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Research Article

Positive Solution of Singular Fractional Differential Equation in Banach Space

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We investigated a singular multipoint boundary value problem for fractional differential equation in Banach space. The nonlinear term $f(t,x,y)$ is positive and singular at $x=\theta$ and (or) $y=\theta$. Employing regularization, sequential techniques, and diagonalization methods, we obtained some new existence results of positive solution.

1. Introduction

Recently, fractional differential equations have been investigated extensively. The motivation for those works rises from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, chemistry, aerodynamics, and electrodynamics of the complex medium. For examples and details, see [1–5] and the references therein.

Prompted by the application of multipoint boundary value problem (BVP for short) to applied mathematics and physics, these problems have provoked a great deal of attention by many authors. Here, for fractional differential equations, we refer the reader to [6–12]. Rehman and Khan [7] studied the problem

$$\begin{gathered} D^{\alpha}y(t)+f\left(t,y(t),D^{\beta}y(t)\right)=0, \quad t\in(0,1), \\ y(0)=0, \qquad D^{\beta}y(1)-\sum_{i=1}^{m-2}b_iD^{\beta}y(\xi_i)=y_0, \end{gathered} \tag{1.1}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq \alpha - 1$, $b_i \geq 0$, $0 < \xi_i < 1$, $(i = 1, 2, \ldots, m-2)$ with $\gamma = \sum_{i=1}^{m-2} b_i \xi_i^{\alpha-\beta-1} < 1$, and D^β represents the standard Riemann-Liouville fractional derivative. The existence and uniqueness of solutions were obtained, by means of Schauder fixed-point theorem and Banach contraction principle. Importantly, they gave the Green function of the multipoint BVP (1.1). But they have not proved the positivity of Green function, so the existence of positive solution is unobtainable. However, only positive solutions are useful for many applications, as some physicists pointed out.

The authors of [13–17] investigated singular problem for fractional differential equations with bounded domain. In particular, Agarwal et al. [13] considered the following Dirichlet problem:

$$\begin{gathered} D^\alpha y(t) + f\left(t, y(t), D^\mu y(t)\right) = 0, \\ y(0) = y(1) = 0, \end{gathered} \tag{1.2}$$

where $1 < \alpha \leq 2$, $0 < \mu \leq \alpha - 1$. $f(t, x, y)$ satisfies the Carathéodory conditions and is singular at $x = 0$. In order to overcome the singularity, they used regularization and sequential techniques for the existence of a positive solution.

When the domain where the problem is considered is unbounded, there are few papers about BVP for fractional differential equations in literatures. This situation has changed recently. One can find some works, for example, see [18–22].

In [21], the following BVP:

$$\begin{gathered} D^\alpha y(t) + f\left(t, y(t)\right) = 0, \quad t \in (0, +\infty), \ \alpha \in (1, 2), \\ y(0) = 0, \qquad \lim_{t \to +\infty} D^{\alpha-1} y(t) = \beta y(\xi) \end{gathered} \tag{1.3}$$

was studied. Using the equicontinuity on any compact intervals and the equiconvergence at infinity of a bounded set, the authors proved that the corresponding operator was completely continuous, then the existence of solutions was obtained by the Leray-Schauder nonlinear alternative theorem.

Let $(E, \|\cdot\|)$ be a real Banach space. P is a cone in E which defines a partial ordering in E by $x \leq y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ denotes the zero element of E, and the smallest N is called the normal constant of P (it is clear that $N \geq 1$). If $x \leq y$ and $x \neq y$, we write $x < y$. Let $P_+ = P \setminus \{\theta\}$. So, $x \in P_+$ if and only if $x > \theta$. For details on cone theory, see [23].

In this paper, we are concerned with the existence of positive solution of a BVP for fractional differential equation with bounded domain

$$D^\alpha y(t) + f\left(t, y(t), D^\beta y(t)\right) = \theta, \quad \text{a.e. } t \in [0, T], \tag{1.4}$$

$$y(0) = \theta, \qquad D^\beta y(T) - \sum_{i=1}^{m-2} a_i y(\xi_i) - \sum_{i=1}^{m-2} b_i D^\beta y(\xi_i) = y_0, \tag{1.5}$$

or with unbounded domain

$$D^{\alpha}y(t) + f\left(t, y(t), D^{\beta}y(t)\right) = \theta, \quad \text{a.e. } t \in [0, \infty), \tag{1.6}$$

$$y(0) = \theta, \qquad \lim_{t \to +\infty} D^{\beta}y(t) - \sum_{i=1}^{m-2} a_i y(\xi_i) - \sum_{i=1}^{m-2} b_i D^{\beta}y(\xi_i) = y_0. \tag{1.7}$$

Here, $1 < \alpha \leq 2$, $0 < \beta \leq \alpha - 1$, $\xi_i > 0$, $a_i, b_i \geq 0$ $(i = 1, 2, \ldots, m-2)$, $y_0 \geq \theta$ are real numbers, and D^{α} is the standard Riemann-Liouville fractional derivative. And $f : [0, +\infty) \times P_+ \times P_+ \to P_+$ is singular at $x = \theta$ and $y = \theta$ and satisfies other conditions which will be specified later. In addition, $f(t, x, y)$ is the Carathéodory function.

We say that f satisfies the Carathéodory conditions on $[0, +\infty) \times B$, $B = P_+ \times P_+ (f \in \mathrm{Car}([0, +\infty) \times B))$ if

(i) $f(\cdot, x, y) : [0, +\infty) \to E$ is measurable for all $(x, y) \in B$,

(ii) $f(t, \cdot, \cdot) : B \to E$ is continuous for a.e. $t \in [0, +\infty)$,

(iii) for each compact set $K \subset B$, there is a function $\phi_K \in L^1[0, +\infty)$ such that

$$\left\| f(t, x, y) \right\| \leq \phi_K(t), \quad \text{for a.e. } t \in [0, +\infty), \ \forall (x, y) \in K. \tag{1.8}$$

No contribution exists, as far as we know, concerning the existence of positive solution of the problems (1.4)-(1.5) and (1.6)-(1.7). In the present paper, we consider, firstly, the case of bounded domain, that is, BVP (1.4)-(1.5), and give some existence results by means of regularization process combined with fixed-point theorem due to Krasnosel'skii. Then we investigate the BVP (1.6)-(1.7). As we know, $[0, \infty)$ is noncompact. In order to overcome these difficulties, based on the results of BVP (1.4)-(1.5), we use diagonalization process to establish the existence of positive solutions for BVP (1.6)-(1.7). Let us mention that this method was widely used for integer-order differential equations, see, for instance, [5, 22]. Using diagonalization process, Agarwal et al. [20] have considered a class of boundary value problems involving Riemann-Liouville fractional derivative on the half line. And Arara et al. [19] continued this study by considering a BVP with the Caputo fractional derivative.

The remainder of this paper is organized as follows. In Section 2, we introduced some notations, definitions, and preliminary facts about the fractional calculus, which are used in the next two sections. In Section 3, the case with bounded domain is considered. In Section 4, we discuss the existence of a positive solution for the BVP (1.6)-(1.7). We end this paper with giving an example to demonstrate the application of our results in Section 5.

2. Preliminaries

Now, we introduce the Riemann-Liouville fractional- (arbitrary)-order integral and derivative as follows.

Definition 2.1. *The fractional- (arbitrary)-order integral of the function $v(t) \in L^1([0, b], \mathbb{R})$ of $\mu \in \mathbb{R}^+$ is defined by*

$$I^{\mu}v(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - s)^{\mu - 1} v(s) ds, \quad t > 0. \tag{2.1}$$

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\mu > 0$ for a function $v(t)$ given in the interval $[0,\infty)$ is defined by*

$$D^{\mu}v(t) = \frac{1}{\Gamma(n-\mu)}\left(\frac{d}{dt}\right)^{n}\int_{0}^{t}(t-s)^{n-\mu-1}v(s)ds \tag{2.2}$$

provided that the right hand side is point wise defined. Here, $n = [\mu] + 1$ and $[\mu]$ means the integral part of the number μ, and Γ is the Euler gamma function.

The following properties of the fractional calculus theory are well known, see, for example, [2, 4]:

(i) $D^{\beta}I^{\beta}v(t) = v(t)$ for a.e. $t \in [0,T]$, where $v(t) \in L^{1}[0,T]$, $\beta > 0$,

(ii) $D^{\beta}v(t) = 0$ if and only if $v(t) = \sum_{j=1}^{n} c_j t^{\beta-j}$, where c_j $(j = 1,2,\ldots,n)$ are arbitrary constants, $n = [\beta] + 1$, $\beta > 0$,

(iii) $I^{\beta} : C([0,T]) \to C([0,T])$, $I^{\beta} : L^{1}([0,T]) \to L^{1}([0,T])$, $\beta > 0$,

(iv) $D^{\beta}I^{\alpha} = I^{\alpha-\beta}$ and $D^{\beta}t^{\alpha} = (\Gamma(\beta+1)/\Gamma(\beta-\alpha+1))t^{\beta-\alpha}$ for $t \in [0,T]$, $\alpha - \beta > 0$.

More details on fractional derivatives and their properties can be found in [2, 4].

For the sake of convenience, we introduce the following assumptions:

(H_0)

$$\Delta = \frac{\Gamma(\alpha)T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \sum_{i=1}^{m-2} a_i\xi_i^{\alpha-1} - \sum_{i=1}^{m-2}\frac{b_i\Gamma(\alpha)\xi_i^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} > 0, \tag{2.3}$$

(H_1) $f \in \mathrm{Car}([0,+\infty)\times B)$, $B = (0,+\infty)\times(0,+\infty)$,

$$\begin{aligned} &\lim_{\|x\|\to 0}\|f(t,x,y)\| = +\infty, \quad \text{for a.e. } t \in [0,+\infty) \text{ and all } y \in P_{+}, \\ &\lim_{\|y\|\to 0}\|f(t,x,y)\| = +\infty, \quad \text{for a.e. } t \in [0,+\infty) \text{ and all } x \in P_{+}, \end{aligned} \tag{2.4}$$

and there exists a positive constant ϖ such that for all $T_0 \geq T$,

$$\|f(t,x,y)\| \geq \varpi\left(1-\frac{t}{T_0}\right)^{2+\beta-\alpha} \quad \text{for a.e. } t \in [0,T_0] \text{ and all } (x,y) \in B, \tag{2.5}$$

(H_2) f fulfills the estimate,

$$\begin{aligned} \|f(t,x,y)\| \leq \gamma(t)\left(\gamma_0(t) + q_1(\|x\|) + p_1(\|y\|) + q(\|x\|) + p(\|y\|)\right) \\ \text{for a.e. } t \in [0,+\infty), \text{ and all } (x,y) \in B, \end{aligned} \tag{2.6}$$

where $\gamma, \gamma_0 \in L^1[0,+\infty)$, $q_1, p_1, q, p \in C((0,+\infty), \mathbb{R}^+)$, q_1, p_1 are nonincreasing, and, for any $T_0 \geq T$,

$$\int_0^{T_0} \gamma(t) q_1\left(\frac{K_1\left(t^{\alpha-1}(T_0-t)^2\right)}{T_0}\right) dt < +\infty, \quad K_1 = \frac{\varpi}{2\Gamma(\alpha)},$$
$$\int_0^{T_0} \gamma(t) p_1\left(\frac{K_2\left(t^{\alpha-\beta-1}(T_0-t)^2\right)}{T_0}\right) dt < +\infty, \quad K_2 = \frac{\varpi}{2\Gamma(\alpha-\beta)}, \tag{2.7}$$

while q, p are nondecreasing and

$$\lim_{\|x\| \to +\infty} \frac{q(\|x\|) + p(\|x\|)}{\|x\|} = 0, \tag{2.8}$$

(H_3) for a.e. $t \in [0,+\infty)$, and for all $D \subset P$, $f(t, D, D)$ is relatively compact.

Rmark 2.3. It follows from (2.4) that under condition (H_2), $\lim_{\|x\| \to 0} q_1(\|x\|) = +\infty$ and $\lim_{\|y\| \to 0} p_1(\|y\|) = +\infty$.

In the sequel, $L^1([0,T], \mathbb{R})$ denote the Banach space of functions $y : [0,T] \to \mathbb{R}$ which are Lebesgue integrable with the norm

$$\|y\|_{L^1} = \int_0^T |y(t)| dt. \tag{2.9}$$

We give now some auxiliary lemmas in scalar space, which will take an important role throughout the paper.

Lemma 2.4. *Suppose that $h(t) \in L^1([0,T])$ and that (H_0) holds, then the unique solution of linear BVP $D^\alpha y(t) + h(t) = 0$, a.e. $t \in [0,T]$ with the boundary condition* (1.5) *is given by*

$$y(t) = \int_0^T G(t,s) h(s) ds + \frac{y_0}{\Delta} t^{\alpha-1}, \tag{2.10}$$

where

$$G(t,s) = \frac{1}{\Delta} \begin{cases} \dfrac{(T-s)^{\alpha-\beta-1} t^{\alpha-1}}{\Gamma(\alpha-\beta)} - \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} \displaystyle\sum_{j=i}^{m-2} a_j (\xi_j - s)^{\alpha-1} - \dfrac{t^{\alpha-1}}{\Gamma(\alpha-\beta)} \displaystyle\sum_{j=i}^{m-2} b_j (\xi_j - s)^{\alpha-\beta-1} - \dfrac{\Delta (t-s)^{\alpha-1}}{\Gamma(\alpha)}, \\ \qquad t \geq s, \quad \xi_{i-1} < s \leq \xi_i, \ i = 1, 2, \ldots, m-1, \\ \dfrac{(T-s)^{\alpha-\beta-1} t^{\alpha-1}}{\Gamma(\alpha-\beta)} - \dfrac{t^{\alpha-1}}{\Gamma(\alpha)} \displaystyle\sum_{j=i}^{m-2} a_j (\xi_j - s)^{\alpha-1} - \dfrac{t^{\alpha-1}}{\Gamma(\alpha-\beta)} \displaystyle\sum_{j=i}^{m-2} b_j (\xi_j - s)^{\alpha-\beta-1}, \\ \qquad t \leq s, \quad \xi_{i-1} < s \leq \xi_i, \ i = 1, 2, \ldots, m-1. \end{cases} \tag{2.11}$$

Proof. The proof is similar to that of [7, Lemma 2.2], so we omit it. □

Lemma 2.5. *Suppose that* (H_0) *holds, then* $G(t,s)$ *defined as* (2.11) *has the following properties:*

(i) $G(t,s)$ *is uniformly continuous about* t *in* $[0,T]$,

(ii) $G(t,s) \geq 0$ *for all* $(t,s) \in [0,T] \times [0,T]$ *and* $G(t,s) \leq E$, *where*

$$E = \frac{T^{2\alpha-\beta-2}}{\Delta\Gamma(\alpha-\beta)}, \tag{2.12}$$

(iii) $\int_0^T G(t,s)R(s)ds \geq t^{\alpha-1}(T-t)^2/2T\Gamma(\alpha)$ *if* $R(s) \geq (1-t/T)^{2+\beta-\alpha}$.

Proof. From (2.3), it is easy to verify (i) and (ii). We now show that (iii) is true. Firstly, if $t \geq s$, then (2.11) gives

$$G(t,s) = \frac{1}{\Delta}\begin{cases} \dfrac{(T-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha-\beta)} - \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}\displaystyle\sum_{j=i}^{m-2} a_j(\xi_j - s)^{\alpha-1} \\ -\dfrac{t^{\alpha-1}}{\Gamma(\alpha-\beta)}\displaystyle\sum_{j=i}^{m-2} b_j(\xi_j - s)^{\alpha-\beta-1} - \dfrac{\Delta(t-s)^{\alpha-1}}{\Gamma(\alpha)}, \\ \qquad (\xi_{i-1} < s \leq \xi_i,\ i = 1,2,\ldots,m-1). \end{cases} \tag{2.13}$$

Then,

$$\begin{aligned} G(t,s) &\geq \frac{1}{\Delta}\left\{\frac{(T-s)^{\alpha-\beta-1}t^{\alpha-1}}{\Gamma(\alpha-\beta)} - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\sum_{j=1}^{m-2} a_j(\xi_j - s)^{\alpha-1}\right. \\ &\qquad \left. -\frac{t^{\alpha-1}}{\Gamma(\alpha-\beta)}\sum_{j=1}^{m-2} b_j(\xi_j - s)^{\alpha-\beta-1} - \frac{\Delta(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right\} \\ &\geq \frac{t^{\alpha-1}}{\Delta}\left\{\frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-1}\left(1-\frac{s}{\xi_j}\right)^{\alpha-1}\right. \\ &\qquad \left. -\frac{1}{\Gamma(\alpha-\beta)}\sum_{j=1}^{m-2} b_j\xi_j^{\alpha-\beta-1}\left(1-\frac{s}{\xi_j}\right)^{\alpha-\beta-1} - \frac{\Delta(1-s/t)^{\alpha-1}}{\Gamma(\alpha)}\right\} \\ &\geq \frac{t^{\alpha-1}(1-s/T)^{\alpha-\beta-1}}{\Delta}\left\{\frac{T^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} - \frac{1}{\Gamma(\alpha-\beta)}\sum_{j=1}^{m-2} b_j\xi_j^{\alpha-\beta-1}\right. \\ &\qquad \left. -\frac{(1-s/T)^{\beta}}{\Gamma(\alpha)}\left(\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-1} + \Delta\right)\right\}. \end{aligned} \tag{2.14}$$

From (2.14) and (2.3), we deduce from the Lagrange mean value theorem that

$$\begin{aligned} G(t,s) &\geq \frac{t^{\alpha-1}(1-s/T)^{\alpha-\beta-1}\left(\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-1}+\Delta\right)}{\Delta\Gamma(\alpha)} \times \left(1-\left(1-\frac{s}{T}\right)^{\beta}\right) \\ &\geq \frac{t^{\alpha-1}(1-s/T)^{\alpha-\beta-1}\left(\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-1}+\Delta\right)}{\Delta\Gamma(\alpha)}\beta\xi^{\beta-1}\frac{s}{T}. \end{aligned} \tag{2.15}$$

In view of $(1-s/T) \leq \xi \leq 1$ and $\beta < 1$, one can obtain for $t \geq s$ that

$$G(t,s) \geq \frac{t^{\alpha-1}(1-s/T)^{\alpha-\beta-1}\left(\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-1}+\Delta\right)}{\Delta\Gamma(\alpha)T}\beta s. \tag{2.16}$$

Analogously, if $t \leq s$, one has

$$G(t,s) \geq \frac{t^{\alpha-1}(1-s/T)^{\alpha-\beta-1}}{\Gamma(\alpha)}. \tag{2.17}$$

It follows from (2.16) and (2.17) that

$$\begin{aligned} \int_0^T G(t,s)R(s)ds &= \int_0^t G(t,s)R(s)ds + \int_t^T G(t,s)R(s)ds \\ &\geq \int_0^t \frac{t^{\alpha-1}\left(\sum_{j=1}^{m-2} a_j\xi_j^{\alpha-1}+\Delta\right)}{\Delta\Gamma(\alpha)T}\beta s\left(1-\frac{s}{T}\right)ds + \int_t^T \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left(1-\frac{s}{T}\right)ds \\ &\geq \frac{t^{\alpha-1}(T-t)^2}{2T\Gamma(\alpha)}. \end{aligned} \tag{2.18}$$

The proof is complete. □

3. Existence Results for BVP (1.4)-(1.5)

In this section, we discuss the uniqueness, existence, and continuous dependence of positive solution for problem (1.4)-(1.5). To this end, we introduce some auxiliary technical lemmas.

Let $\mathbb{E} = \{x \in C([0,T],E) : D^{\beta}x \in C([0,T],E)\}$ equipped with the norm $\|x\|_* = \max\{\|x\|, \|D^{\beta}x\|\}$, then $\mathbb{E}$ is a real Banach space (see [24]).

Since the nonlinear term $f(t,x,y)$ is singular at $x=\theta$ and $y=\theta$, we use the following regularization process. For each $m \in N^+$, define f_m by the formula

$$f_m(t,x,y) = \begin{cases} f(t,x,y) & \text{if } x \geq \dfrac{c}{m},\ y \geq \dfrac{c}{m}, \\ f\left(t, \dfrac{c}{m}, y\right) & \text{if } 0 \leq x < \dfrac{c}{m},\ y \geq \dfrac{c}{m}, \\ f\left(t, x, \dfrac{c}{m}\right) & \text{if } x \geq \dfrac{1}{m},\ 0 \leq y < \dfrac{1}{m}, \\ f\left(t, \dfrac{c}{m}, \dfrac{c}{m}\right) & \text{if } 0 \leq x < \dfrac{c}{m}, 0 \leq y < \dfrac{c}{m}, \end{cases} \tag{3.1}$$

where $c > \theta$ is a given element of $\mathbb{E}$ and $\|c\| = 1$.

Rmark 3.1. The function f_m defined by (3.1) satisfies $f_m \in \mathrm{Car}([0,T] \times B_*)$, $B_* = P \times P$. And conditions (H_1) and (H_2) imply

$$\|f_m(t,x,y)\| \geq \varpi\left(1 - \frac{t}{T}\right)^{1+\beta-\alpha}, \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in B_*, \tag{3.2}$$

$$\|f_m(t,x,y)\| \leq \gamma(t)\left(\gamma_0(t) + q_1\left(\left\|\frac{1}{m}\right\|\right) + p_1\left(\left\|\frac{1}{m}\right\|\right) + q(1) + p(1) + q(\|x\|) + p(\|y\|)\right), \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in B_*, \tag{3.3}$$

$$\|f_m(t,x,y)\| \leq \gamma(t)(\gamma_0(t) + q_1(\|x\|) + p_1(\|y\|) + q(1) + p(1) + q(\|x\|) + p(\|y\|)), \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in B. \tag{3.4}$$

Rmark 3.2. The function f_m defined by (3.1) satisfies $\lim_{m\to+\infty} f_m = f$.

Define operator $Q_m : P_+ \to P_+$ by the following:

$$(Q_m y)(t) = \int_0^T G(t,s) f_m\left(s, y(s), D^\beta y(s)\right) ds + \frac{y_0}{\Delta} t^{(\alpha-1)}. \tag{3.5}$$

Lemma 3.3. *Suppose that* (H_0) *holds, then*

$$\left(D^\beta Q_m y\right)(t) = \int_0^T D^\beta G(t,s) f_m\left(s, y(s), D^\beta y(s)\right) ds + \frac{y_0 \Gamma(\alpha)}{\Delta \Gamma(\alpha-\beta)} t^{(\alpha-\beta-1)}, \tag{3.6}$$

where

$$D^{\beta}G(t,s)=\frac{\Gamma(\alpha)}{\Delta\Gamma(\alpha-\beta)}\begin{cases}\dfrac{(T-s)^{\alpha-\beta-1}t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}-\dfrac{t^{\alpha-\beta-1}}{\Gamma(\alpha)}\displaystyle\sum_{j=i}^{m-2}a_j(\xi_j-s)^{\alpha-1}\\ \quad-\dfrac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\displaystyle\sum_{j=i}^{m-2}b_j(\xi_j-s)^{\alpha-\beta-1}-\dfrac{\Delta(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha)},\\ \qquad t\geq s,\quad \xi_{i-1}<s\leq\xi_i,\ i=1,2,\ldots,m-1,\\ \dfrac{(T-s)^{\alpha-\beta-1}t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}-\dfrac{t^{\alpha-\beta-1}}{\Gamma(\alpha)}\displaystyle\sum_{j=i}^{m-2}a_j(\xi_j-s)^{\alpha-1}\\ \quad-\dfrac{t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\displaystyle\sum_{j=i}^{m-2}b_j(\xi_j-s)^{\alpha-\beta-1},\\ \qquad t\leq s,\quad \xi_{i-1}<s\leq\xi_i,\ i=1,2,\ldots,m-1.\end{cases}\tag{3.7}$$

Proof. The proof is similar to that of [7, Lemma 2.2], and we omit it. □

Lemma 3.4. *Suppose that* (H_0) *holds, then* $D^{\beta}G(t,s)$ *defined as* (3.7) *has the following properties:*

(i) $D^{\beta}G(t,s)$ *is uniformly continuous about* t *in* $[0,T]$,

(ii) $D^{\beta}G(t,s)\geq 0$ *for all* $(t,s)\in[0,T]\times[0,T]$ *and* $D^{\beta}G(t,s)\leq E_D$, *where*

$$E_D=\frac{\Gamma(\alpha)T^{2\alpha-2\beta-2}}{\Delta\left[\Gamma(\alpha-\beta)\right]^2},\tag{3.8}$$

(iii) $\int_0^T D^{\beta}G(t,s)R(s)ds\geq t^{\alpha-\beta-1}(T-t)^2/2T\Gamma(\alpha-\beta)$ *if* $R(s)\geq(1-t/T)^{2+\beta-\alpha}$.

Proof. The proof of this Lemma is similar to that of Lemma 2.5. Hence it is omitted. □

Lemma 3.5. *Suppose that* (H_0) *and following condition* (H_4) *hold:*

(H_4) *there exist positive constants* L, L_D *such that*

$$\left\|f\left(s,x(s),D^{\beta}x(s)\right)-f\left(s,y(s),\ D^{\beta}y(s)\right)\right\|\leq L\|x-y\|+L_D\left\|D^{\beta}x-D^{\beta}y\right\|\tag{3.9}$$

and $\tau=\max\{E(L+L_D),E_D(L+L_D)\}<1$, *then* Q_m *has a unique fixed point.*

Proof. Obviously, f_m defined by the formula (3.1) satisfy also the condition (H_4). By (3.5) and (3.6), it is easy to show that $\|Q_mx-Q_my\|_*<\tau\|x-y\|_*$, then Banach contraction principle implies that the operator Q_m has a unique fixed point, which completes this proof. □

The following fixed-point result of cone compression type is due to Krasnosel'skii, which is fundamental to establish another auxiliary existence result (Lemma 3.8).

Lemma 3.6 (see, e.g., [23, 25]). *Let Y be a Banach space, and let $P \subset Y$ be a cone in Y. Let Ω_1, Ω_2 be bounded open balls of Y centered at the origin with $\overline{\Omega_1} \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator such that*

$$\|Ax\| \geq \|x\| \quad \text{for } x \in P \cap \partial\Omega_1, \qquad \|Ax\| \leq \|x\| \quad \text{for } x \in P \cap \partial\Omega_2 \tag{3.10}$$

hold, then A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 3.7. *Let (H_0)–(H_3) hold, then $Q_m : P \to P$ and Q_m is a completely continuous operator.*

Proof. Firstly, let $y \in P$, because $f_m \in \mathrm{Car}([0,T] \times B_*)$ is positive. It follows from Lemma 2.5 (i) and (ii) that $Q_m y \in C([0,T], \mathbb{E})$ and $(Q_m y)(t) \geq \theta$ for $t \in [0,T]$. Similarly, from Lemma 3.4 (i) and (ii) we can get that $D^\beta Q_m y \in C([0,T], \mathbb{E})$ and $(D^\beta Q_m y)(t) \geq \theta$ for $t \in [0,T]$. To summarize, $Q_m : P \to P$.

Secondly, we prove that Q_m is a continuous operator. Let $\{x_k\} \subset P$ be a convergent sequence and $\lim_{k\to+\infty}\|x_k - x\|_* = 0$, then $x \subset P$ and $\|x_k\|_* \leq S$, where S is a positive constant. In view of $f_m \in \mathrm{Car}([0,T] \times B_*)$, we have $\lim_{k\to+\infty} f_m(t, x_k(t), D^\beta x_k(t)) = f_m(t, x(t), D^\beta x(t))$. Since by (3.2), (3.3),

$$\begin{aligned} 0 &< \left\| f_m\left(t, x_k(t), D^\beta x_k(t)\right) \right\| \\ &\leq \gamma(t)\left(\gamma_0(t) + q_1\left(\frac{1}{m}\right) + p_1\left(\frac{1}{m}\right) + q(1) + p(1) + q(S) + p(S)\right), \end{aligned} \tag{3.11}$$

the Lebesgue dominated convergence theorem gives

$$\lim_{k\to+\infty} \int_0^T \left\| f_m\left(t, x_k(t), D^\beta x_k(t)\right) - f_m\left(t, x(t), D^\beta x(t)\right) \right\| dt = 0. \tag{3.12}$$

Now, from (3.12), Lemma 2.5(ii), Lemma 3.4(ii) and from the inequalities (cf. (3.5), (3.6))

$$\begin{aligned} &\|(Q_m x_k)(t) - (Q_m x)(t)\| \\ &\quad \leq E \int_0^T \left\| f_m\left(t, x_k(t), D^\beta x_k(t)\right) - f_m\left(t, x(t), D^\beta x(t)\right) \right\| dt, \\ &\left\| \left(D^\beta Q_m x_k\right)(t) - \left(D^\beta Q_m x\right)(t) \right\| \\ &\quad \leq E_D \int_0^T \left\| f_m\left(t, x_k(t), D^\beta x_k(t)\right) - f_m\left(t, x(t), D^\beta x(t)\right) \right\| dt, \end{aligned} \tag{3.13}$$

we have that $\lim_{k\to+\infty}\|Q_m x_k - Q_m x\|_* = 0$, which proves that Q_m is a continuous operator.

Thirdly, let $\Omega \subset P$ be bounded in $\mathbb{E}$ and let $\|x\|_* \leq L$ for all $x \in \Omega$, where L is a positive constant. We are in position to prove that $Q_m(\Omega)$ is bounded. Keeping in mind $f_m \in \mathrm{Car}([0,T] \times B_*)$, there exists $\phi \in L^1([0,T])$ such that

$$0 < \left\| f_m\left(t, x_k(t), D^\beta x_k(t)\right) \right\| \leq \phi(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x \in \Omega, \tag{3.14}$$

then (cf. (3.5))

$$\begin{aligned} \|(Q_m x)(t)\| &\leq E \int_0^T \left\| f_m\left(s, x(s), D^\beta x(s)\right) \right\| ds + \frac{y_0}{\Delta} t^{(\alpha-1)} \\ &\leq E\|\phi\|_{L^1} + \frac{\|y_0\|}{\Delta} T^{(\alpha-1)}, \end{aligned} \tag{3.15}$$

and (cf. (3.6))

$$\begin{aligned} \left\| \left(D^\beta Q_m x\right)(t) \right\| &\leq E_D \int_0^T \left\| f_m\left(s, x(s), D^\beta x(s)\right) \right\| ds + \left\| \frac{y_0 \Gamma(\alpha)}{\Delta \Gamma(\alpha-\beta)} t^{(\alpha-\beta-1)} \right\| \\ &\leq E_D \|\phi\|_{L^1} + \frac{\|y_0\| \Gamma(\alpha)}{\Delta \Gamma(\alpha-\beta)} T^{(\alpha-\beta-1)}, \end{aligned} \tag{3.16}$$

for $t \in [0,T]$ and all $x \in \Omega$. Therefore, $Q_m(\Omega)$ is bounded in $\mathbb{E}$.

Fourthly, by (H_3) and (3.5), it is easy to show that $Q_m(\Omega)(t)$ is relatively compact.

Finally, let $0 \leq t_1 < t_2 \leq T$. From Lemma 2.5(i) and the functions $t^{\alpha-1}, t^{\alpha-\beta-1}$ being uniformly continuous on $[0,T]$, for any arbitrary $\epsilon > 0$, there exists a positive number $\delta(\epsilon)$, such that when $|t_1 - t_2| < \delta(\epsilon)$, one has $|G_T(t_1,s) - G_T(t_2,s)| < \epsilon$ and $|t_1^{\alpha-1} - t_2^{\alpha-1}| < \epsilon$, then (cf. (3.14)) the inequality

$$\|(Q_m x)(t_1) - (Q_m x)(t_2)\| < \epsilon \|\phi\|_{L^1} + \frac{\|y_0\|}{\Delta} \epsilon \tag{3.17}$$

holds. Hence the set of functions $Q_m(\Omega)$ is equicontinuous on $[0,T]$.

Therefore, by the Arzelá-Ascoli theorem, $Q_m(\Omega)$ is relatively compact in $\mathbb{E}$. We have proved that Q_m is a completely continuous operator. □

Lemma 3.8. *Suppose that (H_0)–(H_3) hold, then the operator Q_m has at least a fixed point.*

Proof. By Lemma 3.7, $Q_m : P \to P$ is completely continuous. In order to apply Lemma 3.6, we construct two bounded open balls Ω_1, Ω_2 and prove that the conditions (3.10) are satisfied with respect to Q_m.

Firstly, let $\Omega_1 = \{y \in \mathbb{E} : \|y\|_* < r\}$, where $r = \sup_{t\in[0,T]} K_1(t^{\alpha-1}(T-t)^2)/T$ and K_1 is defined as in (H_2). It follows from Lemma 2.5, (H_1), Remark 3.1 and from the definition of Q_m that $\|(Q_m y)(t)\| \geq K_1(t^{\alpha-1}(T-t)^2/T)$. Then $\|Q_m y\| \geq r$. Immediately:

$$\|Q_m y\|_* \geq \|y\|_*, \quad \text{for } y \in P \cap \partial\Omega_1. \tag{3.18}$$

Secondly, (3.3) and Lemma 2.5(ii) imply that, for $x \in P$,

$$\begin{aligned}
\|(Q_m y)(t)\| &\le E\int_0^T \left\| f_m\left(s, y(s), D^\beta y(s)\right)\right\| ds + \frac{\|y_0\|}{\Delta} T^{(\alpha-1)} \\
&\le E\int_0^T \gamma(t)\left[\gamma_0(t) + q_1\left(\frac{1}{m}\right) + p_1\left(\frac{1}{m}\right) + q(1) + p(1)\right. \\
&\qquad \left. + q(y(s)) + p\left(D^\beta y(s)\right)\right] ds + \frac{\|y_0\|}{\Delta} T^{(\alpha-1)} \\
&\le E\int_0^T \gamma(t)\left[\gamma_0(t) + q_1\left(\frac{1}{m}\right) + p_1\left(\frac{1}{m}\right) + q(1) + p(1)\right. \\
&\qquad \left. + q(\|y\|) + p\left(\left\|D^\beta y\right\|\right)\right] ds + \frac{\|y_0\|}{\Delta} T^{(\alpha-1)} \\
&\le E\left\{\|\gamma\gamma_0\|_{L^1} + \left[q_1\left(\frac{1}{m}\right) + p_1\left(\frac{1}{m}\right) + q(1) + p(1)\right.\right. \\
&\qquad \left.\left. + q(\|y\|) + p\left(\left\|D^\beta y\right\|\right)\right]\|\gamma\|_{L^1}\right\} + \frac{\|y_0\|}{\Delta} T^{(\alpha-1)},
\end{aligned} \tag{3.19}$$

because q, p are nondecreasing as stated in (H_2). Analogously, by (3.3) and Lemma 3.4(ii), one can get that for $x \in P$

$$\begin{aligned}
\left\|\left(D^\beta Q_m y\right)(t)\right\| &\le E_D\left\{\|\gamma\gamma_0\|_{L^1} + \left[q_1\left(\frac{1}{m}\right) + p_1\left(\frac{1}{m}\right) + q(1) + p(1) + q(\|y\|) + p\left(\left\|D^\beta y\right\|\right)\right]\right. \\
&\qquad \left. \times\|\gamma\|_{L^1}\right\} + \frac{\|y_0\|\Gamma(\alpha)}{\Delta\Gamma(\alpha-\beta)} T^{(\alpha-\beta-1)}.
\end{aligned} \tag{3.20}$$

Let $W_1 = \max\{E, E_D\}$ and $W_2 = \max\{(\|y_0\|/\Delta)T^{(\alpha-1)}, (\|y_0\|\Gamma(\alpha)/(\Delta\Gamma(\alpha-\beta)))T^{(\alpha-\beta-1)}\}$. Hence for $x \in P$, we have the following inequality

$$\begin{aligned}
\|Q_m y\|_* &\le W_1\left\{\|\gamma\gamma_0\|_{L^1} + \left[q_1\left(\frac{1}{m}\right) + p_1\left(\frac{1}{m}\right) + q(1) + p(1) + q(\|y\|_*) + p(\|y\|_*)\right]\right. \\
&\qquad \left. \times\|\gamma\|_{L^1}\right\} + W_2.
\end{aligned} \tag{3.21}$$

Since $\lim_{\|x\|\to\infty}(q(\|x\|) + p(\|x\|))/\|x\| = 0$ by (H_2), there exists a sufficiently large number $R > r$ such that

$$W_1\left\{\|\gamma\gamma_0\|_{L^1} + \left[q_1\left(\frac{1}{m}\right) + p_1\left(\frac{1}{m}\right) + q(1) + p(1) + q(R) + p(R)\right]\|\gamma\|_{L^1}\right\} + W_2 \le R. \tag{3.22}$$

Let $\Omega_2 = \{y \in \mathbb{E} : \|y\|_* < R\}$, then (cf. (3.21) and (3.22))

$$\|Q_m y\|_* \leq \|y\|_*, \quad \text{for } y \in P \cap \partial\Omega_2. \tag{3.23}$$

Applying Lemma 3.6, we conclude from (3.18) and (3.23) that Q_m has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$. □

Lemma 3.9. *Suppose that (H_0)–(H_3) hold, then the sequences $\{y_m\}_{m=1}^{\infty}$ and $\{D^\beta y_m\}_{m=1}^{\infty}$ are relatively compact in $C([0,T])$, where y_m be a fixed point of operator Q_m defined by* (3.5).

Proof. Let y_m be a fixed point of operator Q_m, that is,

$$\begin{aligned} y_m(t) = &\int_0^T G(t,s) f_m\Big(s, y_m(s), D^\beta y_m(s)\Big) ds \\ &+ \frac{y_0}{\Delta} t^{(\alpha-1)}, \quad t \in [0,T],\ m \in N. \end{aligned} \tag{3.24}$$

And consider (cf. (3.6))

$$\begin{aligned} D^\beta y_m(t) = &\int_0^T D^\beta G(t,s) f_m\Big(s, y_m(s), D^\beta y_m(s)\Big) ds \\ &+ \frac{y_0 \Gamma(\alpha)}{\Delta \Gamma(\alpha-\beta)} t^{(\alpha-\beta-1)}, \quad t \in [0,T],\ m \in N. \end{aligned} \tag{3.25}$$

By Lemma 2.5(iii), Lemma 3.4(iii), and Remark 3.1, we have also

$$\begin{aligned} \|y_m(t)\| &\geq K_1 \frac{t^{\alpha-1}(T-t)^2}{T} + \frac{y_0 \Gamma(\alpha)}{\Delta \Gamma(\alpha-\beta)} t^{(\alpha-\beta-1)} \\ &\geq K_1 \frac{t^{\alpha-1}(T-t)^2}{T}, \quad t \in [0,T],\ m \in N, \end{aligned} \tag{3.26}$$

$$\left\|D^\beta y_m(t)\right\| \geq K_2 \frac{t^{\alpha-\beta-1}(T-t)^2}{T}, \quad t \in [0,T],\ m \in N. \tag{3.27}$$

Hence (cf. (3.4)),

$$\begin{aligned} \left\|f_m\Big(t, y_m(t), D^\beta y_m(t)\Big)\right\| \leq \gamma(t) \Bigg\{ &\gamma_0(t) + q_1\left(K_1 \frac{t^{\alpha-1}(T-t)^2}{T}\right) + p_1\left(K_2 \frac{t^{\alpha-\beta-1}(T-t)^2}{T}\right) \\ &+ q(1) + p(1) + q(y_m(t)) + p\Big(D^\beta y_m(t)\Big) \Bigg\}, \end{aligned} \tag{3.28}$$

for a.e. $t \in [0,T]$, and all $m \in N$. Therefore, by (3.26), (3.27), Lemma 2.5(ii), Lemma 3.4(ii), and Remark 3.1,

$$\|y_m(t)\| \le E\left\{\|\gamma\gamma_0\|_{L^1} + U_1 + U_2 + \left[q(1) + p(1) + q(\|y_m\|) + p\left(\left\|D^\beta y_m\right\|\right)\right]\|\gamma\|_{L^1}\right\} + \frac{y_0}{\Delta}T^{(\alpha-1)}, \tag{3.29}$$

$$\left\|D^\beta y_m(t)\right\| \le E_D\left\{\|\gamma\gamma_0\|_{L^1} + U_1 + U_2 + \left[q(1) + p(1) + q(\|y_m\|) + p\left(\left\|D^\beta y_m\right\|\right)\right]\|\gamma\|_{L^1}\right\} + \frac{y_0\Gamma(\alpha)}{\Delta\Gamma(\alpha-\beta)}T^{(\alpha-\beta-1)}, \tag{3.30}$$

for $t \in [0,T]$, $m \in N$, where

$$U_1 = \int_0^T \gamma(t)q_1\left(K_1\frac{t^{\alpha-1}(T-t)^2}{T}\right)dt < +\infty, \tag{3.31}$$

$$U_2 = \int_0^T \gamma(t)p_1\left(K_2\frac{t^{\alpha-\beta-1}(T-t)^2}{T}\right)dt < +\infty. \tag{3.32}$$

In particular,

$$\|y_m\|_* \le W_1\{\|\gamma\gamma_0\|_{L^1} + U_1 + U_2 + [q(1) + p(1) + q(\|y_m\|_*) + p(\|y_m\|_*)]\|\gamma\|_{L^1}\} + W_2, \quad \forall m \in N, \tag{3.33}$$

where W_1, W_2 are defined in the proof of Lemma 3.8. Since $\lim_{x\to+\infty}(q(x) + p(x))/x = 0$ by (H_2), there exists a constant $W > 0$ such that for each $x > W$,

$$W_1\{\|\gamma\gamma_0\|_{L^1} + U_1 + U_2 + [q(1) + p(1) + q(x) + p(x)]\|\gamma\|_{L^1}\} + W_2 < x. \tag{3.34}$$

Immediately, (cf. (3.33))

$$\|y_m\|_* \le W, \quad \forall m \in N. \tag{3.35}$$

Hence, the sequences $\{y_m\}_{m=1}^\infty$ and $\{D^\beta y_m\}_{m=1}^\infty$ are uniformly bounded.

We will take similar discussions as in Lemma 3.7 to show that $\{y_m\}_{m=1}^{\infty}$ and $\{D^{\beta}y_m\}_{m=1}^{\infty}$ are equicontinuous on $[0,T]$. Let $0 \le t_1 < t_2 \le T$, then we have

$$\begin{aligned}
&\left\|(y_m)(t_1)-(y_m)(t_2)\right\| \\
&\quad \le \int_0^T |G_T(t_1,s)-G_T(t_2,s)|\left\|f_m\left(s,x(s),D^{\beta}x(s)\right)\right\|ds + \frac{\|y_0\|}{\Delta}\left|t_1^{\alpha-1}-t_2^{\alpha-1}\right|, \\
&\left\|\left(D^{\beta}y_m\right)(t_1)-\left(D^{\beta}y_m\right)(t_2)\right\| \\
&\quad \le \int_0^T \left|D^{\beta}G_T(t_1,s)-D^{\beta}G_T(t_2,s)\right|\left\|f_m\left(s,x(s),D^{\beta}x(s)\right)\right\|ds \\
&\qquad + \frac{\|y_0\|\Gamma(\alpha)}{\Delta\Gamma(\alpha-\beta)}\left|t_1^{\alpha-\beta-1}-t_2^{\alpha-\beta-1}\right|.
\end{aligned} \tag{3.36}$$

Using (3.28), (3.35), one can get

$$\begin{aligned}
0 &< \left\|f_m\left(t,y_m(t),D^{\beta}y_m(t)\right)\right\| \\
&\le \gamma(t)\left\{\gamma_0(t)+q_1\left(K_1\frac{t^{\alpha-1}(T-t)^2}{T}\right)+p_1\left(K_2\frac{t^{\alpha-\beta-1}(T-t)^2}{T}\right)\right. \\
&\qquad \left. + q(1)+p(1)+q(W)+p(W)\right\}.
\end{aligned} \tag{3.37}$$

From Lemma 2.5 (i), Lemma 3.4 (i), and the functions $t^{\alpha-1}$, $t^{\alpha-\beta-1}$ being uniformly continuous on $[0,T]$, choosing an arbitrary $\epsilon > 0$, there exists a positive number $\delta(\epsilon)$. When $|t_1-t_2| < \delta(\epsilon)$, we can get $|G_T(t_1,s)-G_T(t_2,s)| < \epsilon$, $|D^{\beta}G_T(t_1,s)-D^{\beta}G_T(t_2,s)| < \epsilon$, $|t_1^{\alpha-1}-t_2^{\alpha-1}| < \epsilon$, and $|t_1^{\alpha-\beta-1}-t_2^{\alpha-\beta-1}| < \epsilon$. Therefore (cf. (3.36) and (3.37)) the inequalities

$$\begin{aligned}
&\left\|(y_m)(t_1)-(y_m)(t_2)\right\| \\
&\quad < \epsilon\left\{\|\gamma\gamma_0\|_{L^1}+U_1+U_2+\left[q(1)+p(1)+q(W)+p(W)\right]\|\gamma\|_{L^1}\right\}+\frac{\|y_0\|}{\Delta}\epsilon, \\
&\left\|\left(D^{\beta}y_m\right)(t_1)-\left(D^{\beta}y_m\right)(t_2)\right\| \\
&\quad < \epsilon\left\{\|\gamma\gamma_0\|_{L^1}+U_1+U_2+\left[q(1)+p(1)+q(W)+p(W)\right]\|\gamma\|_{L^1}\right\}+\frac{\|y_0\|\Gamma(\alpha)}{\Delta\Gamma(\alpha-\beta)}\epsilon,
\end{aligned} \tag{3.38}$$

hold, where U_1, U_2 are defined as (3.31) and (3.32), respectively. As a result, $\{y_m\}_{m=1}^{\infty}$ and $\{D^{\beta}y_m\}_{m=1}^{\infty}$ are equicontinuous on $[0,T]$.

Finally, we prove that $\{y_m(t)\}_{m=1}^{\infty}$ and $\{D^{\beta}y_m(t)\}_{m=1}^{\infty}$ are relatively compact. Because E is a Banach space, we need only to show that $\{y_m(t)\}_{m=1}^{\infty}$ and $\{D^{\beta}y_m(t)\}_{m=1}^{\infty}$ are completely

bounded. For all $\varepsilon > 0$, by the Remark 3.2, there exists a sufficiently large positive integer N, such that if $m > N$,

$$\left\| f_m\left(t, y(t), D^\beta y(t)\right) - f\left(t, y(t), D^\beta y(t)\right)\right\| < \frac{\varepsilon}{\widetilde{E}}, \quad \text{a.e. } t \in [0,T], \tag{3.39}$$

where $\widetilde{E} = \max\{E, E_D\}$.

Hence, by (3.5) and (3.6), we have $\|(Q_m y)(t) - Q_0\| < \varepsilon$ and $\|D^\beta(Q_m y)(t) - D^\beta Q_0\| < \varepsilon$, for $m > N$, where

$$\begin{aligned} Q_0 &= \int_0^T G(t,s) f\left(s, y(s), D^\beta y(s)\right) ds + \frac{y_0}{\Delta} t^{(\alpha-1)}, \\ D^\beta Q_0 &= \int_0^T D^\beta G(t,s) f\left(s, y(s), D^\beta y(s)\right) ds + \frac{y_0 \Gamma(\alpha)}{\Delta \Gamma(\alpha-\beta)} t^{(\alpha-\beta-1)}. \end{aligned} \tag{3.40}$$

This implies that $\{y_m(t)\}_{m=1}^\infty$ and $\{D^\beta y_m(t)\}_{m=1}^\infty$ have an ε-net constituted by finite elements ($\{y_1(t), y_2(t), y_N(t), Q_0\}$ and $\{D^\beta y_1(t), D^\beta y_2(t), D^\beta y_N(t), D^\beta Q_0\}$, resp.) of E, that is, completely bounded.

Therefore, $\{y_m\}_{m=1}^\infty$ and $\{D^\beta y_m\}_{m=1}^\infty$ are relatively compact in $C([0,T])$ by the Arzelá-Ascoli theorem.

Using above results, we now give the existence of positive solution of singular problem (1.4)-(1.5). □

Theorem 3.10. *Suppose that* (H_0)–(H_3) *hold, then problem* (1.4)-(1.5) *has a positive solution* y *and*

$$\begin{aligned} \|y(t)\| &\geq K_1 \frac{t^{\alpha-1}(T-t)^2}{T}, \\ \left\|D^\beta y(t)\right\| &\geq K_2 \frac{t^{\alpha-\beta-1}(T-t)^2}{T}, \quad t \in [0,T]. \end{aligned} \tag{3.41}$$

Moreover, y *is continuous and* $\|y\|_* \leq W$, *where* W *is a constant as in* (3.35).

Proof. From Lemmas 3.8 and 3.9, the operator Q_m has a fixed point y_m satisfying (3.26), (3.27), (3.35). And $\{y_m\}_{m=1}^\infty$ and $\{D^\beta y_m\}_{m=1}^\infty$ are relatively compact in $C([0,T])$. Hence, $\{y_m\}_{m=1}^\infty$ is relatively compact in $\mathbb{E}$. And therefore, there exist $y \in \mathbb{E}$ and a subsequence y_{m_k} of $\{y_m\}_{m=1}^\infty$ such that $\lim_{k\to\infty} y_{m_k} = y$ in $\mathbb{E}$. Consequently, y is positive and continuous. Moreover y satisfies (3.47), $\|y\|_* \leq W$. And

$$\lim_{k\to\infty} f_{m_k}\left(t, y_{m_k}(t), D^\beta y_{m_k}(t)\right) = f\left(t, y(t), D^\beta y(t)\right), \quad \text{for a.e. } t \in [0,T]. \tag{3.42}$$

Keeping in mind (3.35) holding, where W is a positive constant, it follows from inequalities (3.4) and (3.26) and from Lemma 2.5(ii) that

$$\begin{aligned} 0 &\leq \left\| G(t,s) f_m\left(s, y_m(s), D^\beta y_m(s)\right)\right\| \\ &\leq E\gamma(s)\left\{\gamma_0(s) + q_1\left(K_1 \frac{s^{\alpha-1}(T-s)^2}{T}\right) + p_1\left(K_2 \frac{s^{\alpha-\beta-1}(T-s)^2}{T}\right)\right. \\ &\qquad \left. + q(1) + p(1) + q(W) + p(W)\right\}, \end{aligned} \tag{3.43}$$

for a.e. $s \in [0,T]$ and all $t \in [0,T]$, $m \in N$. Hence, by the Lebesgue dominated convergence theorem, we have

$$\lim_{k\to+\infty} \int_0^T G(t,s) f_m\left(s, y_m(s), D^\beta y_m(s)\right) ds = \int_0^T G(t,s) f\left(s, y(s), D^\beta y(s)\right) ds, \tag{3.44}$$

for $t \in [0,T]$. Now, passing to the limit as $k \to +\infty$ in

$$y_{m_k}(t) = \int_0^T G(t,s) f_{m_k}\left(s, y_{m_k}(s), D^\beta y_{m_k}(s)\right) ds + \frac{y_0}{\Delta} t^{(\alpha-1)}, \tag{3.45}$$

we have

$$y(t) = \int_0^T G(t,s) f\left(s, y(s), D^\beta y(s)\right) ds + \frac{y_0}{\Delta} t^{(\alpha-1)}, \quad \text{for } t \in [0,T]. \tag{3.46}$$

Consequently, y is a positive solution of BVP (1.4)-(1.5) by Lemma 2.4. □

By Lemmas 3.5 and 3.9, and Theorem 3.10, we give the following unique result without proof.

Theorem 3.11. *Suppose that* (H_0)–(H_4) *hold, then problem* (1.4)-(1.5) *has a unique positive solution* y *and*

$$\begin{gathered} \|y(t)\| \geq K_1 \frac{t^{\alpha-1}(T-t)^2}{T}, \\ \left\|D^\beta y(t)\right\| \geq K_2 \frac{t^{\alpha-\beta-1}(T-t)^2}{T}, \quad t \in [0,T]. \end{gathered} \tag{3.47}$$

Moreover, y *is continuous and* $\|y\|_* \leq W$*, where* W *is a constant as in* (3.35).

4. Existence Results for BVP (1.6)-(1.7)

We now give the existence of positive solution of BVP (1.6)-(1.7) by using diagonalization process.

Theorem 4.1. *Suppose that* (H_0)–(H_3) *hold, then BVP* (1.6)-(1.7) *has a positive solution* y, *and* $D^{\beta}y$ *is also positive.*

Proof. Firstly, choose

$$\begin{gathered}\{T_n\}_{n=1}^{\infty} \in N^* \text{ to be a subsequence of numbers,} \\ \text{such that } T \le T_1 < T_2 < \cdots < T_n < \cdots \uparrow \infty,\end{gathered} \tag{4.1}$$

then consider the BVP,

$$D^{\alpha}y(t) + f\left(t, y(t), D^{\beta}y(t)\right) = \theta, \quad \text{a.e. } t \in [0, T_n], \tag{4.2}$$

subject to

$$y(0) = \theta, \quad D^{\beta}y(T_n) - \sum_{i=1}^{m-2} a_i y(\xi_i) - \sum_{i=1}^{m-2} b_i D^{\beta}y(\xi_i) = y_0. \tag{4.3}$$

Theorem 3.10 guarantees that BVP (4.2)-(4.3) has a positive continuous solution y_n. And for any $n \in N$,

$$\|y_n\|_* \le W^n, \quad \text{for } t \in [0, T_n], \tag{4.4}$$

where W^n is a constant defined similarly to W.

Secondly, we apply the following diagonalization process. For $n \in N$, let

$$u_n(t) = \begin{cases} y_n(t), & t \in [0, T_n], \\ y_n(T_n), & t \in [T_n, +\infty). \end{cases} \tag{4.5}$$

Here, $\{T_n\}_{n=1}^{\infty}$ is defined in (4.1). Notice that $u_n(t) \in C[0, +\infty)$ with

$$0 \le \|u_n(t)\| \le W^1, \qquad 0 \le \left\|D^{\beta}u_n(t)\right\| \le W^1, \quad \text{for } t \in [0, T_1]. \tag{4.6}$$

Also for $n \in N$ and $t \in [0, T_1]$, we get

$$u_n(t) = \int_0^{T_1} G_{T_1}(t, s) f\left(s, u_n(t), D^{\beta}u_n(t)\right) ds + \frac{y_0}{\Delta} t^{\alpha-1}, \tag{4.7}$$

where $G_{T_n}(t,s)$ are similarly defined as in (2.11), but all of T should be replaced by T_n. Then for $t_1, t_2 \in [0, T_1]$, we have

$$\begin{aligned}
\|u_n(t_1) - u_n(t_2)\| &\le \int_0^{T_1} |G_{T_1}(t_1,s) - G_{T_1}(t_2,s)| \left\| f\left(s, u_{n_k}(t), D^\beta u_{n_k}(t)\right)\right\| ds \\
&\quad + \frac{y_0}{\Delta}\left|t_1^{\alpha-1} - t_2^{\alpha-1}\right|, \\
\left\|D^\beta u_n(t_1) - D^\beta u_n(t_2)\right\| &\le \int_0^{T_1} \left|D^\beta G_{T_1}(t_1,s) - D^\beta G_{T_1}(t_2,s)\right| \times \left\| f\left(s, u_{n_k}(t), D^\beta u_{n_k}(t)\right)\right\| ds \\
&\quad + \frac{\|y_0\|\Gamma(\alpha)}{\Delta\Gamma(\alpha-\beta)}\left|t_1^{\alpha-\beta-1} - t_2^{\alpha-\beta-1}\right|.
\end{aligned} \tag{4.8}$$

Thus, when $|t_1 - t_2| < \delta(\epsilon, 1)$, similarly to (3.38),

$$\begin{aligned}
&\|u_n(t_1) - u_n(t_2)\| \\
&\quad < \epsilon\left\{\|\gamma\gamma_0\|_{L^1} + U_1^1 + U_2^1 + \left[q(1) + p(1) + q\left(W^1\right) + p\left(W^1\right)\right]\|\gamma\|_{L^1}\right\} + \frac{\|y_0\|}{\Delta}\epsilon, \\
&\left\|D^\beta u_n(t_1) - D^\beta u_n(t_2)\right\| \\
&\quad < \epsilon\left\{\|\gamma\gamma_0\|_{L^1} + U_1^1 + U_2^1 + \left[q(1) + p(1) + q\left(W^1\right) + p\left(W^1\right)\right]\|\gamma\|_{L^1}\right\} + \frac{\|y_0\|\Gamma(\alpha)}{\Delta\Gamma(\alpha-\beta)}\epsilon
\end{aligned} \tag{4.9}$$

hold for an arbitrary $\epsilon > 0$, where $\delta(\epsilon, 1)$ is a suitable positive number and U_1^1, U_2^1 are defined similarly to U_1, U_2 as in (3.31) and (3.32), respectively. By using (H_3), we know that, for a.e. $t \in [0, +\infty)$, $f(t, D_{W^1}, D_{W^1})$ is relatively compact, where $D_{W^1} = \{u \in C([0,T_1] : \|u\|_* \le W^1\} \cap P$. Therefore, $\{u_n(t)\}_{n=1}^\infty$ and $\{D^\beta u_n(t)\}_{n=1}^\infty$ are relatively compact. The Arzelá-Ascoli theorem guarantees that there is a subsequence N_1^* of N and a function $z_1 \in C([0,T_1], E)$ with $u_{n_k} \to z_1$ in $C([0,T_1], E)$ as $k \to +\infty$ through N_1^*. Obviously, z_1 is positive. Let $N_1 = N_1^* \setminus \{1\}$, noticing that

$$0 \le \|u_n(t)\| \le W^2, \qquad 0 \le \left\|D^\beta u_n(t)\right\| \le W^2, \quad \text{for } t \in [0, T_2]. \tag{4.10}$$

Similarly to above argumentation, we have that there is a subsequence N_2^* of N_1 and a function $z_2 \in C([0,T_2], E)$ with $u_{n_k} \to z_2$ in $C([0,T_2], E)$ as $k \to +\infty$ through N_2^*. Obviously, z_2 is positive. Note that $z_1 = z_2$ on $[0, T_1]$ since $N_2^* \subset N_1$. Let $N_2 = N_2^* \setminus \{2\}$. Proceed inductively to obtain for $m = \{2, 3, \ldots\}$ a subsequence N_m^* of N_{m-1} and a function $z_m \in C([0,T_m], E)$ with $u_{n_k} \to z_m$ in $C([0,T_m], E)$ as $k \to +\infty$ through N_m^*. Also, z_m is positive. Let $N_m = N_m^* \setminus \{m\}$.

Define a function y as follows. Fix $t \in (0,+\infty)$, and Let $m \in N$ with $s \le T_m$, then define $y(t) = z_m(t)$. Hence $y \in C([0,+\infty),\mathbb{R})$.

Again fix $t \in [0,+\infty)$ and Let $m \in N$ with $s \le T_m$. Then for $n \in N_m$ we get

$$u_{n_k}(t) = \int_0^{T_m} G_{T_m}(t,s) f\left(s, u_{n_k}(s), D^\beta u_{n_k}(s)\right) ds + \frac{y_0}{\Delta} t^{(\alpha-1)}. \tag{4.11}$$

Let $n_k \to +\infty$ through N_m to obtain

$$z_m(t) = \int_0^{T_m} G_{T_m}(t,s) f\left(s, z_m(s), D^\beta z_m(s)\right) ds + \frac{y_0}{\Delta} t^{(\alpha-1)}, \tag{4.12}$$

that is,

$$y(t) = \int_0^{T_m} G_{T_m}(t,s) f\left(s, y(s), D^\beta y(s)\right) ds + \frac{y_0}{\Delta} t^{(\alpha-1)}. \tag{4.13}$$

We can use this method for each $s \in [0,T_m]$ and for each $m \in N$. Hence,

$$D^\alpha y(t) + f\left(t, y(t), D^\beta y(t)\right) = \theta, \quad \text{a.e. } t \in [0,T_m], \tag{4.14}$$

for each $m \in N$. Consequently, the constructed function y is a solution of (1.6)-(1.7). This completes the proof of the theorem. □

Rmark 4.2. In [21], the authors considered the BVP (1.3). Under some suitable conditions, they obtained the existence result of unbounded solution. In nature, BVP (1.3) is a special form of BVP (1.6)-(1.7). In that scalar situation, $\alpha - \beta = 1$, $b_i = 0$ $(i = 1,2,\ldots,m-2)$, $b_1 > 0$, $b_i = 0$ $(i = 2,3,\ldots,m-2)$, $y_0 = 0$, and $f = f(t,y(t))$ are not singular, then our Theorem 4.1 includes the result from [21]. But our approach is different from those of [21].

Proceeding the similar arguments above, we list the following unique result for BVP (1.6)-(1.7), and the proof will be omitted.

Theorem 4.3. *Suppose that* (H_0)–(H_4) *hold, then BVP* (1.6)-(1.7) *has a unique positive solution* y, *and* $D^\beta y$ *is also positive.*

5. Application

We end this paper with giving an example to demonstrate the application of our existence result.

Example 5.1. Consider the following BVP in scalar space:

$$\begin{gathered} D^\alpha y(t) + \left(2 - \sin\frac{1}{3t-1}\right)\left(e^{-t} + \frac{1}{y^{\eta_1}} + \frac{1}{\left(D^\beta y\right)^{\kappa_1}} + y^\eta + \left(D^\beta y\right)^\kappa\right) = 0, \\ y(0) = 0, \qquad \lim_{t\to+\infty} D^\beta y(t) - \frac{1}{4} y(4) - 2y(5) - D^\beta y(6) = 1, \end{gathered} \tag{5.1}$$

where $1 < \alpha \leq 2$, $0 < \beta < \alpha - 1$. We will apply Theorem 4.1 with $\varpi = 1$, $\gamma(t) = 2 - \sin(1/(3t-1))$, $\gamma_0(t) = e^{-t}$, $q_1(x) = 1/x^{\eta_1}$, $p_1(x) = 1/(x)^{\kappa_1}$, $q(x) = x^{\eta}$, $p(t) = (x)^{\kappa}$. Clearly (H_1) holds because $f(t,x,y) \geq 1$ for $t \in [0,+\infty) \setminus \{1/3\}$, $(x,y) \in (0,+\infty) \times (0,+\infty)$. And also (H_2) holds for $\eta, \kappa \in (0,1)$, $\eta_1 \in (0,1/2)$, and $\kappa_1 \in (0,1/2)$. Hence, Theorem 4.1 guarantees the existence of positive solution of (5.1).

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