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Research Article

Convergence of GAOR Iterative Method with Strictly α Diagonally Dominant Matrices

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We discuss the convergence of GAOR method for linear systems with strictly α diagonally dominant matrices. Moreover, we show that our results are better than ones of Darvishi and Hessari (2006), Tian et al. (2008) by using three numerical examples.

1. Introduction

Sometimes we have to solve the following linear system:

$$Hy = f, \tag{1.1}$$

where

$$H = \begin{pmatrix} I - B_1 & D \\ C & I - B_2 \end{pmatrix} \tag{1.2}$$

is invertible. Yuan proposed a generalized SOR(GSOR) method to solve linear system (1.1) in [1]; afterwards, Yuan and Jin [2] established a generalized AOR(GAOR) method to solve linear system (1.1) as follows:

$$y^{(k+1)} = L_{\omega,r} y^{(k)} + \omega k, \tag{1.3}$$

where

$$
\begin{gathered}
L_{\omega,r} = (1-\omega)I + \omega J + \omega r K, \\
k = \begin{pmatrix} I & 0 \\ -rC & I \end{pmatrix} f, \\
J = \begin{pmatrix} B_1 & -D \\ -C & B_2 \end{pmatrix}, \\
K = \begin{pmatrix} 0 & 0 \\ C(I-B_1) & CD \end{pmatrix} = \begin{pmatrix} 0 \\ C \end{pmatrix} \begin{pmatrix} I-B_1 & D \end{pmatrix}.
\end{gathered} \tag{1.4}
$$

In [2–4], some people studied the convergence of GAOR method for solving linear systems $Hy = f$. In [2], Yuan and Jin studied the convergence of GAOR method and show that the GAOR method is better than the GSOR method under certain conditions. In [3], Darvishi and Hessari studied the convergence of GAOR method for diagonally dominant coefficient matrices and gave the regions of convergence. In [4], Tian et al. studied the convergence of GAOR method for strictly diagonally dominant coefficient matrices and gave the regions of convergence.

Sometimes, the coefficient matrices of linear systems $Hy = f$ are not strictly diagonally dominant. In this paper, we will discuss the convergence of GAOR method for linear systems $Hy = f$ with strictly α diagonally dominant matrices which need not to be strictly diagonally dominant.

Throughout this paper, we denote the set of all complex matrices by $C^{n\times n}$, the spectral radius of iterative matrix $L_{\omega,r}$ by $\rho(L_{\omega,r})$. And

$$
N = \{1,2,\ldots,n\}, \quad R_i(A) = \sum_{j\neq i} |a_{ij}|, \quad S_i(A) = \sum_{j\neq i} |a_{ji}|, \quad i,j \in N. \tag{1.5}
$$

Definition 1.1 (see [5]). Let $A = (a_{ij}) \in C^{n\times n}$. If there exists $\alpha \in [0,1]$,

$$
|a_{ii}| \geq \alpha R_i(A) + (1-\alpha) S_i(A) \quad (\forall i \in N) \tag{1.6}
$$

holds, then we call A as α diagonally dominant and denote it as $A \in D_0(\alpha)$. If all the inequalities are strict, then we denote it as $A \in D(\alpha)$.

Obviously, if A is a strictly diagonally dominant matrix ($A \in SD$), then $A \in D(\alpha)$. But not vice versa.

In [3], Darvishi and Hessari obtained the following result.

Theorem 1.2. *Let $H \in SD$ and assume that $\omega \geq r \geq 0$. Then the sufficient condition for convergence of the GAOR method is*

$$
0 < \omega < \frac{2}{1 + \max_i \{J_i + rK_i\}}, \tag{1.7}
$$

where J_i and K_i are the i-row sums of the modulus of the entries of J and K, respectively.

In [4], Tian et al. obtained the following result.

Theorem 1.3. *If $H \in SD$, then the sufficient conditions for the convergence of the GAOR method are either*

(i) $0 \le r \le 1$ *and* $0 < \omega \le 1$ *or*

(ii) $|r| \le \min_i\{(1 - J_i)/K_i\}$ *and* $0 < \omega < 2/(1 + \max_i(J + rK)_i)$.

This work is organized as follows. In Section 2, we obtain bound for the spectral radius of the iterative matrix $L_{\omega,r}$ of GAOR iterative method. In Section 3, we present two convergence theorems of GAOR method. In Section 4, we present three numerical examples to show that our results are better than ones of Theorems 1.2 and 1.3.

2. Upper Bound of the Spectral Radius of $L_{\omega,r}$

In this section, we obtain upper bound of the spectral radius of iterative matrix $L_{\omega,r}$.

Theorem 2.1. *Let $H \in D(\alpha)$, then $\rho(L_{\omega,r})$ satisfies the following inequality:*

$$\rho(L_{\omega,r}) \le \max_i\{|1 - \omega| + (|\omega J| + |\omega r K|)_{ii} + \alpha R_i(|\omega J| + |\omega r K|) + (1 - \alpha)S_i(|\omega J| + |\omega r K|)\}. \tag{2.1}$$

Proof. Let λ be an arbitrary eigenvalue of iterative matrix $L_{\omega,r}$, then

$$\det(\lambda I - L_{\omega,r}) = 0. \tag{2.2}$$

Equation (2.2) holds if and only if

$$\det((\lambda + \omega - 1)I - \omega J - \omega r K) = 0. \tag{2.3}$$

If $(\lambda + \omega - 1)I - \omega J - \omega r K \in D(\alpha)$, that is,

$$|\lambda - (1 - \omega) - (\omega J + \omega r K)_{ii}| > \alpha R_i(\omega J + \omega r K) + (1 - \alpha)S_i(\omega J + \omega r K), \quad \forall i \in N, \tag{2.4}$$

where $(\omega J + \omega r K)_{ii}$ denotes the diagonal element of matrix $\omega J + \omega r K$. From [5], we know that

$$\det((\lambda + \omega - 1)I - \omega J - \omega r K) \ne 0; \tag{2.5}$$

then λ is not an eigenvalue of iterative matrix $L_{\omega,r}$, and when

$$|\lambda| - |1 - \omega| - (|\omega J| + |\omega r K|)_{ii} > \alpha R_i(|\omega J| + |\omega r K|) + (1 - \alpha)S_i(|\omega J| + |\omega r K|), \quad \forall i \in N, \tag{2.6}$$

λ is not an eigenvalue of iterative matrix $L_{\omega,r}$.

When λ is an eigenvalue of iterative matrix $L_{\omega,r}$, then there exists at least one i $(i \in N)$, such that

$$|\lambda| - |1-\omega| - (|\omega J| + |\omega r K|)_{ii} \leq \alpha R_i(|\omega J| + |\omega r K|) + (1-\alpha) S_i(|\omega J| + |\omega r K|), \tag{2.7}$$

that is,

$$|\lambda| \leq |1-\omega| + (|\omega J| + |\omega r K|)_{ii} + \alpha R_i(|\omega J| + |\omega r K|) + (1-\alpha) S_i(|\omega J| + |\omega r K|). \tag{2.8}$$

Hence,

$$\rho(L_{\omega,r}) \leq \max_i \{|1-\omega| + (|\omega J| + |\omega r K|)_{ii} + \alpha R_i(|\omega J| + |\omega r K|) + (1-\alpha) S_i(|\omega J| + |\omega r K|)\}. \tag{2.9}$$

□

3. Convergence of GAOR Method

In this section, we investigate the convergence of GAOR method to solve linear system (1.1). We assume that H is a strictly α diagonally dominant coefficient matrix and get some sufficient conditions for the convergence of GAOR method.

Theorem 3.1. *Let $H \in D(\alpha)$, then GAOR method is convergent if ω, r satisfy either*

(I) $0 < \omega \leq 1$ *and*

$$|r| < \min_i \frac{1 - |J|_{ii} - \alpha R_i(|J|) - (1-\alpha) S_i(|J|)}{\alpha R_i(|K|) + (1-\alpha) S_i(|K|) + |K|_{ii}} \tag{3.1}$$

or

(II) $1 < \omega < \min_i (2/(1 + |J|_{ii} + \alpha R_i(|J|) + (1-\alpha) S_i(|J|)))$ *and*

$$|r| < \min_i \frac{2 - \omega - \omega(|J|_{ii} + \alpha R_i(|J|) + (1-\alpha) S_i(|J|))}{\omega(\alpha R_i(|K|) + (1-\alpha) S_i(|K|) + |K|_{ii})}. \tag{3.2}$$

Proof. Since $H \in D(\alpha)$, then GAOR method is convergent if we have $\rho(L_{\omega,r}) < 1$, that is,

$$\max_i \{|1-\omega| + (|\omega J| + |\omega r K|)_{ii} + \alpha R_i(|\omega J| + |\omega r K|) + (1-\alpha) S_i(|\omega J| + |\omega r K|)\} < 1. \tag{3.3}$$

Firstly, when $0 < \omega \leq 1$, then

$$1 - \omega + (|\omega J| + |\omega r K|)_{ii} + \alpha R_i(|\omega J| + |\omega r K|) + (1-\alpha) S_i(|\omega J| + |\omega r K|) < 1, \quad \forall i \in N. \tag{3.4}$$

That is,

$$|J|_{ii} + |r||K|_{ii} + \alpha R_i(|J|) + |r|\alpha R_i(|K|) + (1-\alpha)S_i(|J|) + |r|(1-\alpha)S_i(|K|) < 1, \tag{3.5}$$

which leads to

$$|r| < \frac{1 - |J|_{ii} - \alpha R_i(|J|) - (1-\alpha)S_i(|J|)}{\alpha R_i(|K|) + (1-\alpha)S_i(|K|) + |K|_{ii}}, \quad \forall i \in N. \tag{3.6}$$

So

$$|r| < \min_i \frac{1 - |J|_{ii} - \alpha R_i(|J|) - (1-\alpha)S_i(|J|)}{\alpha R_i(|K|) + (1-\alpha)S_i(|K|) + |K|_{ii}}. \tag{3.7}$$

Secondly, when $1 < \omega$, then

$$\omega - 1 + (|\omega J| + |\omega r K|)_{ii} + \alpha R_i(|\omega J| + |\omega r K|) + (1-\alpha)S_i(|\omega J| + |\omega r K|) < 1, \quad \forall i \in N. \tag{3.8}$$

That is,

$$\omega(|J|_{ii} + |r||K|_{ii} + \alpha R_i(|J|) + |r|\alpha R_i(|K|) + (1-\alpha)S_i(|J|) + |r|(1-\alpha)S_i(|K|)) < 2 - \omega, \tag{3.9}$$

which leads to

$$|r| < \frac{2 - \omega - \omega(|J|_{ii} + \alpha R_i(|J|) + (1-\alpha)S_i(|J|))}{\omega(\alpha R_i(|K|) + (1-\alpha)S_i(|K|) + |K|_{ii})}, \quad \forall i \in N. \tag{3.10}$$

So

$$|r| < \min_i \frac{2 - \omega - \omega(|J|_{ii} + \alpha R_i(|J|) + (1-\alpha)S_i(|J|))}{\omega(\alpha R_i(|K|) + (1-\alpha)S_i(|K|) + |K|_{ii})}, \tag{3.11}$$

then

$$2 - \omega - \omega(|J|_{ii} + \alpha R_i(|J|) + (1-\alpha)S_i(|J|)) > 0, \quad \forall i \in N, \tag{3.12}$$

so

$$1 < \omega < \min_i \frac{2}{1 + |J|_{ii} + \alpha R_i(|J|) + (1-\alpha)S_i(|J|)}. \tag{3.13}$$

Thus we complete the proof. □

When $r = 0$, we can get the following theorem.

Theorem 3.2. *Let $H \in D(\alpha)$, then $\rho(L_{\omega,0}) < 1$ if*

$$0 < \omega < \min_i \frac{2}{1 + |J|_{ii} + \alpha R_i(|J|) + (1-\alpha) S_i(|J|)}. \tag{3.14}$$

Proof. Because $H \in D(\alpha)$, from Theorem 1.2, when $r = 0$, we have

$$\rho(l_{\omega,r}) \leq \max_i \{|1-\omega| + |\omega J|_{ii} + \alpha R_i(|\omega J|) + (1-\alpha) S_i(|\omega J|)\} < 1. \tag{3.15}$$

Firstly, when $0 < \omega \leq 1$,

$$1 - \omega + \omega |J|_{ii} + \alpha \omega R_i(|J|) + (1-\alpha)\omega S_i(|J|) < 1. \tag{3.16}$$

That is

$$|J|_{ii} + \alpha R_i(|J|) + (1-\alpha) S_i(|J|) < 1. \tag{3.17}$$

Secondly, when $1 < \omega < 2$,

$$\omega - 1 + \omega |J|_{ii} + \alpha \omega R_i(|J|) + (1-\alpha)\omega S_i(|J|) < 1. \tag{3.18}$$

That is

$$\omega + \omega |J|_{ii} + \alpha \omega R_i(|J|) + (1-\alpha)\omega S_i(|J|) < 2, \tag{3.19}$$

which leads to

$$\omega < \frac{2}{1 + |J|_{ii} + \alpha R_i(|J|) + (1-\alpha) S_i(|J|)}. \tag{3.20}$$

So ω must satisfy

$$0 < \omega < \min_i \frac{2}{1 + |J|_{ii} + \alpha R_i(|J|) + (1-\alpha) S_i(|J|)}. \tag{3.21}$$

Thus we complete the proof of Theorem 3.2. □

4. Examples

In the following examples, we give the regions of convergence of GAOR method to show that our results are better than ones of Theorems 1.2 and 1.3.

Example 4.1. Let

$$H = \left(\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{5}{6} \\ \hline \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right) = \begin{pmatrix} I - B_1 & D \\ C & I - B_2 \end{pmatrix}. \tag{4.1}$$

Obviously, H is not a strictly diagonally dominant matrix, so we cannot use the results of Theorems 1.2 and 1.3. But $H \in D(1/2)$, and

$$J = \begin{pmatrix} 0 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{5}{6} \\ -\frac{1}{3} & -\frac{1}{3} & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{4}{9} & \frac{4}{9} & \frac{7}{18} \end{pmatrix}. \tag{4.2}$$

By Theorem 3.1, we obtain the following regions of convergence:

(I) $0 < \omega \leq 1$ and $|r| < 1/10$

or

(II) $1 < \omega < 24/23$ and $|r| < (2 - (23/12)\omega)(6/5\omega)$.

Example 4.2 (Example 1 of paper [4]). Let

$$H = \left(\begin{array}{cc|c} 1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{3} \\ \hline \frac{1}{3} & \frac{1}{3} & 1 \end{array}\right) = \begin{pmatrix} I - B_1 & D \\ C & I - B_2 \end{pmatrix}. \tag{4.3}$$

It is easy to test that $H \in D(1/4)$, and

$$J = \begin{pmatrix} 0 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{4}{9} & \frac{4}{9} & \frac{2}{9} \end{pmatrix}. \tag{4.4}$$

By Theorem 3.1, we obtain the following regions of convergence:

(I) $0 < \omega \leq 1$ and $|r| < 3/4$

or

(II) $1 < \omega < 1.2$ and $|r| < ((18 - 5\omega)/4\omega)$.

In addition, H is a strictly diagonally dominant matrix.

By Theorem 1.3, we obtain the following regions of convergence:

(I) $0 \leq r \leq 1$ and $0 < \omega \leq 1$,

(II) $0 \leq r < 0.3$ and $0 < \omega < 1.2$,

or

(III) $-0.3 < r < 0$ and $0 < \omega < (18/(15 - 10r))$.

By Theorem 1.2, we obtain the following region of convergence:
$0 \leq r \leq \omega$ and $0 < \omega < (18/(15 + 10r))$.

And we get Figure 1, where the regions of convergence got by Theorems 3.1 and 1.3, Theorem 1.2 are bounded by blue lines, red lines, green lines, respectively.

This figure shows that the regions of convergence got by Theorem 3.1 are larger than ones got by Theorems 1.2 and 1.3.

Example 4.3. Consider

$$H = \begin{pmatrix} 1 & \frac{1}{96} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{1}{96} \\ -\frac{1}{2} & 1 & \frac{1}{96} & 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{96} \\ -\frac{1}{96} & -\frac{1}{2} & 1 & \frac{1}{96} & \frac{1}{96} & \cdots & \cdots & \frac{1}{96} & \frac{1}{96} \\ -\frac{1}{96} & 0 & -\frac{1}{2} & 1 & \frac{1}{96} & \frac{1}{96} & \cdots & \frac{1}{96} & \frac{1}{96} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{1}{96} & 0 & \cdots & 0 & -\frac{1}{2} & 1 & \frac{1}{96} & \frac{1}{96} & \frac{1}{96} \\ -\frac{1}{96} & 0 & \cdots & \cdots & 0 & -\frac{1}{2} & 1 & \frac{1}{96} & \frac{1}{96} \\ -\frac{1}{96} & 0 & \cdots & \cdots & \cdots & 0 & -\frac{1}{2} & 1 & \frac{1}{96} \\ -\frac{1}{96} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -\frac{1}{2} & 1 \end{pmatrix}_{51 \times 51}, \tag{4.5}$$

where

$$I - B_1 = \begin{pmatrix} 1 & \frac{1}{96} \\ -\frac{1}{2} & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} & 0 \cdots 0 & \frac{1}{96} \\ \frac{1}{96} & 0 \cdots 0 & \frac{1}{96} \end{pmatrix}. \tag{4.6}$$

Obviously, H is not a strictly diagonally dominant matrix, so we cannot use the results of Theorems 1.2 and 1.3. But $H \in D(1/2)$, and

$$J = \begin{pmatrix} 0 & -\frac{1}{96} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -\frac{1}{96} \\ \frac{1}{2} & 0 & -\frac{1}{96} & 0 & \cdots & \cdots & \cdots & 0 & -\frac{1}{96} \\ \frac{1}{96} & \frac{1}{2} & 0 & -\frac{1}{96} & -\frac{1}{96} & \cdots & \cdots & -\frac{1}{96} & -\frac{1}{96} \\ \frac{1}{96} & 0 & \frac{1}{2} & 0 & -\frac{1}{96} & -\frac{1}{96} & \cdots & -\frac{1}{96} & -\frac{1}{96} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \frac{1}{96} & 0 & \cdots & 0 & \frac{1}{2} & 0 & -\frac{1}{96} & -\frac{1}{96} & -\frac{1}{96} \\ \frac{1}{96} & 0 & \cdots & \cdots & 0 & \frac{1}{2} & 0 & -\frac{1}{96} & -\frac{1}{96} \\ \frac{1}{96} & 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{2} & 0 & -\frac{1}{96} \\ \frac{1}{96} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{1}{2} & 0 \end{pmatrix}_{51\times 51},$$

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \frac{23}{96} & -\frac{4609}{9216} & -\frac{1}{192} & 0 & \cdots & 0 & 0 & 0 & -\frac{49}{9216} \\ -\frac{1}{96} & -\frac{1}{9216} & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{9216} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{1}{96} & -\frac{1}{9216} & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{9216} \\ -\frac{1}{96} & -\frac{1}{9216} & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{9216} \\ -\frac{1}{96} & -\frac{1}{9216} & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{9216} \\ -\frac{1}{96} & -\frac{1}{9216} & 0 & 0 & \cdots & 0 & 0 & 0 & -\frac{1}{9216} \end{pmatrix}_{51\times 51}. \tag{4.7}$$

By Theorem 3.1, we obtain the following regions of convergence:

(I) $0 < \omega \leq 1$ and $|r| < (2208/3481)$

or

(II) $1 < \omega < (192/169)$ and $|r| < ((18432 - 16224\omega)/3481\omega)$.

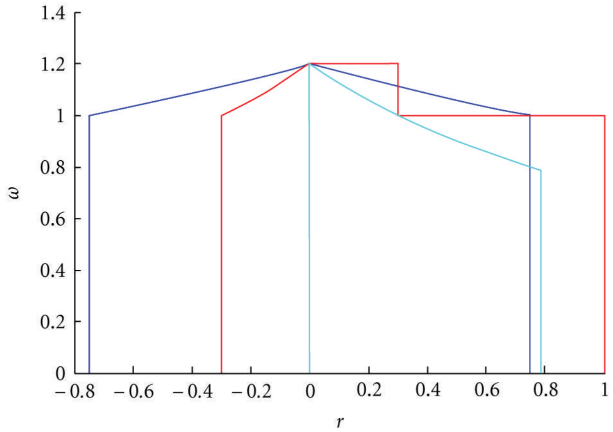

Figure 1

Acknowledgments

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