n}{2^k} \leq t < \dfrac{n+1}{2^k}, \\ 0, & \text{otherwise,} \end{cases} \tag{4.3}$$

where

$$\text{CAS}_m(t) = \cos(2m\pi t) + \sin(2m\pi t). \tag{4.4}$$

The set of CAS wavelets forms an orthonormal basis for $L^2([0,1])$. This implies that any function $f(t)$ defined over $[0,1]$ can be expanded as

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} \sum_{m\in Z} d_{n,m}\psi_{n,m}(t) \\ &\simeq \sum_{n=0}^{2^k-1} \sum_{m=-M}^{M} d_{n,m}\psi_{n,m}(t) = D^T\Psi(t), \end{aligned} \tag{4.5}$$

where

$$d_{n,m} = \left(f(t), \psi_{n,m}(t)\right) = \int_0^1 f(t)\psi_{n,m}(t)dt, \tag{4.6}$$

and $D$ and $\Psi(t)$ are $2^k(2M+1)\times 1$ vectors given by

$$\begin{aligned} D &= \left[d_{0,-M}, d_{0,(-M+1),}, \dots, d_{0,M}, d_{1,-M}, \dots, d_{1,M}, \dots, d_{(2^k-1),-M}, \dots, d_{(2^k-1),M}\right]^T, \\ \Psi(t) &= \left[\Psi_{0,-M}, \Psi_{0,(-M+1)}, \dots, \Psi_{0,M}, \Psi_{1,-M}, \dots, \Psi_{1,M}, \dots, \Psi_{(2^k-1),-M}, \dots, \Psi^T_{(2^k-1),M}\right]. \end{aligned} \tag{4.7}$$

### *4.2. Operational Matrices of Integration*

The integration of the function $\Psi(t)$ in (4.5) is given by

$$\int_0^t \Psi(s)ds = P\Psi(t), \tag{4.8}$$

where $P$ is an $2^k(2M+1)\times 2^k(2M+1)$ matrix, called the operational matrix, and is given by [12]

$$P = \frac{1}{2^{k+1}}\begin{bmatrix} S & F & F & \cdots & F \\ O & S & F & \cdots & F \\ \vdots & O & \ddots & \ddots & \vdots \\ & & & & F \\ O & O & \cdots & O & S \end{bmatrix} \tag{4.9}$$

in which $O$ is a zero matrix and $F$ and $S$ are $(2M+1)\times(2M+1)$ matrices given by

$$F=\begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \\ 0 & \cdots & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix},$$

$$S=\begin{bmatrix} 0 & 0 & \cdots & 0 & -\dfrac{1}{M\pi} & 0 & \cdots & 0 & \dfrac{1}{M\pi} \\ 0 & 0 & \cdots & 0 & -\dfrac{1}{(M-1)\pi} & 0 & \cdots & \dfrac{1}{(M-1)\pi} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -\dfrac{1}{\pi} & \dfrac{1}{\pi} & \cdots & 0 & 0 \\ \dfrac{1}{\pi} & \dfrac{1}{\pi} & \cdots & \dfrac{1}{\pi} & 1 & \dfrac{1}{\pi} & \cdots & \dfrac{1}{\pi} & \dfrac{1}{\pi} \\ 0 & 0 & \cdots & \dfrac{1}{\pi} & \dfrac{1}{\pi} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \dfrac{1}{(M-1)\pi} & \cdots & 0 & \dfrac{1}{(M-1)\pi} & 0 & \cdots & 0 & 0 \\ \dfrac{1}{M\pi} & 0 & \cdots & 0 & \dfrac{1}{M\pi} & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{4.10}$$

The integration of the product of two CAS function vectors is given by

$$\int_0^1 \Psi(t)\Psi(t)^T dt = I. \tag{4.11}$$

The product operational matrix of the CAS wavelet is given by

$$\Psi(t)\Psi(t)^T C \simeq \widetilde{C}\Psi(t), \tag{4.12}$$

where the matrix $C$ is given in (4.7) and $\widetilde{C}$ is an $2^k(2M+1)\times 2^k(2M+1)$ given by [11]

$$\widetilde{C}=\begin{bmatrix} \widetilde{C_1} & 0 \\ 0 & \widetilde{C_2} \end{bmatrix}, \tag{4.13}$$

where $\widetilde{C_i}$, $i = 1, 2$ are $(2M+1) \times (2M+1)$ given by

$$\widetilde{C_i} = \begin{bmatrix} c_{i,0} & c_{i,-1} & 0 \\ c_{i,-1} & c_{i,0} & c_{i,1} \\ 0 & c_{i,1} & c_{i,0} \end{bmatrix} \quad \text{for } i = 1, 2. \tag{4.14}$$

## 5. CAS-Wavelets-Based Approach

To redefine the wavelet functions over the interval $[0, t_f]$, we let $t = t_f\tau$. Then (3.3), (3.4), and (3.5), respectively, become

$$\frac{d^2}{d\tau^2} z_i(\tau) + w_i^2 t_f^2 a(\tau) z_i(\tau) = 0, \quad \text{for } 0 < \tau < 1,\ i = 1, 2, \dots, N, \tag{5.1}$$

$$\begin{gathered} z_i(0) = z_{i0}, \quad \frac{d}{d\tau} z_i(0) = t_f z_{i1}, \quad i = 1, \dots, N, \\ J_N(a(\tau)) = \sum_{i=1}^{N} \left[ \mu_1 z_i^2(t_f) + t_f^2 \mu_2 \frac{d}{d\tau} z_i^2(t_f) \right] + t_f \mu_3 \int_0^1 a^2(\tau) d\tau. \end{gathered} \tag{5.2}$$

Using the expansion in (4.5) gives

$$\begin{aligned} \frac{d^2}{d\tau^2} z_i(\tau) &= \vec{B_i}^T \vec{\Psi}(\tau) = \vec{\Psi}^T(\tau) \vec{B_i}, \\ a(\tau) &= \vec{C}^T \vec{\Psi}(\tau) = \vec{\Psi}^T(\tau) \vec{C}, \\ z_i(0) &= \vec{\Phi}_i^T \vec{\Psi}(\tau) = \vec{\Psi}^T(\tau) \vec{\Phi}_i, \\ \frac{d}{d\tau} z_i(0) &= \vec{F}_i^T \vec{\Psi}(\tau) = \vec{\Psi}^T(\tau) \vec{F}_i, \end{aligned} \tag{5.3}$$

where $\vec{B_i}$, $\vec{C}$, $\vec{\Phi}_i$, $\vec{F}_i$, and $\vec{\Psi}(\tau)$ are $2^k(2M+1) \times 1$ vectors defined as in (4.7). Furthermore,

$$\begin{aligned} \frac{d}{d\tau} z_i(\tau) &= \int_0^\tau \frac{d^2}{ds^2} z_i(s) ds + \frac{d}{d\tau} z_i(0) \\ &= \int_0^\tau \vec{B_i}^T \vec{\Psi}(s) ds + \vec{F}_i^T \vec{\Psi}(\tau) \\ &= \vec{B_i}^T P \Psi(\vec{\tau}) + \vec{F}_i^T \vec{\Psi}(\tau), \end{aligned} \tag{5.4}$$

$$\begin{aligned} z_i(\tau) &= \int_0^{\tau} \frac{d}{d\tau} z_i(s)\,ds + z_i(0) \\ &= \int_0^{\tau} \left( \vec{B}_i^{\,T} P \vec{\Psi(s)} + \vec{F}_i^{\,T} \vec{\Psi}(s) \right) ds + \vec{\Phi}_i^T \vec{\Psi}(\tau) \\ &= \vec{B}_i^{\,T} P^2 \vec{\Psi(\tau)} + \vec{F}_i^{\,T} P \vec{\Psi}(\tau) + \vec{\Phi}_i^T \vec{\Psi}(\tau) \\ &= \Psi^T(\tau) \left(\vec{P^2}\right)^T \vec{B}_i + \vec{\Psi}^T(\tau) P^T \vec{F}_i + \vec{\Psi}^T(\tau) \vec{\Phi}_i. \end{aligned} \tag{5.5}$$

Substituting (5.5) in (5.1) yields

$$\vec{B}_i^{\,T} \vec{\Psi}(\tau) + w_i^2 t_f^2 \vec{C}^T \vec{\Psi}(\tau) \left[ \Psi^T(\tau) \left(\vec{P^2}\right)^T \vec{B}_i + \vec{\Psi}^T(\tau) P^T \vec{F}_i + \Psi^T(\tau) \vec{\Phi}_i \right] = 0 \tag{5.6}$$

and hence

$$\begin{aligned} &\vec{B}_i^{\,T} \vec{\Psi}(\tau) + w_i^2 t_f^2 \vec{C}^T \vec{\Psi}(\tau) \Psi^T(\tau) \left(P^2\right)^T \vec{B}_i \\ &\quad + w_i^2 t_f^2 \vec{C}^T \vec{\Psi}(\tau) \vec{\Psi}^T(\tau) P^T \vec{F}_i + w_i^2 t_f^2 \vec{C}^T \vec{\Psi}(\tau) \Psi^T(\tau) \vec{\Phi}_i = 0. \end{aligned} \tag{5.7}$$

Using (4.12) leads to

$$\vec{B}_i^{\,T} \vec{\Psi}(\tau) + w_i^2 t_f^2 \vec{\Psi}^T(\tau) \tilde{C}^T \left(P^2\right)^T \vec{B}_i + w_i^2 t_f^2 \vec{\Psi}^T(\tau) \tilde{C}^T P^T \vec{F}_i + w_i^2 t_f^2 \vec{\Psi}^T(\tau) \tilde{C}^T \vec{\Phi}_i = 0. \tag{5.8}$$

Multiplying (5.8) by $\vec{\Psi}(\tau)$, integrating, and using (4.11) give

$$\vec{B}_i + w_i^2 t_f^2 \tilde{C}^T \left(P^2\right)^T \vec{B}_i + w_i^2 t_f^2 \tilde{C}^T P^T \vec{F}_i + w_i^2 t_f^2 \tilde{C}^T \vec{\Phi}_i = 0 \tag{5.9}$$

or

$$\vec{B}_i = -G^{-1} \left( w_i^2 t_f^2 \tilde{C}^T P^T \vec{F}_i + w_i^2 t_f^2 \tilde{C}^T \vec{\Phi}_i \right) \tag{5.10}$$

provided that

$$G = \left( I + w_i^2 t_f^2 \tilde{C}^T \left(P^2\right)^T \right)^{-1} \tag{5.11}$$

exists. Substituting equations (5.4) and (5.5) into equation (3.5) convert the performance index $J_N[a(t)]$ into a function of $\tilde{C}$ and hence to optimize $J_N[a(t)]$, we solve

$$\frac{\partial J_N}{\partial c_i} = 0. \tag{5.12}$$

**Table 1:** Comparison between uncontrolled and controlled performance indices.

| Controller $a(t)$ | $J_1[a(t)]$ |
|---|---|
| $a^*(t)$ | 0.02451 |
| $t$ | 0.50858 |
| $t^2$ | 0.36656 |
| $\sin t$ | 0.44015 |
| $\cos t$ | 3.62556 |

## 6. Numerical Example

Consider the wave equation

$$\frac{d^2}{d\tau^2}z(\tau) + w^2 t_f^2 a(\tau) z(\tau) = 0, \quad \text{for } 0 < \tau < 1 \tag{6.1}$$

with initial conditions

$$z(0) = 1, \qquad \frac{d}{d\tau}z(0) = 0. \tag{6.2}$$

For the sake of illustration, the following parameters were assumed:

$$\begin{gathered} w_1 = \pi, \qquad t_f = 1, \qquad \mu_1 = \mu_2 = \mu_3 = 1, \qquad \Omega_x = (0,1), \\ M = 1, \qquad k = 1 \ (6 \text{ wavelet expansions}). \end{gathered} \tag{6.3}$$

The performance index was computed for the optimal control $a^*(t)$ and compared with the performance index for the controllers $a(t) = t$, $a(t) = t^2$, $a(t) = \sin t$, and $a(t) = \cos t$. The results are summarized in Table 1.

It is observed that the proposed control is effective in significantly reducing the performance index of the problem.

## 7. Conclusion

A control for a wave equation where the control is a time dependent coefficient is considered. A modal space technique simplifies the optimal control of a distributed parameter system into the optimal control of a bilinear time-invariant lumped-parameter system. A Galerkin CAS wavelet-based method was developed to solve this bilinear optimal control problem. The main aspect of the proposed approach resides in converting the optimization problem into a mathematical programming problem where the necessary conditions of optimality are derived as a system of algebraic equations . A test example, which includes a variable coefficient and one-dimensional hyperbolic equation, demonstrates the capability of the proposed Galerkin-Wavelet approach for solving optimal control problems governed by bilinear systems. Moreover, the numerical simulations show that the optimal control procedure led to a substantial damping in the bilinear system energy.

This method may be extended to treat a more general setting where the coefficients are $x$ and $t$ dependent. That is, the wave speed function is a controllable function of the form $a(x,t)$ [13].

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