Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 340192, 12 pages
doi:10.1155/2011/340192

Research Article

An Improved Predictor-Corrector Interior-Point Algorithm for Linear Complementarity Problems with $O(\sqrt{n}L)$-Iteration Complexity

Debin Fang[1] and Qian Yu[2]

[1] *School of Economics and Management, Wuhan University, Wuhan 430072, China*
[2] *School of Economics, Wuhan University of Technology, Wuhan 430070, China*

Correspondence should be addressed to Qian Yu, yuqian365@126.com

Received 18 September 2011; Revised 23 November 2011; Accepted 24 November 2011

Academic Editor: Chong Lin

This paper proposes an improved predictor-corrector interior-point algorithm for the linear complementarity problem (LCP) based on the Mizuno-Todd-Ye algorithm. The modified corrector steps in our algorithm cannot only draw the iteration point back to a narrower neighborhood of the center path but also reduce the duality gap. It implies that the improved algorithm can converge faster than the MTY algorithm. The iteration complexity of the improved algorithm is proved to obtain $O(\sqrt{n}L)$ which is similar to the classical Mizuno-Todd-Ye algorithm. Finally, the numerical experiments show that our algorithm improved the performance of the classical MTY algorithm.

1. Introduction

Since Karmarkar published the first paper on interior point method [1] in 1984, the interior point methodologies have yielded rich theories and algorithms in the fields of linear programming (LP), quadratic programming (QP), and linear complementarity problems (LCP). Among these interior point methods, predictor-corrector interior-point methods play a special role due to their best polynomial complexity and superlinear convergence.

In 1993, Mizuno et al. [2] proposed the classical representative of predictor-corrector method for linear programming. The Mizuno-Todd-Ye (MTY) algorithm has $O(\sqrt{n}L)$-iteration complexity which is the best iteration complexity obtained so far for all the interior-point method [3, 4]. Moreover, Ye et al. [5] proved the duality gap of classical MTY algorithm converges to zero quadratically, which dedicated that MTY algorithm has superlinear convergence. The classical MTY algorithm is the first algorithm for LP has both polynomial complexity and superlinear convergence. So the classical MTY algorithm therefore was considered as the most efficient interior point methods for LP.

Afterwards, the MTY algorithm was extended to LCP [6, 7] for its excellent performance. In recent papers, Potra [8, 9] presented several predictor-corrector methods for linear complementarity problems acting in a wide neighborhood of the central path, and Stoer et al. [10] proposed a predictor-corrector algorithm with arbitrarily high order of convergence to degenerate sufficient linear complementarity problems.

Subsequently, Potra proposed a generalization of the MTY algorithm for infeasible starting points for monotone LCP [11]. Based on the so-called "fast step-safe step" strategy, Wright proposed an infeasible interior point method with polynomial and quadratic convergence for nondegenerate LCP [12]. And Ai and Zhang presented an $O(\sqrt{n}L)$-iteration primal-dual path-following method for monotone LCP based on wide neighborhoods and large updates [13]. These workings improved the MTY algorithm in various aspects. In this paper, the MTY algorithm for LCP will be improved by modifying the corrector of the algorithm to obtain a larger iteration reduce factor than the original, which can guarantee a faster convergence.

The typical iterations of the MTY algorithm operate between two neighborhoods of the central path, $\rho(1/4)$ and $\rho(1/2)$. Before starting the MTY algorithm, an arbitrary initial point $\omega \in \rho(1/4)$ will be given. And then, the predictor produces a point $\overline{\omega} \in \rho(1/2)$ by carefully choosing a steplength along the affine-scaling direction at ω and reduces the primal dual gap by a factor of at least $(1 - \widehat{\chi}/\sqrt{n})$ where $\widehat{\chi} = 1/\sqrt[4]{8} \approx 0.5946$. In the corrector steps, the corrector produces a point $\widehat{\omega} \in \rho(1/4)$ by taking a unit steplength along the centering direction at $\overline{\omega}$. It is shown that the corrector put the iteration points back to $\rho(1/4)$ which is the narrower neighborhood of the central path and maintains the same duality gap. Thus predictor-corrector reduces the duality gap μ^k by a factor of at least $(1 - \widehat{\chi}/\sqrt{n})$ and holds the iteration points in the neighborhood $\rho(1/4)$. It follows that the corresponding iterative procedure has $O(\sqrt{n}L)$-iteration complexity.

The modified algorithm in this paper has only one corrector step in neighborhood of the central path $\rho(1/2)$ and one predictor step in a narrow neighborhood $\rho(1/4)$ same as the MTY algorithm. Moreover, the modified iteration direction of corrector step can make a reduction for duality gap. This improvement attained a larger iteration reduce factor and results in a faster convergence than the classical MTY algorithm.

Section 2 describes the procedure of our predictor-corrector algorithm. Section 3 gives the convergence analysis of our algorithm. It shows that the improved algorithm can generate a sequence $\{(x^k, y^k)\}$ in the neighborhood $\rho(1/4)$ from an arbitrary initial point $(x^0, y^0) \in \rho(1/4)$ and converge to optimal solution (x^k, y^k) with $O(\sqrt{n}L)$-iteration complexity. Finally, the performances of our algorithm are evaluated by numerical experiments in Section 4.

2. Description of the Algorithm

In this paper, LCP is to find a vector pair $(x, y) \in R^n \times R^n$ such that $y = Mx + h$, $(x, y) \geq (0, 0)$ and $x^T y = 0$, where $h \in R^n$ and M is a $n \times n$ positive semidefinite matrix.

Denote the set of all feasible points of LCP by $S = \{(x, y) \mid Mx - y + h = 0,\ x \geq 0,\ y \geq 0\}$ and the solution set of LCP by $S^* = \{(x^*, y^*) \mid (x^*, y^*) \in S,\ x^{*T} y^* = 0\}$. The relative interior of S, $S^+ = \{(x, y) \mid (x, y) \in S,\ x > 0,\ y > 0\}$ is called the set of strictly feasible points, or the set of interior points. We assume $S^+ \neq \Phi$ in this paper. It is known that if S^+ is nonempty, then for any parameter $\tau > 0$ the nonlinear system $Mx - y + h = 0,\ xy = \tau e$ has a unique positive solution [14]. The set of all such solutions defines the central path C of LCP. So the assumption is sufficient to establish the existence of the central path and the existence of a solution to LCP.

The mean value of $x^T y$ is denoted as $\mu = x^T y/n$ throughout this paper. Then $\rho(\beta) = \{(x, y) \in S^+ \mid \|Xy - \mu e\| \le \beta\mu,\ \mu = x^T y/n\}$ is considered as the neighborhood of central path, where $\beta > 0$, e denotes the vector of ones, $X = \operatorname{diag}(x)$ and $\|\cdot\|$ express the Euclidean norm.

The improved predictor-corrector interior-point algorithm (IPCIP) is as follows.

2.1. Main Steps of IPCIP Algorithm

Step 0. Choose an initial pair of interior point (x^0, y^0) with $(x^0, y^0) \in \rho(1/4)$, and set the accuracy parameter $\varepsilon > 0$. Let $k = 0$.

Step 1. If $(x^k)^T y^k/n \le \varepsilon$, then stop.

Step 2. Compute the predictor direction $(\Delta x^p, \Delta y^p)$ by solving the system (2.1),

$$\begin{aligned} M\Delta x^p - \Delta y^p &= 0, \\ Y^k \Delta x^p + X^k \Delta y^p &= \frac{2}{3}\mu^k e - X^k y^k. \end{aligned} \tag{2.1}$$

Step 3. Set $(\hat{x}^k, \hat{y}^k) = (x^k, y^k) + Q^k(\Delta x^p, \Delta y^p)$ where $Q^k = \min\{1/8\sqrt{n}, \sqrt{\mu^k/8\|\Delta X^p \Delta y^p\|}\}$. (We will show later that $\|\hat{X}^k \hat{y}^k - \overline{\mu}^k e\| \le (1/2)\overline{\mu}^k$, where $\overline{\mu}^k = (1 - Q^k)\mu^k$ holds for every $(\hat{x}^k, \hat{y}^k)$.)

Step 4. Set $r = 1 - 1/4\sqrt{2n}$, and compute the corrector direction $(\Delta x^c, \Delta y^c)$ by solving the system (2.2),

$$\begin{aligned} M\Delta x^c - \Delta y^c &= 0, \\ \hat{Y}^k \Delta x^c + \hat{X}^k \Delta y^c &= r\overline{\mu}^k e - \hat{X}^k \hat{y}^k. \end{aligned} \tag{2.2}$$

Step 5. Set $(x^{k+1}, y^{k+1}) = (\hat{x}^k, \hat{y}^k) + (\Delta x^c, \Delta y^c)$, update $k = k + 1$, and return to Step 1.

3. Convergence and Complexity Analysis

Lemma 3.1. *If $b, c \in R^n$ and $b + c = h$, $b^T c \ge 0$, $B = \operatorname{diag}(b)$, then*

(i) $\|Bc\| \le (\sqrt{2}/4)\|h\|^2$;

(ii) $b^T c \le (1/2)\|h\|^2$;

(iii) $\|b\|^2 \le \|h\|^2$.

Proof. The proof of (i) can be seen in Lemma 1 of [2].

From $\|h\|^2 = \|b + c\|^2 = \|b\|^2 + \|c\|^2 + 2b^T c$ and $b^T c \ge 0$, we can obtain $b^T c \le (1/2)\|h\|^2$ and $\|b\|^2 \le \|h\|^2$. Thus inequalities (ii) and (iii) hold. This completes the proof. □

Lemma 3.2. *If the point $(\hat{x}^k, \hat{y}^k)$ was generated by the algorithm, then one has $\hat{\mu}^k = (\hat{x}^k)^T \hat{y}^k/n \ge \overline{\mu}^k$, where $\overline{\mu}^k = (1 - Q^k)\mu^k$.*

Proof. From Step 2 of our algorithm and M is a $n \times n$ positive semidefinite matrix, we have $(\Delta x^p)^T \Delta y^p = (\Delta x^p)^T M \Delta x^p \geq 0$. Hence

$$\begin{aligned}
\widehat{\mu}^k &= \frac{(\widehat{x}^k)^T \widehat{y}^k}{n} \\
&= \frac{(x^k + Q^k \Delta x^p)^T (y^k + Q^K \Delta y^p)}{n} \\
&= \frac{(x^k)^T y^k + Q^k \left[(x^k)^T \Delta y^p + (\Delta x^p)^T y^k \right] + (Q^k)^2 (\Delta x^p)^T \Delta y^p}{n} \\
&\geq \frac{(x^k)^T y^k}{n} + \frac{Q^k}{n} \left(n \cdot \frac{2}{3} \mu^k - \left(x^k \right)^T y^k \right) \\
&= \mu^k + \frac{2}{3} Q^k \mu^k - Q^k \mu^k \\
&\geq \left(1 - Q^k \right) \mu^k \\
&= \overline{\mu}^k.
\end{aligned} \tag{3.1}$$

This completes the proof. □

Lemma 3.3. *If $(x^k, y^k) \in \rho(1/4)$, then the point $(\widehat{x}^k, \widehat{y}^k)$ generated by the algorithm satisfies $\widehat{x}^k > 0$, $\widehat{y}^k > 0$, $\|\widehat{X}^k \widehat{y}^k - \overline{\mu}^k e\| \leq (1/2)\overline{\mu}^k$, and $(\widehat{x}^k, \widehat{y}^k) \in \rho(1/2)$.*

Proof. Let us define $(x(Q), y(Q)) = (x^k + Q\Delta x^p, y^k + Q\Delta y^p)$, where

$$0 \leq Q \leq Q^k = \min \left\{ \frac{1}{8\sqrt{n}}, \sqrt{\frac{\mu^k}{8\|\Delta X^p \Delta y^p\|}} \right\}. \tag{3.2}$$

Note that $0 \leq Q \leq Q^k = \min\{1/8\sqrt{n}, \sqrt{\mu^k/8\|\Delta X^p \Delta y^p\|}\}$, then we have $(2/3)\sqrt{n}Q \leq 1/12$ from $Q \leq 1/8\sqrt{n}$ and $Q^2\|\Delta X^p \Delta y^p\| \leq (1/8)\mu^k$ from $Q \leq \sqrt{\mu^k/8\|\Delta X^p \Delta z^p\|}$.

Furthermore $\|X^k y^k - \mu^k e\| \leq (1/4)\mu^k$ holds since $(x^k, y^k) \in \rho(1/4)$. Therefore we have

$$\begin{aligned}
\left\| X(Q)y(Q) - (1-Q)\mu^k e \right\| &= \left\| X^k y^k + Q(X^p \Delta y^p + \Delta X^p y^p) + Q^2 \Delta X^p \Delta y^p - (1-Q)\mu^k e \right\| \\
&= \left\| X^k y^k + Q\left(\frac{2}{3}\mu^k e - X^k y^k \right) + Q^2 \Delta X^p \Delta y^p - (1-Q)\mu^k e \right\| \\
&\leq (1-Q)\left\| X^k y^k - \mu^k e \right\| + \frac{2}{3}\sqrt{n} Q \mu^k + Q^2 \left\| \Delta X^p \Delta y^p \right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-Q)\frac{1}{4}\mu^k + \frac{1}{12}\mu^k + \frac{1}{8}\mu^k \\
&= \left[\frac{1}{4} + \frac{1}{1-Q}\left(\frac{1}{12}+\frac{1}{8}\right)\right](1-Q)\mu^k \\
&\leq \left(\frac{1}{4}+\frac{5}{21}\right)(1-Q)\mu^k \\
&\leq \frac{1}{2}(1-Q)\mu^k.
\end{aligned}
\tag{3.3}
$$

So $x_i(Q)y_i(Q) \geq (1/2)(1-Q)\mu^k > 0$ $(i = 1, 2, \ldots, n)$ for any Q $(0 \leq Q \leq Q^K)$.

Since $(x(Q), y(Q)) = (x^k + Q\Delta x^p, y^k + Q\Delta y^p)$ is continuous with respect to $Q, x^k > 0$ and $y^k > 0$, it is easily seen that $\hat{x}^k = x(Q^k) > 0, \hat{y}^k = y(Q^k) > 0$.

Note that $X(Q^k) = \hat{X}^k, Y(Q^k) = \hat{Y}^k, (1-Q^k)\mu^k = \overline{\mu}^k$, the above argument implies $\|\hat{X}^k\hat{y}^k - \overline{\mu}^k e\| \leq (1/2)\overline{\mu}^k$.

It's well known that the inequality $\|Xz - (x^T z/n)e\| \leq \|Xz - ce\|$ holds for arbitrary vector x, z and arbitrary constant c.

So we have $\|\hat{X}^k\hat{y}^k - \hat{\mu}^k e\| \leq \|\hat{X}^k\hat{y}^k - \overline{\mu}^k e\| \leq (1/2)\overline{\mu}^k \leq (1/2)\hat{\mu}$ from the proof above and Lemma 3.2, thus $(\hat{x}^k, \hat{y}^k) \in \rho(1/2)$.

This completes the proof. □

Remark 3.4. So Lemma 3.3 dedicates that the predictors of our algorithm operate in a wide neighborhood of the central path $\rho(1/2)$.

Lemma 3.5. *If* $\|\hat{X}^k\hat{y}^k - \overline{\mu}^k e\| \leq (1/2)\overline{\mu}^k$ *then* $\|(\hat{X}^k\hat{Y}^k)^{-1/2}(r\overline{\mu}^k e - \hat{X}^k\hat{y}^k)\|^2 \leq 2(1/4 + (1-r)^2 n)\overline{\mu}^k$

Proof.

$$
\begin{aligned}
\left\|\left(\hat{X}^k\hat{Y}^k\right)^{-1/2}\left(r\overline{\mu}^k e - \hat{X}^k\hat{y}^k\right)\right\|^2 &\leq \frac{1}{\min\limits_{1\leq i\leq n}\hat{x}_i^k\hat{y}_i^k}\left\|r\overline{\mu}^k e - \hat{X}^k\hat{y}^k\right\|^2 \\
&\leq \frac{1}{(1-1/2)\overline{\mu}^k}\left\|\hat{X}^k\hat{y}^k - \overline{\mu}^k e + (1-r)\overline{\mu}^k e\right\|^2 \\
&\leq \frac{2}{\overline{\mu}^k}\left(\left\|\hat{X}^k\hat{y}^k - \overline{\mu}^k e\right\|^2 + \left\|(1-r)\overline{\mu}^k e\right\|^2\right) \\
&\leq 2\left(\frac{1}{4} + (1-r)^2 n\right)\overline{\mu}^k.
\end{aligned}
\tag{3.4}
$$

This completes the proof. □

Lemma 3.6. *If* $n > 2$ *and the point* (x^{k+1}, y^{k+1}) *was generated by the algorithm, then* $\mu^{k+1} \leq (1 - Q^k)\mu^k$ *always hold.*

Proof. Denote $D = (\widehat{X}^k)^{1/2}(\widehat{Y}^k)^{-1/2}$ then

$$\begin{gathered} D^{-1}\Delta x^c + D\Delta y^c = \left(\widehat{X}^k\widehat{Y}^k\right)^{-1/2}\left(r\overline{\mu}^k e - \widehat{X}^k\widehat{y}^k\right) \\ \left(D^{-1}\Delta x^c\right)^T(D\Delta y^c) = (\Delta x^c)^T\Delta y^c = (\Delta x^c)^T M\Delta x^c \geq 0. \end{gathered} \tag{3.5}$$

From Lemmas 3.1 and 3.5 we have

$$\begin{aligned} (\Delta x^c)^T\Delta y^c &\leq \frac{1}{2}\left\|\left(\widehat{X}^k\widehat{y}^k\right)^{-1/2}\left(r\overline{\mu}^k e - \widehat{X}^k\widehat{y}^k\right)\right\|^2 \leq \left(\frac{1}{4} + (1-r)^2 n\right)\overline{\mu}^k \\ \therefore \mu^{k+1} &= \frac{\left(x^{k+1}\right)^T y^{k+1}}{n} \\ &= \frac{\left(\widehat{x}^k + \Delta x^c\right)^T\left(\widehat{y}^k + \Delta y^c\right)}{n} \\ &= \frac{\left(\widehat{x}^k\right)^T\widehat{y}^k + \left(\widehat{x}^k\right)^T\Delta y^c + (\Delta x^c)^T\widehat{y}^k + (\Delta x^c)^T\Delta y^c}{n} \\ &= \frac{nr\overline{\mu}^k + (\Delta x^c)^T\Delta y^c}{n} \\ &\leq r\overline{\mu}^k + \left(\frac{1}{4n} + (1-r)^2\right)\overline{\mu}^k. \end{aligned} \tag{3.6}$$

Note that $r = 1 - 1/4\sqrt{2n}$, we have

$$\left(r + \frac{1}{4n} + (1-r)^2\right) = 1 - \frac{1}{4\sqrt{2n}} + \frac{1}{4n} + \frac{1}{32n} = 1 - \frac{4\sqrt{2n} - 9}{32n}. \tag{3.7}$$

It is obvious that $(r + 1/4n + (1-r)^2) < 1$ when $n > 2$, so $\mu^{k+1} \leq \overline{\mu}^k = (1 - Q^k)\mu^k$. This completes the proof. □

Remark 3.7. From Lemmas 3.2 and 3.6, we can obtain immediately that $\mu^{k+1} \leq \overline{\mu}^k \leq \widehat{\mu}^k$. That means the corrector reduces $\widehat{\mu}^k$ to μ^{k+1}. Because the corrector in the classical MTY algorithm just maintains the same duality gap, the improvement of the corrector in our algorithm will make a faster reduction of μ^k than MTY algorithm.

Then the polynomial convergence of the algorithm could be established.

Theorem 3.8. *Suppose that the sequence $\{(x^{k+1}, y^{k+1})\}$ generated by the algorithm, then one has $(x^{k+1}, y^{k+1}) \in \rho(1/4)$.*

Proof. Let us define that $x^{k+1}(\alpha) = \widehat{x}^k + \alpha\Delta x^c$, $y^{k+1}(\alpha) = \widehat{y}^k + \alpha\Delta y^c$, and denote

$$\mu^{k+1}(\alpha) = \frac{\left(x^{k+1}(\alpha)\right)^T y^{k+1}(\alpha)}{n}. \tag{3.8}$$

As discussed in Lemma 3.6, we also have $(D^{-1}\Delta x^c)^T(D\Delta y^c) = (\Delta x^c)^T\Delta y^c = (\Delta x^c)^T M\Delta x^c \geq 0$.

From Lemmas 3.1 and 3.5 and $r = 1 - 1/4\sqrt{2n}$, then

$$\begin{aligned}
\|\Delta X^c \Delta y^c\| &\leq \frac{\sqrt{2}}{4}\left\|\left(\widehat{X}^k\widehat{Y}^k\right)^{-1/2}\left(r\overline{\mu}^k e - \widehat{X}^k\widehat{y}^k\right)\right\|^2 \\
&\leq \frac{\sqrt{2}}{4}2\left(\frac{1}{4} + (1-r)^2 n\right)\overline{\mu}^k \\
&= \frac{1}{4}\left(\frac{\sqrt{2}}{2} + 2\sqrt{2}n(1-r)^2\right)\overline{\mu}^k \\
&\leq \frac{1}{4}\left(\frac{\sqrt{2}}{2} + \frac{1}{8}\right)\overline{\mu}^k \\
&\leq \frac{7}{32}\overline{\mu}^k, \\
\mu^{k+1}(\alpha) &= \frac{\left(x^{k+1}(\alpha)\right)^T y^{k+1}(\alpha)}{n} \\
&= \frac{\left(\widehat{x}^k + \alpha\Delta x^c\right)^T\left(\widehat{y}^k + \alpha\Delta y^c\right)}{n} \\
&= \frac{\left(\widehat{x}^k\right)^T\widehat{y}^k + \alpha\left(\left(\widehat{x}^k\right)^T\Delta y^c + (\Delta x^c)^T\widehat{y}^k\right) + \alpha^2(\Delta x^c)^T\Delta y^c}{n} \\
&\geq \frac{\left(\widehat{x}^k\right)^T\widehat{y}^k + \alpha\left(\left(\widehat{x}^k\right)^T\Delta y^c + (\Delta x^c)^T\widehat{y}^k\right)}{n} \\
&= \frac{\left(\widehat{x}^k\right)^T\widehat{y}^k + \alpha\left(nr\overline{\mu}^k - \left(\widehat{x}^k\right)^T\widehat{y}^k\right)}{n} \\
&= \widehat{\mu}^k + \alpha\left(r\overline{\mu}^k - \widehat{\mu}^k\right) \\
&= (1-\alpha)\widehat{\mu}^k + \alpha r\overline{\mu}^k \\
&\geq (1-(1-r)\alpha)\overline{\mu}^k.
\end{aligned} \tag{3.9}$$

So we can write $\mu^{k+1}(\alpha) \geq (1-(1-r)\alpha)\overline{\mu}^k$, that is, $\overline{\mu}^k \leq \mu^{k+1}(\alpha)/(1-(1-r)\alpha)$. Hence

$$\begin{aligned}
X^{k+1}(\alpha)y^{k+1}(\alpha) - \mu^{k+1}(\alpha)e &= \left(\widehat{X}^k + \alpha\Delta X^c\right)\left(\widehat{y}^k + \alpha\Delta y^c\right) - \frac{\left(\widehat{x}^k + \alpha\Delta x^c\right)^T\left(\widehat{y}^k + \alpha\Delta y^c\right)}{n}e \\
&= \widehat{X}^k\widehat{y}^k + \alpha\left(\widehat{X}^k\Delta y^c + \Delta X^c\widehat{y}^k\right) + \alpha^2\Delta X^c\Delta y^c \\
&\quad - \frac{\left(\widehat{x}^k\right)^T\widehat{y}^k + \alpha\left(\left(\widehat{x}^k\right)^T\Delta y^c + (\Delta x^c)^T\widehat{y}^k\right) + \alpha^2(\Delta x^c)^T\Delta y^c}{n}e
\end{aligned}$$

$$
\begin{aligned}
&= \widehat{X}^k \widehat{y}^k + \alpha\left(r\overline{\mu}^k e - \widehat{X}^k \widehat{y}^k\right) + \alpha^2 \Delta X^c \Delta y^c \\
&\quad - \frac{\left(\widehat{x}^k\right)^T \widehat{y}^k + \alpha\left(nr\overline{\mu}^k - \left(\widehat{x}^k\right)^T \widehat{y}^k\right) + \alpha^2 (\Delta x^c)^T \Delta y^c}{n} e \\
&= (1-\alpha)\left(\widehat{X}^k \widehat{y}^k - \frac{\left(\widehat{x}^k\right)^T \widehat{y}^k}{n} e\right) + \alpha^2\left(\Delta X^c \Delta y^c - \frac{(\Delta x^c)^T \Delta y^c}{n} e\right).
\end{aligned}
\tag{3.10}
$$

Since the inequality $\|Xz - (x^T z/n)e\| \le \|Xz - ce\|$ holds for arbitrary vector x, z and arbitrary constant c, thus

$$
\begin{aligned}
\left\|X^{k+1}(\alpha) y^{k+1}(\alpha) - \mu^{k+1}(\alpha) e\right\| &\le (1-\alpha)\left\|\widehat{X}^k \widehat{y}^k - \frac{\left(\widehat{x}^k\right)^T \widehat{y}^k}{n} e\right\| + \alpha^2 \left\|\Delta X^c \Delta y^c - \frac{(\Delta x^c)^T \Delta y^c}{n} e\right\| \\
&\le (1-\alpha)\left\|\widehat{X}^k \widehat{y}^k - \overline{\mu}^k e\right\| + \alpha^2 \left\|\Delta X^c \Delta y^c\right\| \\
&\le (1-\alpha)\frac{1}{2}\overline{\mu}^k + \alpha^2 \frac{7}{32}\overline{\mu}^k \\
&\le \frac{(1/2)(1-\alpha) + (7/32)\alpha^2}{1-(1-r)\alpha}\mu^{k+1}(\alpha).
\end{aligned}
\tag{3.11}
$$

Let us define $f(\alpha) = ((1/2)(1-\alpha) + (7/32)\alpha^2)/(1-(1-r)\alpha)$, then $f'(\alpha) = (-(1/2)r + (7/16)\alpha - (7/32)\alpha^2(1-r))/[1-(1-r)\alpha]^2$.

When $n > 2$, we have $r \ge 7/8$, this results in $f'(\alpha) \le 0$ and $f(\alpha)$ decreases monotonously, when $0 \le \alpha \le 1$. Because $f(\alpha) \le f(0) = 1/2$ holds for every $\alpha \in [0,1]$ thus

$$
\left\|X^{k+1}(\alpha) y^{k+1}(\alpha) - \mu^{k+1}(\alpha) e\right\| \le \frac{1}{2}\mu^{k+1}(\alpha). \tag{3.12}
$$

Since $\mu^k > \varepsilon > 0$ (otherwise the algorithm will be terminated), it follows that

$$
\begin{aligned}
X_i^{k+1}(\alpha) y_i^{k+1}(\alpha) &\ge \frac{1}{2}\mu^{k+1}(\alpha) \\
&\ge \frac{1}{2}(1-(1-r)\alpha)\overline{\mu}^k \\
&= \frac{1}{2}(1-(1-r)\alpha)\left(1-Q^k\right)\mu^k > 0.
\end{aligned}
\tag{3.13}
$$

We have proved that $\widehat{x}^k = x(Q^k) > 0$, $\widehat{y}^k = y(Q^k) > 0$ in Lemma 3.3. So we have

$$
x^{k+1}(0) = \widehat{x}^k > 0, \quad y^{k+1}(0) = \widehat{y}^k > 0, \quad \text{when } \alpha = 0. \tag{3.14}
$$

Note that $x^{k+1}(\alpha)$ and $y^{k+1}(\alpha)$ are continuous with respect to α. It implies that $x^{k+1}(\alpha) > 0$ and $y^{k+1}(\alpha) > 0$ hold for any $\alpha \in [0,1]$.

Let $\alpha = 1$, then we have $x^{k+1} > 0$ and $y^{k+1} > 0$.

Furthermore, when $\alpha = 1$, $f(\alpha) = 7/32r$ thus $\|X^{k+1}y^{k+1} - \mu^{k+1}e\| \leq (7/32r)\mu^{k+1} \leq (1/4)\mu^{k+1}$.

Consider that we can obtain $Mx^{k+1} - y^{k+1} + h = 0$ directly from the steps in our algorithm, so $(x^{k+1}, y^{k+1}) \in \rho(1/4)$ holds for every (x^{k+1}, y^{k+1}) generated by the algorithm.

This completes the proof. □

Theorem 3.9. *The iteration complexity of the algorithm is $O(\sqrt{n}L)$.*

Proof. Let $D' = (X^k)^{1/2}(Y^k)^{-1/2}$ then

$$\begin{gathered}(D')^{-1}\Delta x^p + D'\Delta y^p = \left(X^kY^k\right)^{-1/2}\left(\frac{2}{3}\mu^k e - X^k y^k\right),\\ \left((D')^{-1}\Delta x^p\right)^T(D'\Delta y^p) = (\Delta x^p)^T\Delta y^p = (\Delta x^p)^T M\Delta y^p \geq 0.\end{gathered} \tag{3.15}$$

From Lemma 3.1, we have

$$\begin{aligned}\|\Delta X^p\Delta y^p\| &= \left\|(D')^{-1}\Delta X^p D'\Delta y^p\right\|\\ &\leq \frac{\sqrt{2}}{4}\cdot\left\|\left(X^ky^k\right)^{-1/2}\left(\frac{2}{3}\mu^k e - X^ky^k\right)\right\|^2\\ &\leq \frac{\sqrt{2}}{4}\cdot\left(\sqrt{\sum_{i=1}^{n}\left(\left(\frac{2}{3}\mu^k\right)^2\left(x_i^k\right)^{-1}\left(y_i^k\right)^{-1} - \frac{4}{3}\mu^k + x_i^ky_i^k\right)}\right)^2.\end{aligned} \tag{3.16}$$

From $\|X^ky^k - \mu^k e\| \leq (1/4)\mu^k$, we have $x_i^ky_i^k \geq (3/4)\mu^k$, then $(x_i^k)^{-1}(y_i^k)^{-1} \leq 4/3\mu^k$,

$$\begin{aligned}\|\Delta X^p\Delta y^p\| &\leq \frac{\sqrt{2}}{4}\cdot\sum_{i=1}^{n}\left(\frac{4}{9}\cdot\frac{4}{3}\cdot\mu^k - \frac{4}{3}\mu^k\right) + n\mu^k\\ &= \frac{\sqrt{2}}{4}\cdot\frac{7}{27}n\mu^k\\ &= \frac{7\sqrt{2}}{108}n\mu^k.\end{aligned} \tag{3.17}$$

Hence $\sqrt{\mu^k/8\|\Delta X^p\Delta y^p\|} \geq \sqrt{27/14\sqrt{2}n}$, and we can obtain $\min\{1/8\sqrt{n}, \sqrt{\mu^k/8\|\Delta X^p\Delta y^p\|}\} \geq \sqrt{27/14\sqrt{2}n}$.

So it implies that $\mu^{k+1} \leq (1 - Q^k)\mu^k \leq (1 - \sqrt{27/14\sqrt{2}n})\mu^k$ for each k. This means that μ^k will decrease at least by a constant factor of $(1 - \sqrt{27/14\sqrt{2}n})$ at each iteration, which guarantees a $O(\sqrt{n}L)$-iteration complexity for the algorithm.

This completes the proof. □

Remark 3.10. Note that the reduce factor in the MTY algorithm is $(1 - \hat{\chi}/\sqrt{n})$ where $\hat{\chi} = 1/\sqrt[4]{8} \approx 0.5946$, but the reduce factor in our algorithm is $(1 - \sqrt{27/14\sqrt{2}n}) = (1 - \lambda/\sqrt{n})$ where $\lambda = \sqrt{27/14\sqrt{2}} \approx 1.168$, which is larger than $\hat{\chi}$. It implies that our algorithm will converge faster than the MTY algorithm, although both have the same iteration complexity.

4. Examples and Numerical Results

Finally, the numerical experiments were carried out to evaluate the performance and practical efficiency of the algorithm,

Test Problem 1

Find a vector pair $(x, y) \in R^3 \times R^3$ such that $y = M_1 x + h_1$, $(x, y) \geq (0, 0)$, $x^T y = 0$, where $h_1 \in R^3$ and M_1 is a 3×3 positive semidefinite matrix,

$$M_1 = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 0 \\ 1 & 1 & 5 \end{pmatrix}, \qquad h_1 = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}. \tag{4.1}$$

Test Problem 2

Find a vector pair $(x, y) \in R^5 \times R^5$ such that $y = M_2 x + h_2$, $(x, y) \geq (0, 0)$, $x^T y = 0$, where $h_2 \in R^5$ and M_2 is a 5×5 positive semidefinite matrix,

$$M_2 = \begin{pmatrix} 7 & 0 & 0 & 2 & 0 \\ 2 & 8 & 3 & 5 & 9 \\ 0 & 0 & 3 & 0 & 3 \\ 0 & 1 & 4 & 6 & 1 \\ 8 & 0 & 0 & 2 & 5 \end{pmatrix}, \qquad h_2 = \begin{pmatrix} -9.5 \\ -36.5 \\ -5 \\ -14 \\ -18.5 \end{pmatrix}. \tag{4.2}$$

Test Problem 3

This example is a general test problem used by Noor et al. [15]. The problem is to find a vector pair $(x, y) \in R^n \times R^n$ such that $y = M_3 x + h_3$, $(x, y) \geq (0, 0)$, $x^T y = 0$, where $h_3 \in R^n$ and M_3 is a $n \times n$ positive semidefinite matrix. To test the efficiency of our algorithm by large-scale problem, we set the dimension of test problem 3 as 100, that is, $n = 100$,

$$M_3 = \begin{pmatrix} 4 & -2 & 0 & \cdots & 0 \\ 1 & 4 & -2 & \cdots & 0 \\ 0 & 1 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 \end{pmatrix}, \qquad h_3 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}. \tag{4.3}$$

Table 1: Computational results of test problem 1.

	MTY	IPCIP
x^0	$(1, 1, 1)^T$	$(1, 1, 1)^T$
y^0	$(5, 5, 5)^T$	$(5, 5, 5)^T$
ε	$1e-06$	$1e-06$
Iterations	12	11
μ^k	$3.76e-07$	$5.73e-07$
x^k	$(0.0, 0.0004, 0.3999)^T$	$(0.0, 0.0009, 0.3998)^T$
y^k	$(0.2002, 0.0009, 9.29e-07)^T$	$(0.2004, 0.0018, 1.81e-07)^T$
Run time (s)	0.078220	0.049975

Table 2: Computational results of test problem 2.

	MTY	IPCIP
x^0	$(1.5, 1.5, 1.5, 1.5, 1.5)^T$	$(1.5, 1.5, 1.5, 1.5, 1.5)^T$
y^0	$(4, 4, 4, 4, 4)^T$	$(4, 4, 4, 4, 4)^T$
ε	$1e-06$	$1e-06$
Iterations	10	9
μ^k	$2.61e-08$	$2.18e-07$
x^k	$(0.8402, 1.3726, 0.0347, 1.8094, 1.6320)^T$	$(0.8402, 1.3726, 0.0347, 1.8094, 1.6320)^T$
y^k	$(0, 0, 0, 0, 0)^T$	$(0, 0, 0, 0, 0)^T$
Run time (s)	0.071335	0.037116

Table 3: Computational results of test problem 3.

	MTY	IPCIP
x^0	$(1, 1, 1, \ldots, 1, 1)^T$	$(1, 1, 1, \ldots, 1, 1)^T$
y^0	$(3, 4, 4, \ldots, 4, 6)^T$	$(3, 4, 4, \ldots, 4, 6)^T$
ε	$1e-05$	$1e-05$
Iterations	8	7
μ^k	$1.00e-08$	$1.92e-08$
x^k	$(0, 0, 0, \ldots, 0, 0)^T$	$(0, 0, 0, \ldots, 0, 0)^T$
y^k	$(1, 1, 1, \ldots, 1, 1)^T$	$(1, 1, 1, \ldots, 1, 1)^T$
Run time (s)	0.217375	0.177515

These test problems are solved by our IPCIP algorithm and the classical MTY algorithm. The experiments are run on a PC (2.6 GHz CPU, 2 G DDR RAM) using MATLAB 7. The results including the corresponding numbers of iterations, μ^k, the approximate solutions, and computing times are shown in Tables 1, 2, and 3.

5. Conclusion

This paper modified the MTY algorithm for solving monotone LCPs to strengthen the convergence results. Although the iteration complexity of the improved algorithm was proved to be $O(\sqrt{n}L)$ which is similar to the classical MTY algorithm, but the reduce factor of duality gap was enhanced to 1.168 and results in a faster convergence than the classical MTY algorithm.

The numerical results show that our IPCIP algorithm is more efficient than the MTY algorithm. The number of iterations was decreased, and the computing times could be reduced by nearly 20% to 50%. That indicated IPCIP algorithm has a better performance than the MTY algorithm.

Acknowledgments

This paper was supported by the Humanities and Social Science Research Projects of the Ministry of Education of China (no. 11YJCZH221) and the National Natural Science Foundation of China (no. 71103135).

References

[1] N. Karmarkar, "A new polynomial-time algorithm for linear programming," *Combinatorica*, vol. 4, no. 4, pp. 373–395, 1984.

[2] S. Mizuno, M. J. Todd, and Y. Ye, "On adaptive-step primal-dual interior-point algorithms for linear programming," *Mathematics of Operations Research*, vol. 18, no. 4, pp. 964–981, 1993.

[3] T. Illés, M. Nagy, and T. Terlaky, "A polynomial path-following interior point algorithm for general linear complementarity problems," *Journal of Global Optimization*, vol. 47, no. 3, pp. 329–342, 2010.

[4] Y. Q. Bai, G. Lesaja, and C. Roos, "A new class of polynomial interior-point algorithms for linear complementary problems," *Pacific Journal of Optimization. An International Journal*, vol. 4, no. 1, pp. 19–41, 2008.

[5] Y. Ye, O. Güler, R. A. Tapia, and Y. Zhang, "A quadratically convergent $O(\sqrt{nL})$-iteration algorithm for linear programming," *Mathematical Programming*, vol. 59, no. 2, pp. 151–162, 1993.

[6] Y. Ye and K. Anstreicher, "On quadratic and $O(\sqrt{nL})$ convergence of a predictor-corrector algorithm for LCP," *Mathematical Programming*, vol. 62, no. 3, pp. 537–551, 1993.

[7] J. Ji, F. A. Potra, and S. Huang, "Predictor-corrector method for linear complementarity problems with polynomial complexity and superlinear convergence," *Journal of Optimization Theory and Applications*, vol. 85, no. 1, pp. 187–199, 1995.

[8] F. A. Potra, "The Mizuno-Todd-Ye algorithm in a larger neighborhood of the central path," *European Journal of Operational Research*, vol. 143, no. 2, pp. 257–267, 2002.

[9] F. A. Potra, "A superlinearly convergent predictor-corrector method for degenerate LCP in a wide neighborhood of the central path with $O(\sqrt{nL})$-iteration complexity," *Mathematical Programming A*, vol. 100, no. 2, pp. 317–337, 2004.

[10] J. Stoer, M. Wechs, and S. Mizuno, "High order infeasible-interior-point methods for solving sufficient linear complementarity problems," *Mathematics of Operations Research*, vol. 23, no. 4, pp. 832–862, 1998.

[11] F. A. Potra, "An $O(\sqrt{nL})$ infeasible-interior-point algorithm for LCP with quadratic convergence," *Annals of Operations Research*, vol. 62, pp. 81–102, 1996.

[12] S. J. Wright, "A path-following interior-point algorithm for linear and quadratic problems," *Annals of Operations Research*, vol. 62, pp. 103–130, 1996.

[13] W. Ai and S. Zhang, "An $O(\sqrt{nL})$ iteration primal-dual path-following method, based on wide neighborhoods and large updates, for monotone LCP," *SIAM Journal on Optimization*, vol. 16, no. 2, pp. 400–417, 2005.

[14] M. Kojima, N. Megiddo, T. Noma, and A. Yoshise, "A unified approach to interior point algorithms for linear complementarity problems," *Lecture Notes in Computer Science*, vol. 538, 1991.

[15] M. A. Noor, Y. J. Wang, and N. Xiu, "Projection iterative schemes for general variational inequalities," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 3, no. 3, pp. 1–8, 2002.