Research Article

On Generalized Transitive Matrices

Jing Jiang,[1,2] Lan Shu,[1] and Xinan Tian[3]

[1] *School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan 611731, China*

[2] *Mathematics and Statistics College, Chongqing University of Arts and Sciences, Chongqing 402160, China*

[3] *Modern Education Technology Center, Chongqing University of Arts and Sciences, Chongqing 402160, China*

Correspondence should be addressed to Jing Jiang, jiangjingstu@163.com

Transitivity of generalized fuzzy matrices over a special type of semiring is considered. The semiring is called incline algebra which generalizes Boolean algebra, fuzzy algebra, and distributive lattice. This paper studies the transitive incline matrices in detail. The transitive closure of an incline matrix is studied, and the convergence for powers of transitive incline matrices is considered. Some properties of compositions of incline matrices are also given, and a new transitive incline matrix is constructed from given incline matrices. Finally, the issue of the canonical form of a transitive incline matrix is discussed. The results obtained here generalize the corresponding ones on fuzzy matrices and lattice matrices shown in the references.

1. Introduction

Generalized transitive matrices [1] over a special type of semiring are introduced. The semiring is called incline algebra. Boolean algebra, fuzzy algebra, and distributive lattice are inclines. And the Boolean matrices, the fuzzy matrices, and the lattice matrices are the prototypical examples of the incline matrices. Inclines are useful tools in diverse areas such as design of switching circuits, automata theory, medical diagnosis, information systems, and clustering. Besides inclines are applied to nervous system, probable reasoning, dynamical programming, and decision theory.

Transitive matrices are an important type of generalized matrices which represent transitive relation (see, e.g., [2–6]). Transitive relation plays an important role in clustering, information retrieval, preference, and so on [5, 7, 8]. The transitivity problems of matrices over some special semirings have been discussed by many authors (see, e.g., [9–17]). In 1982, Kim [18] introduced the concept of transitive binary Boolean matrices. Hashimoto [11]

presented the concept of transitive fuzzy matrices and considered the convergence of powers of transitive fuzzy matrices. Kołodziejczyk [10] gave the concept of s-transitive fuzzy matrices and considered the convergence of powers of s-transitive fuzzy matrices. Tan [17, 19] discussed the convergence of powers of transitive lattice matrices. Han and Li [1] studied the convergence of powers of incline matrices. In [12, 13], the canonical form of a transitive matrix over fuzzy algebra was established, and, in [14, 15, 17], the canonical form of a transitive matrix over distributive lattice was characterized. In [9, 16, 20], some properties of compositions of generalized fuzzy matrices and lattice matrices were examined.

In this paper, we continue to study transitive incline matrices. In Section 3, the transitive closure of an incline matrix is discussed. In Section 4, the convergence of powers of transitive incline matrices is considered. In Section 5, some properties of compositions of incline matrices are given and a new transitive incline matrix is constructed from given incline matrices. In Section 6, the issue of the canonical form of an incline matrix is further discussed. Some results in this paper generalize the corresponding results in [14, 17, 20].

2. Definitions and Preliminary Lemmas

In this section, we give some definitions and lemmas.

Definition 2.1 (see [1]). A nonempty set L with two binary operations $+$ and $\cdot$ is called an incline if it satisfies the following conditions:

(1) $(L, +)$ is a semilattice;

(2) $(L, \cdot)$ is a commutative semigroup;

(3) $x(y + z) = xy + xz$ for all $x, y, z \in L$;

(4) $x + xy = x$ for all $x, y \in L$.

In an incline L, define a relation $\leq$ by $x \leq y \Leftrightarrow x + y = y$. Obviously, $xy \leq x$ for all $x, y \in L$. It is easy to see that $\leq$ is a partial order relation over L and satisfies the following properties.

Proposition 2.2 (see [21]). *Let L be an incline and $a, b, c \in L$. Then,*

(1) $0 \leq a \leq 1$;

(2) *if $a \leq b$, then $a + c \leq b + c$, $ac \leq bc$, $ca \leq cb$;*

(3) $a \leq a + b$, *and $a + b$ is the least upper bound of a and b;*

(4) $ab \leq a$, $ab \leq b$. *In other words, ab is a lower bound of a and b;*

(5) $acb \leq ab$;

(6) $a + b = 0$ *if and only if $a = b = 0$;*

(7) $ab = 1$ *if and only if $a = b = 1$.*

Boolean algebra $(\{0, 1\}, \vee, \wedge)$, fuzzy algebra $([0, 1], \vee, T)$ (T is a t-norm) and distributive lattice are inclines. Let $(L, \leq)$ be a poset and $a, b \in L$. If $a \leq b$ or $b \leq a$, then a and b are called comparable. Otherwise, a and b are called incomparable, in notation, $a \parallel b$. If for any $a, b \in L$, a and b are comparable, then L is linear and L is called a chain. An unordered poset is a poset in which $a \parallel b$ for all $a \neq b$. A chain B in a poset L is a nonempty subset of L, which,

as a subposet, is a chain. An antichain B in a poset L is nonempty subset which, as a subposet, is unordered. The width of a poset L, denoted by $w(L)$, is n, where n is a natural number, iff there is an antichain in L of n elements and all antichains in L have $\leq n$ elements. A poset $(L, \leq)$ is called an incline if L satisfies Definition 2.1. It is clear that any chain is an incline, which is called a linear incline.

An element a of an incline L is said to be idempotent if $a^2 = a$. The set of all idempotent elements in L is denoted by $I(L)$, that is, $I(L) = \{a \in L \mid a^2 = a\}$.

A matrix is called an incline matrix if its entries belong to an incline. In this paper, the incline $(L, \leq, +, \cdot)$ is always supposed to be a commutative incline with the least and greatest elements 0 and 1, respectively. Let $M_n(L)$ be the set of all $n \times n$ matrices over L. For any A in $M_n(L)$, we will denote by a_{ij} or A_{ij} the element of L which stands in the (i, j)th entry of A. For convenience, we will use N to denote the set $\{1, 2, \ldots, n\}$, and Z_+ denotes the set of all positive integers.

For any A, B, C in $M_n(L)$ and a in L, we define

$A + B = C$ iff $c_{ij} = a_{ij} + b_{ij}$ for all i, j in N;

$AB = C$ iff $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ for all i, j in N;

$A^T = C$ iff $c_{ij} = a_{ji}$ for all i, j in N;

$aA = C$ iff $c_{ij} = a a_{ij}$ for all i, j in N;

$A \leq B$ iff $a_{ij} \leq b_{ij}$ for all i, j in N and $A \geq B$ iff $B \leq A$;

$I_n = (t_{ij})$, where

$$t_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad \text{for } i, j \in N. \tag{2.1}$$

For any A in $M_n(L)$, the powers of A are defined as follows:
$A^0 = I_n$, $A^l = A^{l-1} A$, $l \in Z_+$.
The (i, j)th entry of A^l is denoted by $a_{ij}^l (l \in Z_+)$, and obviously

$$a_{ij}^l = \sum_{1 \leq i_1, i_2, \ldots, i_{l-1} \leq n} a_{i i_1} a_{i_1 i_2} \cdots a_{i_{l-1} j}. \tag{2.2}$$

The following properties will be used in this paper.

(1) $M_n(L)$ is a semigroup with the identity element I_n with respect to the multiplication;

(2) $(M_n(L), +, \cdot)$ is a semiring.

If $A^2 \leq A$, then A is called transitive; if $A^2 = A$, then A is called idempotent; if $A^T = A$, then A is called symmetric; if $A \leq A^2$, then A is called increasing; if $A \geq I_n$, then A is called reflexive; if $a_{ii} = 0$ for all $i \in N$, then A is called irreflexive; if $A^m = 0$ ($m \in Z_+$), then A is called nilpotent; if $a_{ij} = 0$ for $i, j = 1, 2, \ldots, n$, then A is called the zero matrix and denoted by 0_n; A is called a permutation matrix if exactly one of the elements of its every row and every column is 1 and the others are 0.

Let $B \in M_n(L)$. The matrix B is called the transitive closure of A if B is transitive and $A \leq B$, and, for any transitive matrix C in $M_n(L)$ satisfying $A \leq C$, we have $B \leq C$. The transitive closure of A is denoted by A^+. It is clear that if A has a transitive closure, then it is unique.

For any $A \in M_n(L)$ with index, the sequence

$$A, A^2, A^3, \ldots, A^l, \ldots \tag{2.3}$$

is of the form

$$A, A^2, \ldots, A^{k-1} \mid A^k, \ldots, A^{k+d-1} \mid A^k, \ldots, A^{k+d-1} \mid \cdots, \tag{2.4}$$

where $k = k(A)$ is the least integer such that $A^k = A^{k+d}$ for some $d > 0$. The least integers $k = k(A)$, $d = d(A)$ are called the index and the period of A, respectively.

The following definition will be used in this paper.

Definition 2.3. A matrix $A = (a_{ij}) \in M_n(L)$ is said to be

(1) row diagonally dominant if $a_{ij} = a_{ii}a_{ij}$ for all $i, j \in N$;

(2) column diagonally dominant if $a_{ij} = a_{ij}a_{jj}$ for all $i, j \in N$;

(3) weakly diagonally dominant if for any $i \in N$, either $a_{ii}a_{ij} = a_{ij}$ for all $j \in N$ or $a_{ii}a_{ji} = a_{ji}$ for all $j \in N$;

(4) strongly diagonally dominant if $a_{ij} = a_{ii}a_{ij} = a_{ij}a_{jj}$ for all $i, j \in N$;

(5) nearly irreflexive if $a_{ii}a_{ij} = a_{ii}$ for all $i, j \in N$.

Lemma 2.4. $I(L)$ *is a distributive lattice, where* $I(L) = \{a \in L \mid a^2 = a\}$.

The proof can be seen in [1].

3. Transitive Closure of an Incline Matrix

In this section, some properties of the transitive closure of an incline matrix are given and an algorithm for computing the transitive closure of an incline matrix is posed.

Lemma 3.1. *For any A in $M_n(L)$, we have* $A^+ = \sum_{k=1}^{n} A^k$.

The proof can be seen in [21].

Lemma 3.2. *Let $A \in M_n(L)$. Then,*

(1) *for any $i, j \in N$ with $i \neq j$ and any $k \geq n$, there exists s ($s \in \{1, 2, \ldots, n-1\}$) such that* $a_{ij}^s \geq a_{ij}^k$;

(2) *for any $i \in N$ and any $k \geq n$, there exists $t \in N$ such that $a_{ii}^t \geq a_{ii}^k$.*

Proof. (1) Let $T = a_{ii_1}a_{i_1i_2}\cdots a_{i_{k-1}j}$ be any term of a_{ij}^k, where $k \geq n$ and $1 \leq i, i_1, i_2, \ldots, i_{k-1}, j \leq n$. Since the number of indices in T is greater than n, a repetition among them must occur. Let us call the sequence of entries between two occurrences of one index a cycle. If we drop

the cycle, a new expression T_1 with $m_1 \leq k$ entries is obtained. If $m_1 \geq n$, there must be a cycle in T_1, then we delete the cycle and obtain a new expression T_2 with $m_2 \leq m_1$ entries. The deleting method can be applied repeatedly until the new expression T_s contains $m_s < n$ entries. According to properties of the operation "$\cdot$", $T \leq T_1 \leq \cdots \leq T_s$, but T_s is a term of the (i,j)th entry a_{ij}^s of A^s for some $s < n$, we have $T \leq T_1 \leq \cdots \leq T_s \leq a_{ij}^s$, and so $a_{ij}^k \leq a_{ij}^s$. This completes the proof.

(2) Let $T = a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} i}$ be any term of a_{ii}^k, where $k \geq n+1$ and $1 \leq i, i_1, i_2, \ldots, i_{k-1} \leq n$. Since the number of indices in T is greater than n, there must be two indices i_u and i_v such that $i_u = i_v$ for some u, v ($u < v$). Then, we delete $a_{i_u i_{u+1}} \cdots a_{i_{v-1} i_v}$ from T and obtain a new expression $T_1 = a_{ii_1} a_{i_1 i_2} \cdots a_{i_{u-1} i_u} a_{i_v i_{v+1}} \cdots a_{i_{k-1} i}$ with $m_1 \leq k$ entries. If $m_1 \geq n + 1$, there are still two identical numbers in the subscripts $i, i_1, \ldots, i_{u-1}, i_u, i_{v+1}, \ldots, i_{k-1}$, then we apply the deleting method used in the above. The method can be applied repeatedly until the subscripts left are pairwise different. Finally, we can get a new term T_t with $m_t \leq n$ entries. According to properties of the operation "$\cdot$", $T \leq T_1 \leq \cdots \leq T_t$, but T_t is a term of the (i,i)th entry a_{ii}^t of A^t for some $t \leq n$, we have $T \leq T_1 \leq \cdots \leq T_t \leq a_{ii}^t$, and so $a_{ii}^k \leq a_{ii}^t$. This completes the proof. □

Lemma 3.3. *Let $A \in M_n(L)$. Then,*

(1) $\sum_{l=1}^{n} A^l \geq A^k$ *for any $k \geq 1$;*
(2) $\sum_{l=1}^{n} A^l = \sum_{l=1}^{m} A^l$ *for any $m \geq n$.*

Proof. From Lemma 3.2, the proof is obvious. □

Proposition 3.4. *If A^+ is reflexive, then $(A^+)^+ = A^+$.*

Proof. Since $A^+ \geq I_n$, we have $A^+ \leq (A^+)^2$. On the other hand, by Lemmas 3.1 and 3.3, we see that $(A^+)^2 \leq A^+$. Hence, $(A^+)^+ = A^+$. □

Proposition 3.5. *For any $A = (a_{ij}) \in M_n(L)$ with index, if A is column (or row) diagonally dominant, then $A^+ = A^s$, where A^s is transitive.*

Proof. We only consider the case A is column diagonally dominant.

For any integer $l > 0$, we have $A^{ls} \leq A^s$ since A^s is transitive, and so $a_{ij}^{ls} \leq a_{ij}^s$, for all $i, j \in N$. Since A is column diagonally dominant, we have $a_{ij}^l = a_{ij}^l a_{jj} a_{jj} \cdots a_{jj} \leq a_{ij}^{ls}$ (because $a_{ij}^l a_{jj} a_{jj} \cdots a_{jj}$ is the sum of some term in a_{ij}^{ls}) $\leq a_{ij}^s$. Hence, $\sum_{l=1}^{n} a_{ij}^l \leq a_{ij}^s$, then $A^+ \leq A^s$. On the other hand, since $A^+ = \sum_{l=1}^{n} A^l = \sum_{l=1}^{m} A^l$ for any $m \geq n$, we have $A^+ \geq A^s$. Therefore, $A^+ = A^s$. □

Corollary 3.6. *For any $A = (a_{ij}) \in M_n(L)$ with index, if A is strongly diagonally dominant, then $A^+ = A^s$, where A^s is transitive.*

Proof. Obviously, any strongly diagonally dominant matrix is column (or row) diagonally dominant. Hence, the conclusion follows from Proposition 3.5. □

Proposition 3.7. *For any $A = (a_{ij}) \in M_n(L)$ with index, if A is weakly diagonally dominant, then $A^+ = A^s$, where A^s is transitive.*

Proof. Since A^s is transitive, we have $A^s \geq A^{ls}$, and so $a_{ij}^s \geq a_{ij}^{ls}$, for any $i, j \in N$ and any integer $l > 0$. Since A is weakly diagonally dominant, we have for any $i \in N$, either $a_{ii} a_{ij} = a_{ij}$ for all $j \in N$ or $a_{ii} a_{ji} = a_{ji}$ for all $j \in N$.

Let

$$a_{ij}^{l} = \sum_{1 \leq i_1,\dots,i_{l-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{l-1} j} \quad (\forall i, j \in N). \tag{3.1}$$

Case 1. If $a_{i_1 i_1} a_{i_1 j} = a_{i_1 j}$ for all $j \in N$, we have $a_{i_1 i_1} a_{i_1 i_2} = a_{i_1 i_2}$. Then,

$$
\begin{aligned}
a_{ij}^{l} &= \sum_{1 \leq i_1,\dots,i_{l-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{l-1} j} \\[4pt]
&= \sum_{1 \leq i_1,\dots,i_{l-1} \leq n} a_{ii_1} a_{i_1 i_1} \cdots a_{i_1 i_1} a_{i_1 i_2} \cdots a_{i_{l-1} j} \\[4pt]
&\leq a_{ij}^{ls} \left(\text{since} \sum_{1 \leq i_1,\dots,i_{l-1} \leq n} a_{ii_1} a_{i_1 i_1} \cdots a_{i_1 i_1} a_{i_1 i_2} \cdots a_{i_{l-1} j} \text{ is the sum of some term in } a_{ij}^{ls} \right) \\[4pt]
&\leq a_{ij}^{s}.
\end{aligned}
\tag{3.2}
$$

Case 2. If $a_{j i_1} a_{i_1 i_1} = a_{j i_1}$ for all $j \in N$, we have $a_{ii_1} a_{i_1 i_1} = a_{ii_1}$. Then,

$$
\begin{aligned}
a_{ij}^{l} &= \sum_{1 \leq i_1,\dots,i_{l-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{l-1} j} \\[4pt]
&= \sum_{1 \leq i_1,\dots,i_{l-1} \leq n} a_{ii_1} a_{i_1 i_1} \cdots a_{i_1 i_1} a_{i_1 i_2} \cdots a_{i_{l-1} j} \\[4pt]
&\leq a_{ij}^{ls} \left(\text{since} \sum_{1 \leq i_1,\dots,i_{l-1} \leq n} a_{ii_1} a_{i_1 i_1} \cdots a_{i_1 i_1} a_{i_1 i_2} \cdots a_{i_{l-1} j} \text{ is the sum of some term in } a_{ij}^{ls} \right) \\[4pt]
&\leq a_{ij}^{s}.
\end{aligned}
\tag{3.3}
$$

From above, we see that $a_{ij}^{l} \leq a_{ij}^{ls} \leq a_{ij}^{s}$ for any $i, j \in N$. Hence, $\sum_{l=1}^{n} a_{ij}^{l} \leq a_{ij}^{s}$, then $A^{+} \leq A^{s}$. On the other hand, since $A^{+} = \sum_{l=1}^{n} A^{l} = \sum_{l=1}^{+\infty} A$, we have $A^{+} \geq A^{s}$. Therefore, $A^{+} = A^{s}$. $\qquad \square$

Proposition 3.8. *Let $A = (a_{ij}) \in M_n(L)$. If the entries of A satisfy $a_{ii} a_{jk} = a_{jk}$ $(1 \leq i, j, k \leq n)$, then*

(1) $A^2 \leq A^3$;

(2) $a_{ii} = a_{ii}^2 = \cdots = a_{ii}^n$ for all $i \in N$;

(3) A converges to $A^{k(A)}$ with $k(A) \leq n - 1$.

Proof. (1) Since any term T of the (i,j)th entry a_{ij}^2 of A^2 is of the form $a_{ii_1}a_{i_1j}$, we have $a_{ij}^2 = \sum_{k=1}^n a_{ik}a_{kj}$ $(1 \le i,j,k \le n)$. Because the hypothesis $a_{ii}a_{jk} = a_{jk}$, we can get $a_{ij}^2 = \sum_{k=1}^n a_{ik}a_{kk}a_{kj}$. Since $a_{ik}a_{kk}a_{kj}$ is a term of a_{ij}^3, we have $a_{ij}^2 = \sum_{k=1}^n a_{ik}a_{kk}a_{kj} \le a_{ij}^3$.

Thus, $A^2 \le A^3$.

(2) Because the hypothesis $a_{ii}a_{jk} = a_{jk}$, we have $a_{ii}a_{ii}\cdots a_{ii} = a_{ii}$. Hence,

$$a_{ii} \le a_{ii}^s = \sum_{1 \le i_1,\ldots,i_{s-1} \le n} a_{ii_1}a_{i_1i_2}\cdots a_{i_{s-1}i} \le \sum_{k \in N} a_{ik} = \sum_{k \in N} a_{ii}a_{ik} \le \sum_{k \in N} a_{ii} = a_{ii}, \qquad (3.4)$$

then $a_{ii} = a_{ii}^s$ (for all $s \in N$).

(3) Since $A^2 \le A^3$, we have $A^{k-1} \le A^k$ (for any integer $k \ge 3$), and so $A^{n-1} \le A^n$. Now we prove that $A^{n-1} \ge A^n$. By (2), it is sufficient only to show that $a_{ij}^{n-1} \ge a_{ij}^n$ for $i \ne j$. Let

$$a_{ij}^n = \sum_{1 \le i_1,\ldots,i_{n-1} \le n} a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j} \qquad (\forall\, i,j \in N). \qquad (3.5)$$

Since the number of indices in $a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j}$ is $n+1$, there must be two indices i_u and i_v such that $i_u = i_v$ for some $u,v(u < v)$. Then, $a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j} \le a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j}$. Since $a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j}$ is a term of $a_{ij}^{n-(v-u)}$, we have $a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j} \le a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \le a_{ij}^{n-(v-u)}$ and so $a_{ij}^n \le a_{ij}^{n-(v-u)}$, that is, $A^n \le A^m(m = n-(v-u) < n) \le A^{n-1}$ (because $A^{k-1} \le A^k$ for any integer $k \ge 3$). Therefore, $A^{n-1} = A^n$. This proves the proposition. $\qquad \square$

Lemma 3.9. *Let $A = (a_{ij}) \in M_n(L)$. If the entries of A satisfy $a_{ii}a_{jk} = a_{jk}$ $(1 \le i,j,k \le n)$, then $adj(A) = A^{n-1}$.*

The proof can be seen in [22].

Proposition 3.10. *Let $A = (a_{ij}) \in M_n(L)$. If the entries of A satisfy $a_{ii}a_{jk} = a_{jk}$ $(1 \le i,j,k \le n)$, then $(adj\,A)^+ = A^{n-1}$.*

Proof. By the Proposition 3.8, we have $A^{n-1} = A^n$. By Lemma 3.9, we can get $adj(A) = A^{n-1}$. Thus, $(adj\,A)^+ = (A^{n-1})^+ = A^{n-1}$. $\qquad \square$

Corollary 3.11. *Let $A = (a_{ij}) \in M_n(L)$. If A is reflexive, then*

(1) $A \le A^2$;

(2) $a_{ii} = a_{ii}^2 = a_{ii}^3 = \cdots = a_{ii}^n = 1$ for all $i \in N$;

(3) A converges to $A^{k(A)}$ with $k(A) \le n-1$;

(4) $(adj\,A)^+ = A^{n-1}$;

(5) $((adj\,A)^+)^+ = A^{n-1}$.

Proof. From Propositions 3.10 and 3.8, the proof is obvious. $\qquad \square$

Proposition 3.12. *Let $A = (a_{ij}) \in M_n(L)$. If L has no nilpotent elements and A is nilpotent, then*

(1) $(adj A)^2 = 0$;

(2) $(adj A)^+ = adj A$.

Proof. (1) The proof can be seen in [23].

(2) Form (1), the proof is obvious. □

Proposition 3.13. *Let $A = (a_{ij}) \in M_n(L)$, $A^+ = (t_{ij})_{n \times n}$, $(I_n + A)^+ = (\tilde{t}_{ij})_{n \times n}$, then*

(1) $(I_n + A)^+ = (I_n + A)^n = I_n + \sum_{l=1}^{n} A^l$;

(2) $(I_n + A)^+ = (I_n + A)^{n-1} = I_n + \sum_{l=1}^{n-1} A^l$;

(3) $t_{ij} = \sum_{l=1}^{n} a_{ij}^l = \tilde{t}_{ij}$ *(for all $i, j \in N$ and $i \neq j$).*

Proof. (1) Since $(I_n + A)$ is reflexive, we have $(I_n + A)$ is increasing. By Lemma 3.1, we can get the conclusion.

(2) By Corollary 3.11, $(I_n + A)$ converges to $(I_n + A)^{k(I_n+A)}$ with $k(I_n + A) \leq n - 1$. Thus, the conclusion is obtained.

(3) From (1) and (2), the proof is obvious. □

At the end of this section, we may establish the following algorithm to find the $(A^+)_{ij}$ $(i \neq j)$.

Let $A \in M_n(L)$, $A^+ = (t_{ij})_{n \times n}$, $(I_n + A)^+ = (\tilde{t}_{ij})_{n \times n}$, and $B = I_n + A$.

Step 1. Compute successively

$$B^2, B^{2^2}, \ldots, B^{2^p}, \ldots, \quad (p \in Z_+). \tag{3.6}$$

In (3.6), find $p \leq k - 1$ such that $B^{2^{p-1}} \neq B^{2^p}$, but $B^{2^p} = B^{2^{p+1}}$. Then, $(I_n + A)^+ = B^{2^p}$. Go to Step 2.

Step 2. Find $(A^+)_{ij}$ $(i \neq j)$.

Let $t_{ij} = \tilde{t}_{ij}$ $(i \neq j)$. Stop.

4. Convergence of Powers of Transitive Incline Matrices

In this section, the convergence of powers of transitive incline matrices in $M_n(L)$ will be discussed.

Definition 4.1. Let $A = (a_{ij}) \in M_n(L)$. For any $i, j \in N$, if $a_{ij}a_{ik} \neq a_{ik} \Rightarrow a_{ij}a_{kj} = a_{kj}$ holds for all $k \in N$, then A is called a strongly transitive matrix.

Theorem 4.2. *If $A = (a_{ij}) \in M_n(L)$ is a strongly transitive matrix, then A is transitive.*

Proof. Since A is strongly transitive, for any $i, j \in N$, we have $a_{ij}a_{ik} = a_{ik}$ for all $k \in N$ or $a_{ij}a_{ik} \neq a_{ik}$ and $a_{ij}a_{kj} = a_{kj}$ for all $k \in N$.

Case 1. For any $i, j \in N$, suppose $a_{ij}a_{ik} = a_{ik}$ for all $k \in N$, then we can get $a_{ij} \geq a_{ij}a_{ik} = a_{ik} \geq a_{ik}a_{kj}$, so $a_{ij} \geq \sum_{k=1}^{n} a_{ik}a_{kj} = a_{ij}^2$, which means that A is transitive.
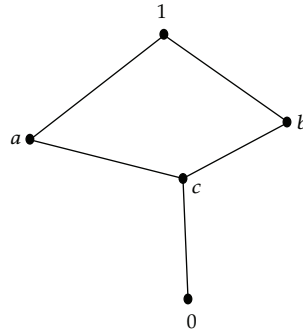
Figure 1

Case 2. For any $i, j \in N$, suppose $a_{ij}a_{ik} \neq a_{ik}$ and $a_{ij}a_{kj} = a_{kj}$ for all $k \in N$, then we can get $a_{ij} \geq a_{ij}a_{kj} = a_{kj} \geq a_{ik}a_{kj}$, so $a_{ij} \geq \sum_{k=1}^{n} a_{ik}a_{kj} = a_{ij}^2$, which means that $A \geq A^2$. This completes the proof.

$\square$

Remark 4.3. *If A is transitive, but A is not necessary to be a strongly transitive matrix.*

Example 4.4. Consider the cline $L = \{0, a, b, c, 1\}$ whose diagram is as Figure 1.
Let now

$$A = \begin{pmatrix} c & a \\ b & c \end{pmatrix}. \tag{4.1}$$

Then, $A^2 \leq A$ which means that A is transitive. But $a_{11}a_{12} = ca \neq a_{12} = a$ (because $ca < a$) $\nRightarrow a_{11}a_{21} = cb = a_{21} = b$ (because $cb < b$), which means that A is not a strongly transitive matrix.

Theorem 4.5. *Let $A = (a_{ij}) \in M_n(L)$. If $A \geq A^2$ and $a_{ii} \in I(L)$ (for all $i \in N$), then*

(1) $a_{ii} = a_{ii}^2 = \cdots = a_{ii}^n$ *for all $i \in N$;*

(2) *A converges to $A^{k(A)}$ with $k(A) \leq n$.*

Proof. (1) By the hypothesis $A \geq A^2$, it follows that $A^k \geq A^{k+1}$ (for all $k \in N$). Hence, $A^n \geq A^{n+1}$ and $a_{ii} \geq a_{ii}^s$ (for all $s \in N$). Since $a_{ii} \in I(L)$, we can get $a_{ii} = a_{ii} \cdots a_{ii} \leq a_{ii}^s$ (for all $s \in N$). Thus $a_{ii} = a_{ii}^s$ and $a_{ii}^s \in I(L)$.

(2) By (1), it is sufficient to verify $A^n \leq A^{n+1}$.

Now, any term T of the (i, j)th entry a_{ij}^n of A^n is of the form $a_{ii_1}a_{i_1i_2} \cdots a_{i_{n-1}j}$, where $1 \leq i_1, i_2, \ldots, i_{n-1} \leq n$. Since the number of indices in T is greater than n, there must be two indices i_u and i_v such that $i_u = i_v$ for some u, v ($0 \leq u < v \leq n$, $i_0 = i$, $i_n = j$). Then, $a_{ii_1}a_{i_1i_2} \cdots a_{i_{n-1}j} = a_{ii_1}a_{i_1i_2} \cdots a_{i_{u-1}i_u}a_{i_ui_{u+1}} \cdots a_{i_{v-1}i_v}a_{i_vi_{v+1}} \cdots a_{i_{n-1}j} \leq a_{ii_1}a_{i_1i_2} \cdots a_{i_{u-1}i_u}a_{i_ui_u}^{v-u}a_{i_vi_{v+1}} \cdots a_{i_{n-1}j}$ (because $a_{i_ui_{u+1}} \cdots a_{i_{v-1}i_v}$ is a term of $a_{i_ui_u}^{v-u}$) $= a_{ii_1}a_{i_1i_2} \cdots a_{i_{u-1}i_u}a_{i_ui_u}^{v-u}a_{i_ui_u}^{v-u}a_{i_vi_{v+1}} \cdots a_{i_{n-1}j}$ (because (1) $a_{ii} = a_{ii}^s$ (for all $i, s \in N$) and $a_{ii} \in I(L)$) $\leq a_{ij}^{n+(v-u)}$ ($v - u \geq 1$) (because $a_{ii_1}a_{i_1i_2} \cdots a_{i_{u-1}i_u}a_{i_ui_u}^{v-u}a_{i_ui_u}^{v-u}a_{i_vi_{v+1}} \cdots a_{i_{n-1}j}$ is the sum of some term in $a_{ij}^{n+(v-u)}$) $\leq a_{ij}^{n+1}$ (because

$A^k \geq A^{k+1}$ for any $k \in N$). Thus, $T \leq a_{ij}^{n+1}$, and so $a_{ij}^n \leq a_{ij}^{n+1}$. Therefore, $A^n \leq A^{n+1}$. This completes the proof. $\qquad\square$

Corollary 4.6. *If $A = (a_{ij}) \in M_n(L)$ is strongly transitive, then A converges to $A^{k(A)}$ with $k(A) \leq n$.*

Proof. Since A is strongly transitive, by Theorem 4.2, we have $A \geq A^2$ and $a_{ii} \in I(L)$ (for all $i \in N$), the conclusion is obtained. $\qquad\square$

Theorem 4.7. *Let $A = (a_{ij}) \in M_n(L)$ be transitive. If $B = (b_{ij}) \in M_n(I(L))$ and $\mathrm{diag}(A) \leq B \leq A$, where $\mathrm{diag}(A) = (c_{ij})$ with $c_{ii} = a_{ii}$ (for all $i \in N$) and $c_{ij} = 0$ ($i \neq j, i, j \in N$). Then,*

(1) *B converges to $B^{k(B)}$ with $k(B) \leq n$;*

(2) *if A satisfies $a_{kk} \geq \sum_{i=1}^n a_{ki}$ (or $a_{kk} \geq \sum_{i=1}^n a_{ik}$) for some k in N, then B converges to $B^{k(B)}$ with $k(B) \leq n - 1$;*

(3) *if B satisfies $b_{kk} \geq \sum_{i=1}^n b_{ki}$ (or $b_{kk} \geq \sum_{i=1}^n b_{ik}$) for some k in N, then B converges to $B^{k(B)}$ with $k(B) \leq n - 1$.*

Proof. (1) Firstly, $b_{ii} = a_{ii}$ (for all $i \in N$) since $\mathrm{diag}(A) \leq B \leq A$. Since $b_{ii} \in I(L)$, we have $a_{ii} \in I(L)$. By Theorem 4.5, we can get $a_{ii} = a_{ii}^2 = \cdots = a_{ii}^n$ for all $i \in N$.

It follows that any term T of the (i, j)th entry b_{ij}^n of B^n is of the form $b_{ii_1}b_{i_1i_2}\cdots b_{i_{n-1}j}$, where $1 \leq i_1, i_2, \ldots, i_{n-1} \leq n$. Since $\{i, i_1, i_2, \ldots, i_{n-1}, j\} \subset \{1, 2, \ldots, n\}$ and $n + 1 > n$, there are u, v such that $i_u = i_v$ ($0 \leq u < v \leq n$, $i_0 = i, i_n = j$). Then, $T \leq b_{i_ui_{u+1}}\cdots b_{i_{v-1}i_v}$ and $T \leq b_{ii_1}b_{i_1i_2}\cdots b_{i_{u-1}i_u}b_{i_vi_{v+1}}\cdots b_{i_{n-1}j}$. Since A is transitive, we have $A \geq A^k$ for all $k \geq 1$, and so $a_{ij} \geq a_{ij}^k$ for all $i, j \in N$. Thus, $b_{i_ui_u}^{v-u-1} \geq b_{i_ui_u}$ (because $B \in I(L)$) $= a_{i_ui_u} = a_{i_ui_u}^{v-u} \geq b_{i_ui_u}^{v-u}$ (because $A \geq B$) $\geq b_{i_ui_{u+1}}\cdots b_{i_{v-1}i_v} \geq T$. Since $b_{ii_1}b_{i_1i_2}\cdots b_{i_{u-1}i_u}b_{i_ui_u}^{v-u-1}b_{i_vi_{v+1}}\cdots b_{i_{n-1}j}$ is the sum of some term in b_{ij}^{n-1}, we have $b_{ij}^{n-1} \geq b_{ii_1}b_{i_1i_2}\cdots b_{i_{u-1}i_u}b_{i_ui_u}^{v-u-1}b_{i_vi_{v+1}}\cdots b_{i_{n-1}j} \geq T$ (because $B \in I_n(L)$), and so $b_{ij}^{n-1} \geq b_{ij}^n$ (because T is any term of b_{ij}^n). Thus, $B^{n-1} \geq B^n$.

Certainly, $B^n \geq B^{n+1} \geq B^{n+2} \geq \cdots$.

On the other hand, for any term $T = b_{ii_1}\cdots b_{i_{n-1}j}$ of b_{ij}^n, there must be two indices i_u and i_v such that $i_u = i_v$ for some u, v ($0 \leq u < v \leq n$, $i_0 = i$, $i_n = j$). Then, $T = b_{ii_1}b_{i_1i_2}\cdots b_{i_{n-1}j} = b_{ii_1}b_{i_1i_2}\cdots b_{i_{u-1}i_u}b_{i_ui_{u+1}}\cdots b_{i_{v-1}i_v}b_{i_vi_{v+1}}\cdots b_{i_{n-1}j} \leq b_{ii_1}b_{i_1i_2}\cdots b_{i_{u-1}i_u}b_{i_ui_u}^{v-u}b_{i_vi_{v+1}}\cdots b_{i_{n-1}j}$ (because $b_{i_ui_{u+1}}\cdots b_{i_{v-1}i_v}$ is a term of $b_{i_ui_u}^{v-u}$) $= b_{ii_1}b_{i_1i_2}\cdots b_{i_{u-1}i_u}b_{i_ui_u}^{v-u}b_{i_ui_u}^{v-u}b_{i_vi_{v+1}}\cdots b_{i_{n-1}j}$ (because $B \in M_n(I(L))$) $\leq b_{ij}^{n+(v-u)}$ (because $b_{ii_1}b_{i_1i_2}\cdots b_{i_{u-1}i_u}b_{i_ui_u}^{v-u}b_{i_ui_u}^{v-u}b_{i_vi_{v+1}}\cdots b_{i_{n-1}j}$ is the sum of some term in $b_{ij}^{n+(v-u)}$; $v - u \geq 1$) $\leq b_{ij}^{n+1}$ (because $B^k \geq B^{k+1}$ for any $k \geq n - 1$). Thus, $T \leq b_{ij}^{n+1}$, and so $b_{ij}^n \leq b_{ij}^{n+1}$. Therefore, $B^n \leq B^{n+1}$.

Consequently, we have $B^n = B^{n+1}$. This completes the proof.

(2) By the proof of (1), we have $B^n \leq B^{n-1}$. Hence, $B^{n-1} \geq B^n \geq B^{n+1} \geq B^{n+2} \geq \cdots$. In the following, we will show that $B^{n-1} \leq B^n$. It is clear that any term T of the (i, j)th entry b_{ij}^{n-1} of B^{n-1} is of the form $b_{ii_1}b_{i_1i_2}\cdots b_{i_{n-2}j}$, where $1 \leq i, i_i, i_2, \ldots, i_{n-2}, j \leq n$. Let $i_0 = i$ and $i_{n-1} = j$.

Case 1. If $i_u = i_v$ for some u and v ($u < v, v - u \geq 1$), then $T = b_{ii_1}b_{i_1i_2}\cdots b_{i_{n-2}j} = b_{ii_1}b_{i_1i_2}\cdots b_{i_{u-1}i_u}b_{i_ui_{u+1}}\cdots b_{i_{v-1}i_v}b_{i_vi_{v+1}}\cdots b_{i_{n-2}j} \leq b_{ii_1}b_{i_1i_2}\cdots b_{i_{u-1}i_u}b_{i_ui_u}^{v-u}b_{i_vi_{v+1}}\cdots b_{i_{n-2}j}$ (because $b_{i_ui_{u+1}}\cdots b_{i_{v-1}i_v}$ is a term of $b_{i_ui_u}^{v-u}$) $= b_{ii_1}b_{i_1i_2}\cdots b_{i_{u-1}i_u}b_{i_ui_u}^{v-u}b_{i_ui_u}^{v-u}b_{i_vi_{v+1}}\cdots b_{i_{n-2}j}$ (because $B \in M_n(I(L))$) $\leq b_{ij}^{n-1+(v-u)}$ (because $b_{ii_1}b_{i_1i_2}\cdots b_{i_{u-1}i_u}b_{i_ui_u}^{v-u}b_{i_ui_u}^{v-u}b_{i_vi_{v+1}}\cdots b_{i_{n-2}j}$ is the sum of some

term in $b_{ij}^{n-1+(v-u)}) \leq b_{ij}^n$ (because $B^{n-1} \geq B^n \geq B^{n+1} \geq \cdots$). Thus, $T \leq b_{ij}^n$, and so $b_{ij}^{n-1} \leq b_{ij}^n$. Therefore, $B^{n-1} \leq B^n$.

Case 2. Suppose that $i_u \neq i_v$ for all $u \neq v$. By the hypothesis $a_{kk} \geq \sum_{i=1}^n a_{ki}$ for some k, we have $b_{i_u i_u} = a_{i_u i_u} \geq \sum_{i=1}^n a_{i_u i} \geq \sum_{i=1}^n b_{i_u i}$ (because $A \geq B$) $\geq b_{i_u i_{u+1}}$. Then, $T = b_{i i_1} b_{i_1 i_2} \cdots b_{i_{n-2} j} = b_{i i_1} b_{i_1 i_2} \cdots b_{i_{u-1} i_u} b_{i_u i_{u+1}} b_{i_{u+1} i_{u+2}} \cdots b_{i_{n-2} j} = b_{i i_1} b_{i_1 i_2} \cdots b_{i_{u-1} i_u} b_{i_u i_u} b_{i_u i_{u+1}} b_{i_{u+1} i_{u+2}} \cdots b_{i_{n-2} j}$ (because $B \in M_n(I(L))$) $\leq b_{i i_1} b_{i_1 i_2} \cdots b_{i_{u-1} i_u} b_{i_u i_u} b_{i_u i_u} b_{i_{u+1} i_{u+2}} \cdots b_{i_{n-2} j}$ (because $b_{i_u i_u} \geq b_{i_u i_{u+1}}$) $\leq b_{ij}^n$ (because $b_{i i_1} b_{i_1 i_2} \cdots b_{i_{u-1} i_u} b_{i_u i_u} b_{i_u i_u} b_{i_{u+1} i_{u+2}} \cdots b_{i_{n-2} j}$ is a term of b_{ij}^n). Thus, $T \leq b_{ij}^n$, and so $b_{ij}^{n-1} \leq b_{ij}^n$. Therefore, $B^{n-1} \leq B^n$.

The case of $a_{kk} \geq \sum_{i=1}^n a_{ik}$ is similar to that of the hypothesis $a_{kk} \geq \sum_{i=1}^n a_{ki}$. Consequently, we have $B^{n-1} = B^n$.

(3) The proof of (3) is similar to that of (2). This completes the proof. $\qquad\square$

Theorem 4.7 generalizes and develops Theorem 4.1 of Tan [17].

5. Compositions of Transitive Matrices

In this section, we construct a transitive matrix from given incline matrices and construct a new incline matrix with some special properties from transitive matrices. We know that any idempotent matrix is also a transitive matrix.

Theorem 5.1. *Let $A = (a_{ij}) \in M_n(L)$. If A is transitive and row diagonally dominant, then $B = (b_{ij})$ is idempotent, where $b_{ij} = a_{ij} a_{ji}$.*

Proof. For any i, j in N, $b_{ij}^2 = \sum_{k=1}^n b_{ik} b_{kj} = \sum_{k=1}^n a_{ik} a_{ki} a_{kj} a_{jk} = \sum_{k=1}^n (a_{ik} a_{kj})(a_{jk} a_{ki}) \leq \sum_{k=1}^n (a_{ij} a_{ji})$ (because $A \geq A^2$) $= \sum_{k=1}^n b_{ij} = b_{ij}$. Thus, $B^2 \leq B$. On the other hand, since $b_{ij}^2 = \sum_{k=1}^n b_{ik} b_{kj} \geq b_{ii} b_{ij} = a_{ii} a_{ii} a_{ij} a_{ji}$ (because A is row diagonally dominant) $= a_{ij} a_{ji}$ $= b_{ij}$. Thus, $B^2 \geq B$. Therefore, $B^2 = B$. This completes the proof. $\qquad\square$

Definition 5.2. $A \circ D = C$ iff $c_{ij} = \prod_{k=1}^n (a_{ik} + d_{kj})$ for any $i, j \in N$.

Theorem 5.3. *Let $A = (a_{ij}) \in M_n(L)$ be a symmetric and nearly irreflexive matrix. Then, the matrix $M = C \circ A$ is idempotent, where $C = (c_{ij})$ with $c_{ii} = c_i \in L$, $c_{ij} = 0$ $(i \neq j)$.*

Proof. By the definitions of the matrices $A, C,$ and M, we have

$$
\begin{aligned}
m_{ij} &= \left(\prod_{l \neq i} a_{lj} \right)(c_i + a_{ij}) \\
&= \left(\prod_{l \neq i} a_{jl} \right)(c_i + a_{ij}) \\
&= \begin{cases} a_{ii} + \prod_{l \neq i} a_{li} c_i, & i = j, \\ a_{jj}, & i \neq j. \end{cases}
\end{aligned}
\tag{5.1}
$$

The (i, j)th entry of M^2 is $m_{ij}^2 = \sum_{k=1}^n m_{ik} m_{kj}$.

Case 1 $(i \neq j)$. In this case, $m_{ij}^2 = \sum_{k \neq i,j} m_{ik} m_{kj} + m_{ii} m_{ij} + m_{ij} m_{jj} = \sum_{k \neq i,j} a_{kk} a_{jj} + (a_{ii} + \prod_{l \neq i} a_{li} c_i) a_{jj} + a_{jj}(a_{jj} + \prod_{l \neq j} a_{lj} c_j) = a_{jj}(\sum_{k=1}^{n} a_{kk} + \prod_{l \neq i} a_{li} c_i + \prod_{l \neq j} a_{lj} c_j) = a_{jj}$ (because A is nearly irreflexive) $= m_{ij}$.

Case 2 $(i = j)$. In this case, $m_{ii}^2 = \sum_{l \neq i} m_{il} m_{li} + m_{ii} = \sum_{l \neq i} a_{ll} a_{ii} + a_{ii} + \prod_{k \neq i} a_{ki} c_i = a_{ii} + \prod_{k \neq i} a_{ki} c_i$ (because A is nearly irreflexive) $= m_{ii}$.

Consequently, we can get $M^2 = M$. Therefore, $M = C \circ A$ is idempotent. □

Theorem 5.3 generalizes Theorem 3.5 of Tan [20].

Lemma 5.4. *Let* $A, B \in M_{m \times n}(L)$, $C \in M_{n \times l}(L)$ *and* $D \in M_{p \times m}(L)$. *Then,*

(1) $(B \circ C)^T = C^T \circ B^T$;

(2) *If* $A \leq B$, *then* $D \circ A \leq D \circ B$ *and* $A \circ C \leq B \circ C$.

The proof is trivial.

Lemma 5.5. *Let* $A = (a_{ij}) \in M_{m \times n}(L)$ *be a nearly irreflexive matrix, then* $A \circ A^T$ *is nearly irreflexive and symmetric.*

Proof. Let $S = A \circ A^T$. Then, $s_{ii} = \prod_{l=1}^{m}(a_{il} + a_{il}) = \prod_{l=1}^{m} a_{il} = a_{ii}$ (because A is nearly irreflexive). Thus, $s_{ii} s_{ij} = a_{ii}(\prod_{l=1}^{m}(a_{il} + a_{jl})) = a_{ii} = s_{ii}$, that is, $S = A \circ A^T$ is nearly irreflexive. By Lemma 5.4, we have $(A \circ A^T)^T = (A^T)^T \circ A^T = A \circ A^T$. □

Lemma 5.5 generalizes Theorem 3.3 of Tan [20].

Corollary 5.6. *Let* $A = (a_{ij}) \in M_n(L)$ *be a nearly irreflexive matrix. Then,* $C \circ (A \circ A^T)$ *is idempotent, where* $C = (c_{ij})$ *with* $c_{ii} = c_i \in L$, $c_{ij} = 0$ $(i \neq j)$.

Proof. It follows from Theorem 5.3 and Lemma 5.5. □

Proposition 5.7. *Let* $A = (a_{ij}) \in M_n(L)$ *be a symmetric and nearly irreflexive matrix. Then,*

(1) $A \circ A \leq A$;

(2) $A \circ A$ *is symmetric and nearly irreflexive.*

Proof. (1) Let $R = A \circ A$. Then, $r_{ij} = \prod_{l=1}^{n}(a_{il} + a_{lj}) \leq a_{ii} + a_{ij} = a_{ii} a_{ij} + a_{ij}$ (because A is nearly irreflexive) $= a_{ij}$, so that $A \circ A \leq A$.

(2) Let $R = A \circ A$. Since $r_{ji} = \prod_{l=1}^{n}(a_{jl} + a_{li}) = \prod_{l=1}^{n}(a_{il} + a_{lj}) = r_{ij}$, we have R is symmetric. Since $r_{ii} = \prod_{l=1}^{n}(a_{il} + a_{li}) = \prod_{l=1}^{n} a_{il} = a_{ii}$ (because A is nearly irreflexive), we can get $r_{ii} r_{ij} = a_{ii}(\prod_{l=1}^{n}(a_{il} + a_{lj})) = a_{ii} = r_{ii}$. Thus, R is nearly irreflexive. □

Proposition 5.7 generalizes Proposition 3.1 of Tan [20].

Corollary 5.8. *Let* $A = (a_{ij}) \in M_n(L)$ *be a symmetric and nearly irreflexive matrix. Then,* $C \circ (A \circ A)$ *is idempotent, where* $C = (c_{ij})$ *with* $c_{ii} = c_i \in L$, $c_{ij} = 0$ $(i \neq j)$.

Proof. It follows from Theorem 5.3 and Proposition 5.7. □

Corollary 5.8 generalizes Corollary 3.7 (1) of Tan [20].

Corollary 5.9. *Let $A = (a_{ij}) \in M_n(L)$ be a nearly irreflexive matrix. Then, $(A \circ A^T) \circ C$ is idempotent, where $C = (c_{ij})$ with $c_{ii} = c_i \in L$, $c_{ij} = 0$ $(i \neq j)$.*

The proof is trivial.
Corollary 5.9 generalizes Corollary 3.8 of Tan [20].

Proposition 5.10. *Let $A \in M_n(L)$ be irreflexive and transitive. Then,*

(1) $A \circ A^T = 0, 0 \in M_n(L)$;

(2) $A^T \circ A = 0, 0 \in M_n(L)$.

Proof. (1) Let $R = A \circ A^T$. Then, $r_{ij} = \prod_{l=1}^{n}(a_{il} + a_{jl}) \leq (a_{ii} + a_{ji})(a_{ij} + a_{jj})$ (because A is irreflexive) $= a_{ij}a_{ji} \leq a_{ii}$ (because A is transitive) $= 0$. Therefore, $R = 0$.
The proof of (2) is similar to that of (1). This proves the Proposition. □

Proposition 5.10 generalizes Proposition 3.9 of Tan [20].

Definition 5.11. An incline $\tilde{L}$ is said to be a Brouwerian incline if for any $a, b \in L$, there exists an element $b \rightarrow a \in \tilde{L}$ such that $bx \leq a \Leftrightarrow x \leq b \rightarrow a$.

Obviously, $b \rightarrow a$ is the largest element $x \in \tilde{L}$ satisfying $bx \leq a$.

Definition 5.12. $A \leftarrow D = C$ iff $c_{ij} = \prod_{k=1}^{n}(d_{kj} \rightarrow a_{ik})$ for any $i, j \in N$.

Lemma 5.13. *Let $\tilde{L}$ be a Brouwerian incline. Then, for any $a \in \tilde{L}$,*

(1) $a \rightarrow a = 1$;

(2) $1 \rightarrow a = a$.

The proof is trivial.

Theorem 5.14. *Let $A \in M_{m \times n}(\tilde{L})$. Then, $A \leftarrow A^T$ is reflexive and transitive.*

Proof. Let $R = A \leftarrow A^T$. Then, $r_{ij} = \prod_{k=1}^{n}(a_{jk} \rightarrow a_{ik})$. Obviously, $r_{ii} = \prod_{k=1}^{n}(a_{ik} \rightarrow a_{ik}) = 1$ (by Lemma 5.13 (1)). Thus, R is reflexive. Furthermore, since $a_{jk}(a_{lk} \rightarrow a_{ik})(a_{jk} \rightarrow a_{lk}) = (a_{lk} \rightarrow a_{ik})(a_{jk}(a_{jk} \rightarrow a_{lk}) \leq (a_{lk} \rightarrow a_{ik})a_{lk} \leq a_{ik}$, we have $(a_{lk} \rightarrow a_{ik})(a_{jk} \rightarrow a_{lk}) \leq a_{jk} \rightarrow a_{ik}$. Therefore, $r_{il}r_{lj} = \prod_{k=1}^{n}(a_{lk} \rightarrow a_{ik})(a_{jk} \rightarrow a_{lk}) \leq \prod_{k=1}^{n}(a_{jk} \rightarrow a_{ik}) = r_{ij}$, and so $R^2 \leq R$. This proves the Theorem. □

Theorem 5.14 generalizes Lemma 4.1 of Tan [20].

Theorem 5.15. *Let $A \in M_n(I(\tilde{L}))$. Then, the following conditions are equivalent.*

(1) *A is reflexive and transitive;*

(2) *$A \leftarrow A^T = A$.*

Proof. (1) $\Rightarrow$ (2). Let $R = A \leftarrow A^T$. Then, $r_{ij} = \prod_{k=1}^{n}(a_{jk} \rightarrow a_{ik}) \leq a_{jj} \rightarrow a_{ij} = 1 \rightarrow a_{ij} = a_{ij}$ (by Lemma 5.13(2)), and so $R \leq A$. On the other hand, since $a_{ik} \geq a_{ij}a_{jk}$ for all $i, j, k \in N$, we

have $a_{ij} \leq a_{jk} \rightarrow a_{ik}$, and so $a_{ij} \leq \prod_{k=1}^{n}(a_{jk} \rightarrow a_{ik})$ (because $A \in M_n(I(\widetilde{L}))) = r_{ij}$, that is, $A \leq R$. Consequently, we have $R = A \leftarrow A^T = A$.

(2) $\Rightarrow$ (1). It follows from Theorem 5.14. □

Theorem 5.15 generalizes Proposition 4.2 of Tan [20].

Theorem 5.16. *Let $A \in M_{m \times n}(\widetilde{L})$. Then, $(A \leftarrow A^T)A = A$.*

Proof. Let $R = (A \leftarrow A^T)A$. Then, $r_{ij} = \sum_{k=1}^{m} \prod_{l=1}^{n}(a_{kl} \rightarrow a_{il})a_{kj} \leq \sum_{k=1}^{m} a_{kj}(a_{kj} \rightarrow a_{ij}) \leq \sum_{k=1}^{m} a_{ij} = a_{ij}$, so that $R \leq A$. On the other hand, since $A \leftarrow A^T$ is reflexive (by Theorem 5.14), we have $R = (A \leftarrow A^T)A \geq A$. Therefore, $(A \leftarrow A^T)A = A$. □

Theorem 5.16 generalizes Lemma 4.3 of Tan [20].

6. On Canonical Form of an Incline Matrix

In this section, we will discuss the canonical form of an incline matrix. Let A be an $n \times n$ incline matrix. If there exists an $n \times n$ permutation matrix P such that $F = PAP^T = (f_{ij})$ satisfies $f_{ij} \nleq f_{ji}$ for $i > j$ then, F is called a canonical form of A. The main results obtained here generalize the previous results on canonical form of a lattice matrix (see, e.g., [17]) and a fuzzy matrix (see, e.g., [12]).

Definition 6.1. Let $A = (a_{ij}) \in M_n(L)$. For any $i, j, k \in N$ with $i \neq j$, $i \neq k$, $j \neq k$, if $a_{ik} > a_{ki}$ and $a_{kj} > a_{jk}$, we have $a_{ij} > a_{ji}$. Then, A is called a especially strongly transitive matrix.

Lemma 6.2. *If $A = (a_{ij}) \in M_n(L)$ is especially strongly transitive matrix and A is put in the block forms*

$$A = \begin{pmatrix} a_{11} & \alpha_1 \\ \alpha_2 & A_1 \end{pmatrix} = \begin{pmatrix} A_2 & \beta_1 \\ \beta_2 & a_{nn} \end{pmatrix}, \tag{6.1}$$

where $\alpha_1, \alpha_2^T, \beta_1, \beta_2^T \in L^{1 \times (n-1)}$ and $A_1, A_2 \in M_{n-1}(L)$, then A_1, A_2, and PAP^T are especially strongly transitive matrices for any permutation matrix P.

The proof is similar to that of Lemma 3.1 in [14].

Lemma 6.3. *Let $A = (a_{ij}) \in M_{m \times n}(L)$ and $B = (b_{ij}) \in M_{n \times m}(L)$. If $A \nleq B^T$, then $AP^T \nleq B^T P^T$ holds for any $n \times n$ permutation matrix P.*

The proof is omitted.

Theorem 6.4. *If $A = (a_{ij}) \in M_n(L)$ is especially strongly transitive, then A has a canonical form.*

The proof is similar to that of Theorem 3.1 in [14].

Theorem 6.5. *Let L be an incline and n an integer with $n \geq 4$. Then, for any $n \times n$ transitive matrix A over L, there exists an $n \times n$ permutation matrix P such that $F = PAP^T$ satisfies $f_{ij} \nleq f_{ji}$ for $i > j$ only if L is a linear incline.*

Proof. Suppose that L is not a linear incline. Then, $w(L) \geq 2$, and so that there must be two elements a and c in L such that $a \| c$. Therefore, $ac < a < a + c$ and $ac < c < a + c$. Now, let $A = aE_{12} + cE_{23} + aE_{34} + cE_{41} + acJ_n \in M_n(L)$, where $J_n = (1)_{n \times n}$.

It is easy to see that $A^2 \leq A$. This means A is transitive. Let P be any $n \times n$ permutation matrix. Then, there exists a unique permutation τ of the set $\{1, 2, \ldots, n\}$ such that $P = \sum_{i=1}^{n} E_{\tau(i)i}$, and so $P^T = \sum_{i=1}^{n} E_{i\tau(i)}$. Therefore, $F = (f_{ij})_{n \times n} = PAP^T = aE_{\tau(1)\tau(2)} + cE_{\tau(2)\tau(3)} + aE_{\tau(3)\tau(4)} + cE_{\tau(4)\tau(1)} + acJ_n$. Thus, $f_{\tau(1)\tau(2)} = f_{\tau(3)\tau(4)} = a$, $f_{\tau(2)\tau(3)} = f_{\tau(4)\tau(1)} = c$ and $f_{\tau(2)\tau(1)} = f_{\tau(4)\tau(3)} = f_{\tau(3)\tau(2)} = f_{\tau(1)\tau(4)} = ac$. Since τ is a permutation, we have $\tau(i) \neq \tau(j) (i \neq j)$. By the hypothesis $F = PAP^T$ satisfies $f_{ij} \nleq f_{ji}$ for $i > j$, we have $\tau(1) > \tau(2)$, $\tau(2) > \tau(3)$, $\tau(3) > \tau(4)$, and $\tau(4) > \tau(1)$. This implies $\tau(1) > \tau(1)$, which leads to a contradiction. This proves the Theorem. □

Theorem 6.5 generalizes Theorem 5.2 of Tan [17].

Remark 6.6. *It is easy to verify that A always has a canonical form for any transitive matrix $A \in M_2(L)$.*

Acknowledgments

References

[1] S.-C. Han and H.-X. Li, "Indices and periods of incline matrices," *Linear Algebra and its Applications*, vol. 387, pp. 143–165, 2004.

[2] D. Dubois and H. Prade, *Fuzzy Sets and Systems*, vol. 144 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1980.

[3] J. A. Goguen, "L-fuzzy sets," *Journal of Mathematical Analysis and Applications*, vol. 18, pp. 145–174, 1967.

[4] A. Kaufmann, *Introduction to the Theory of Fuzzy Subsets*, Academic Press, New York, NY, USA, 1975.

[5] S. V. Ovchinnikov, "Structure of fuzzy binary relations," *Fuzzy Sets and Systems*, vol. 6, no. 2, pp. 169–195, 1981.

[6] L. A. Zadeh, "Similarity relations and fuzzy orderings," *Information Sciences*, vol. 3, pp. 177–200, 1971.

[7] V. Tahani, "A fuzzy model of document retrieval systems," *Information Processing Management*, vol. 12, pp. 177–187, 1976.

[8] S. Tamura, S. Higuchi, and K. Tanaka, "Pattern classification based on fuzzy relations," *IEEE Transictions on Systems, Man, and Cybernetics*, vol. 1, no. 1, pp. 61–66, 1971.

[9] H. Hashimoto, "Transitivity of generalized fuzzy matrices," *Fuzzy Sets and Systems*, vol. 17, no. 1, pp. 83–90, 1985.

[10] W. Kołodziejczyk, "Convergence of powers of s-transitive fuzzy matrices," *Fuzzy Sets and Systems*, vol. 26, no. 1, pp. 127–130, 1988.

[11] H. Hashimoto, "Convergence of powers of a fuzzy transitive matrix," *Fuzzy Sets and Systems*, vol. 9, no. 2, pp. 153–160, 1983.

[12] H. Hashimoto, "Canonical form of a transitive fuzzy matrix," *Fuzzy Sets and Systems*, vol. 11, no. 2, pp. 157–162, 1983.

[13] W. Kołodziejczyk, "Canonical form of a strongly transitive fuzzy matrix," *Fuzzy Sets and Systems*, vol. 22, no. 3, pp. 297–302, 1987.

[14] C. G. Hao, "Canonical form of strongly transitive matrices over lattices," *Fuzzy Sets and Systems*, vol. 45, no. 2, pp. 219–222, 1992.

[15] X. T. Peng, "A property of matrices over ordered sets," *Fuzzy Sets and Systems*, vol. 19, no. 1, pp. 47–50, 1986.

[16] H. Hashimoto, "Transitivity of fuzzy matrices under generalized connectedness," *Fuzzy Sets and Systems*, vol. 29, no. 2, pp. 229–234, 1989.

[17] Y.-J. Tan, "On the transitive matrices over distributive lattices," *Linear Algebra and its Applications*, vol. 400, pp. 169–191, 2005.

[18] K. H. Kim, *Boolean Matrix Theory and Applications*, vol. 70 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1982.

[19] Y.-j. Tan, "On the powers of matrices over a distributive lattice," *Linear Algebra and its Applications*, vol. 336, pp. 1–14, 2001.

[20] Y.-j. Tan, "On compositions of lattice matrices," *Fuzzy Sets and Systems*, vol. 129, no. 1, pp. 19–28, 2002.

[21] J.-S. Duan, "The transitive closure, convergence of powers and adjoint of generalized fuzzy matrices," *Fuzzy Sets and Systems*, vol. 145, no. 2, pp. 301–311, 2004.

[22] M. P. Zhou and Y. J. Tan, "An open problem on incline matrices," *Journal of Fuzhou University*, vol. 34, no. 3, pp. 313–315, 2006.

[23] S.-C. Han, H.-X. Li, and J.-Y. Wang, "On nilpotent incline matrices," *Linear Algebra and its Applications*, vol. 406, pp. 201–217, 2005.