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Research Article

Generalized Hyers-Ulam Stability of the Second-Order Linear Differential Equations

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We prove the generalized Hyers-Ulam stability of the 2nd-order linear differential equation of the form $y'' + p(x)y' + q(x)y = f(x)$, with condition that there exists a nonzero $y_1 : I \to X$ in $C^2(I)$ such that $y_1'' + p(x)y_1' + q(x)y_1 = 0$ and I is an open interval. As a consequence of our main theorem, we prove the generalized Hyers-Ulam stability of several important well-known differential equations.

1. Introduction

The *stability problem* of functional equations started with the question concerning stability of group homomorphisms proposed by Ulam [1] during a talk before a Mathematical Colloquium at the University of Wisconsin, Madison. In 1941, Hyers [2] gave a partial solution of *Ulam's* problem for the case of approximate additive mappings in the context of Banach spaces. In 1978, Rassias [3] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences $\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p)$, $(\epsilon > 0,\ p \in [0,1))$. This phenomenon of stability that was introduced by Rassias [3] is called the Hyers-Ulam-Rassias stability (or the generalized Hyers-Ulam stability).

Let X be a normed space over a scalar field $\mathbb{K}$, and let I be an open interval. Assume that for any function $f : I \longrightarrow X$ satisfying the differential inequality

$$\left\| a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) + h(t) \right\| \le \epsilon \qquad (1.1)$$

for all $t \in I$ and for some $\epsilon \geq 0$, there exists a function $f_0 : I \longrightarrow X$ satisfying

$$\begin{gathered} a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) + h(t) = 0, \\ \left\| f(t) - f_0(t) \right\| \leq K(\epsilon) \end{gathered} \tag{1.2}$$

for all $t \in I$; here $K(t)$ is an expression for ϵ with $\lim_{\epsilon \longrightarrow 0} K(\epsilon) = 0$. Then, we say that the above differential equation has the Hyers-Ulam stability.

If the above statement is also true when we replace ϵ and $K(\epsilon)$ by $\varphi(t)$ and $\phi(t)$, where $\varphi,\ \phi : I \longrightarrow [0, \infty)$ are functions not depending on f and f_0 explicitly, then we say that the corresponding differential equation has the Hyers-Ulam-Rassias stability (or the generalized Hyers-Ulam stability).

The Hyers-Ulam stability of differential equation $y' = y$ was first investigated by Alsina and Ger [4]. This result has been generalized by Takahasi et al. [5] for the Banach space-valued differential equation $y' = \lambda y$. In [6], Miura et al. also proved the Hyers-Ulam-Rassias stability of linear differential of first order, $y' + g(t)y(t) = 0$, where $g(t)$ is a continuous function, while the author [7] proved the Hyers-Ulam-Rassias stability of linear differential of the form $c(t)y'(t) = y(t)$. Jung [8] proved the Hyers-Ulam-Rassias stability of linear differential of first order of the form $c(t)y'(t) + g(t)y(t) + h(t) = 0$.

In this paper, we investigate the generalized Hyers-Ulam stability of differential equations of the form

$$y'' + p(x)y' + q(x)y = f(x). \tag{1.3}$$

We assume that X is a complex Banach space, $I = (a, b)$ is an arbitrary interval, and $y_1 : I \longrightarrow X$ is a nonzero solution of corresponding homogeneous equation of (1.3), where

$$y_1'' + p(x)y_1' + q(x)y_1 = 0. \tag{1.4}$$

2. Main Results

Taking some idea from [8], we are going to investigate the stability of the 2nd-order linear differential equations. For the sake of convenience, all the integrals and derivations will be viewed as existing and $\Re(\omega)$ denotes the real part of complex number ω. Moreover, let $I = (a, b)$ be an open interval, where $a, b \in \mathbb{R} \bigcup \{\pm\infty\}$ are arbitrarily given with $a < b$.

Theorem 2.1. *Let X be a complex Banach space. Assume that $p, q : I \longrightarrow \mathbb{C}$ and $f : I \longrightarrow X$ are continuous functions and $y_1 : I \longrightarrow X$ is a nonzero twice continuously differentiable function which satisfies the differential equation* (1.4)*. If a twice continuously differentiable function $y : I \longrightarrow X$ satisfies*

$$\left\| y'' + p(x)y' + q(x)y - f(x) \right\| \leq \varphi(x) \tag{2.1}$$

for all $x \in I$, *where* $k = y(a)/y_1(a) \in X$ *and* $\varphi : I \longrightarrow (0, \infty)$ *is a continuous function, then there exists a unique* $x_0 \in X$ *such that*

$$\begin{aligned}
&\Bigg\| y(x) - y_1(x) \cdot \Bigg(\int_a^x \Bigg(\exp\Bigg\{ - \int_a^s \Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u) \Bigg) du \Bigg\} \\
&\qquad \cdot \Bigg[x_0 + \int_a^s \frac{f(v)}{y_1(v)} \exp\Bigg\{ \int_a^v \Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u) \Bigg) du \Bigg\} dv \Bigg] \Bigg) ds + k \Bigg) \Bigg\| \\
&\quad \le \|y_1(x)\| \cdot \int_a^x \Bigg(\exp\Bigg\{ -\Re \int_a^s \Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u) \Bigg) du \Bigg\} \\
&\qquad \cdot \Bigg| \int_s^b \exp\Bigg\{ \Re\Bigg(\int_a^t \Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u) du \Bigg) \Bigg) \cdot \frac{\varphi(t)}{\|y_1(t)\|} dt \Bigg\} \Bigg| \Bigg) ds.
\end{aligned} \tag{2.2}$$

Proof. We assume that

$$v(x) = \frac{y(x)}{y_1(x)} \tag{2.3}$$

for all $x \in I$. It follows from (1.4), (2.1), and (2.3) that

$$\begin{aligned}
&\Big\| (v(x)y_1(x))'' + p(x)(v(x)y_1(x))' + q(x)(v(x)y_1(x)) - f(x) \Big\| \\
&\quad = \Big\| (v(x)'y_1(x) + v(x)y_1(x)')' + p(x)(v(x)'y_1(x) + v(x)y_1(x)') \\
&\qquad + q(x)v(x)y_1(x) - f(x) \| \\
&\quad = \| v(x)''y_1(x) + v(x)'(2y_1(x)' + p(x)y_1(x)) \\
&\qquad + v(x)(y_1(x)'' + p(x)y_1(x)' + q(x)y_1(x)) - f(x) \| \\
&\quad = \| v(x)''y_1(x) + v(x)'(2y_1(x)' + p(x)y_1(x)) - f(x) \| \\
&\quad = \|y_1(x)\| \Bigg\| v(x)'' + v(x)' \Bigg(\frac{2y_1(x)'}{y_1(x)} + p(x) \Bigg) - \frac{f(x)}{y_1(x)} \Bigg\| \\
&\quad \le \varphi(x),
\end{aligned} \tag{2.4}$$

so, we have

$$\Bigg\| v(x)'' + v(x)' \Bigg(\frac{2y_1(x)'}{y_1(x)} + p(x) \Bigg) - \frac{f(x)}{y_1(x)} \Bigg\| \le \frac{\varphi(x)}{\|y_1(x)\|}. \tag{2.5}$$

For simplicity, we use the following notation:

$$\begin{aligned} z(s) :&= \exp\left\{\int_a^s\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\right\}\cdot\left(\frac{y(s)}{y_1(s)}\right)' \\ &\quad -\int_a^s\left(\frac{f(v)}{y_1(v)}\exp\left\{\int_a^v\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\right\}\right)dv \end{aligned} \tag{2.6}$$

for all $s \in I$. By making use of this notation and by (2.5), we get

$$\begin{aligned} \|z(s)-z(l)\| &= \Bigg\| \exp\left\{\int_a^s\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\right\}\cdot\left(\frac{y(s)}{y_1(s)}\right)' \\ &\quad -\int_a^s\left(\frac{f(v)}{y_1(v)}\exp\left\{\int_a^v\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\right\}\right)dv \\ &\quad -\exp\left\{\int_a^l\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\right\}\cdot\left(\frac{y(l)}{y_1(l)}\right)' \\ &\quad +\int_a^l\left(\frac{f(v)}{y_1(v)}\exp\int_a^v\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\,dv\right)\Bigg\| \\ &= \Bigg\|\int_l^s dt\left(\exp\int_a^t\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\cdot\left(\frac{y(t)}{y_1(t)}\right)'\right. \\ &\quad \left.-\int_a^t\left(\frac{f(v)}{y_1(v)}\exp\left\{\int_a^v\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\right\}\right)dv\right)\Bigg\| \\ &= \Bigg\|\int_l^s\left(\left(\frac{2y_1(t)'}{y_1(t)}+p(t)\right)\cdot\exp\left\{\int_a^t\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\right\}\right. \\ &\quad \cdot\left(\frac{y(t)}{y_1(t)}\right)'+\exp\left\{\int_a^t\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\right\}\cdot\left(\frac{y(t)}{y_1(t)}\right)''\Bigg) \\ &\quad \left.+\left(-\frac{f(t)}{y_1(t)}\exp\left\{\int_a^t\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\right\}\right)dt\right\| \\ &= \Bigg\|\int_l^s \exp\left\{\int_a^t\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\right\} \\ &\quad \cdot\left(\left(\frac{2y_1(t)'}{y_1(t)}+p(t)\right)\cdot\left(\frac{y(t)}{y_1(t)}\right)'+\left(\frac{y(t)}{y_1(t)}\right)''-\frac{f(t)}{y_1(t)}\right)dt\Bigg\| \\ &\le \left|\int_l^s \exp\left\{\Re\left(\int_a^t\left(\frac{2y_1(u)'}{y_1(u)}+p(u)du\right)\right)\cdot\frac{\varphi(t)}{\|y_1(t)\|}dt\right\}\right| \end{aligned} \tag{2.7}$$

for all $l, x \in I$. Since $\exp\{\Re(\int_a^t((2y_1(u)'/y_1(u)) + p(u)du)) \cdot (\varphi(t)/\|y_1(t)\|)$ is assumed to be integrable on I, we may select $l_0 \in I$, for any given $\epsilon > 0$, such that $l, x \geq l_0$ implies $\|z(x) - z(l)\| < \epsilon$. That is, $\{z(l)\}_{l\in I}$ is a Cauchy net. By completeness of X, there exists an $x_0 \in X$ such that $z(l)$ converges to x_0 as $l \longrightarrow b$. It follows from (2.7) and the previous argument that, for any $x \in I$,

$$\begin{aligned}
&\Bigg\| y(x) - y_1(x)\Bigg(\int_a^x\Bigg(\exp\Bigg\{-\int_a^s\Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u)\Bigg)du\Bigg\} \\
&\quad\times\Bigg[x_0 + \int_a^s \frac{f(v)}{y_1(v)}\exp\Bigg\{\int_a^v\Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u)\Bigg)du\Bigg\}dv\Bigg]\Bigg)ds + k\Bigg)\Bigg\| \\
&\qquad = \Bigg\| y_1(x)\cdot\Bigg(\int_a^x\Bigg(\exp\Bigg\{-\int_a^s\Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u)\Bigg)du\Bigg\}\cdot(z(s) - x_0)\Bigg)ds\Bigg)\Bigg\| \\
&\qquad \leq \|y_1(x)\|\cdot\int_a^x\Bigg(\exp\Bigg\{-\Re\int_a^s\Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u)\Bigg)du\Bigg\}\cdot\|z(s) - z(l)\|\Bigg)ds \\
&\qquad\quad + \|y_1(x)\|\cdot\int_a^x\Bigg(\exp\Bigg\{-\Re\int_a^s\Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u)\Bigg)du\Bigg\}\cdot\|z(l) - x_0\|\Bigg)ds \\
&\qquad \leq \|y_1(x)\|\cdot\int_a^x\Bigg(\exp\Bigg\{-\Re\int_a^s\Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u)\Bigg)du\Bigg\} \\
&\qquad\qquad\cdot\Bigg|\int_l^s \exp\Bigg\{\Re\Bigg(\int_a^t\Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u)du\Bigg)\Bigg)\cdot\frac{\varphi(t)}{\|y_1(t)\|}dt\Bigg|\Bigg)ds \\
&\qquad\quad + \|y_1(x)\|\cdot\int_a^x\Bigg(\exp\Bigg\{-\Re\int_a^s\Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u)\Bigg)du\Bigg\}\cdot\|z(l) - x_0\|\Bigg)ds \\
&\qquad \longrightarrow \|y_1(x)\|\cdot\int_a^x\Bigg(\exp\Bigg\{-\Re\int_a^s\Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u)\Bigg)du\Bigg\} \\
&\qquad\qquad\cdot\Bigg|\int_s^b \exp\Bigg\{\Re\Bigg(\int_a^t\Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u)du\Bigg)\Bigg)\cdot\frac{\varphi(t)}{\|y_1(t)\|}dt\Bigg|\Bigg)ds
\end{aligned} \tag{2.8}$$

as $l \longrightarrow b$. Moreover,

$$\begin{aligned}
y_0(x) = y_1(x)\cdot\Bigg(\int_a^x\Bigg(\exp\Bigg\{\int_a^s\Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u)\Bigg)du\Bigg\} \\
\cdot\Bigg[x_0 + \int_a^s\frac{f(v)}{y_1(v)}\exp\Bigg\{\int_a^v\Bigg(\frac{2y_1(u)'}{y_1(u)} + p(u)\Bigg)du\Bigg\}dv\Bigg]\Bigg)ds + k\Bigg)
\end{aligned} \tag{2.9}$$

is a solution of (1.3).

Now, we prove the uniqueness property of x_0. Assume that $x_1, x_2 \in X$ satisfy inequality (2.2) in place of x_0. Then, we have

$$\begin{aligned}
&\left\| y_1(x)\cdot\int_a^x\left(\exp\left\{-\int_a^s\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\right\}\cdot(x_2-x_1)\right)ds\right\| \\
&\quad\le 2\|y_1(x)\|\cdot\int_a^x\left(\exp\left\{-\Re\int_a^s\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\right\}\right. \\
&\qquad\left.\cdot\left|\int_s^b\exp\left\{\Re\left(\int_a^t\left(\frac{2y_1(u)'}{y_1(u)}+p(u)du\right)\right)\right\}\cdot\frac{\varphi(t)}{\|y_1(t)\|}dt\right|\right)ds,
\end{aligned}\tag{2.10}$$

thus,

$$\|x_2-x_1\| \le \frac{2\cdot\int_a^x\left(\exp\left\{-\Re\int_a^s\mathcal{A}du\right\}\cdot\left|\int_s^b\exp\left\{\Re\left(\int_a^t(\mathcal{A}du)\right)\right\}\cdot(\varphi(t)/\|y_1(t)\|)dt\right|\right)ds}{\left|\int_a^x\left(\exp\left\{-\Re\int_a^s\mathcal{A}du\right\}\right)ds\right|},\tag{2.11}$$

where $\mathcal{A}$ denotes $((2y_1(u)'/y_1(u))+p(u))$.

It follows from the integrability hypothesis that

$$\left|\int_s^b\exp\left\{\Re\left(\int_a^t\left(\frac{2y_1(u)'}{y_1(u)}+p(u)du\right)\right)\right\}\cdot\frac{\varphi(t)}{\|y_1(t)\|}dt\right|\longrightarrow 0\tag{2.12}$$

as $s \longrightarrow b$. This implies that $x_1 = x_2$ and the proof is complete. □

Remark 2.2. It follows from Theorem 2.1 that

$$\begin{aligned}
y(x) = y_1(x)\cdot&\left(\int_a^x\left(\exp\left\{\int_a^s\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\right\}\right.\right. \\
&\left.\left.\cdot\left[c_1+\int_a^s\frac{f(v)}{y_1(v)}\exp\left\{\int_a^v\left(\frac{2y_1(u)'}{y_1(u)}+p(u)\right)du\right\}dv\right]\right)ds+c_2\right)
\end{aligned}\tag{2.13}$$

is the general solution of the differential equation (1.3), where c_1, c_2 are arbitrary elements of X and $y_1(x)$ is a nonzero solution of the corresponding homogeneous equation (1.3).

Remark 2.3. If we replace $\mathbb{C}$ by $\mathbb{R}$ in the proof of Theorem 2.1 and we assume that p, q are real-valued continuous functions, then we can see that Theorem 2.1 is true for a real Banach space X.

Hence, every 2nd-order linear differential equation has the generalized Hyers-Ulam stability with the condition that there exists a solution of corresponding homogeneous equation or there exists a general solution in the ordinary differential equations.

Example 2.4. Consider the second-order linear differential equation with constant coefficients

$$y'' + by' + cy = f(x). \tag{2.14}$$

Let $b^2 - 4c \geq 0$, $m = (-b \pm \sqrt{b^2 - 4c})/2$, and let $f : I \longrightarrow \mathbb{R}$, $\varphi : I \longrightarrow [0, \infty)$ be continuous functions. Assume that $y : I \longrightarrow \mathbb{R}$ is a twice continuously differential function satisfying the differential inequality

$$\left| y'' + by' + cy - f(x) \right| \leq \varphi(x) \tag{2.15}$$

for all $x \in I$. On the other hand, by ordinary differential equations, we know that $y_1(x) = \exp(mx)$ is a solution of corresponding homogeneous equation of (2.14). It follows from Theorem 2.1, Remark 2.3, and (2.14) that there exists a solution $y_0 : I \longrightarrow \mathbb{R}$ of (2.14) such that

$$\begin{aligned} y_0(x) = \exp(mx) \cdot \bigg(\int_a^x & (\exp(-(2m+b)(s-a)) \\ & \cdot \left[x_0 + \int_a^s f(v) \cdot \exp(v(m+b) - a(2m+b)) dv \right] \bigg) ds + k \bigg) \end{aligned} \tag{2.16}$$

for all $x \in I$ and that

$$\begin{aligned} \left| y(x) - y_0(x) \right| \leq \left| \exp(mx) \right| \cdot \int_a^x \Bigg(& \exp(-(2m+b)(s-a)) \\ & \cdot \left| \int_s^b \exp((2m+b)(t-a)) \cdot \frac{\varphi(t)}{\left| \exp(mx) \right|} dt \right| \Bigg) ds. \end{aligned} \tag{2.17}$$

Example 2.5. Consider (2.14). Let $b^2 - 4c < 0$, $m = (-b \pm \sqrt{b^2 - 4c})/2 = \alpha \pm i\beta$, and let $f : I \longrightarrow \mathbb{R}, \varphi : I \longrightarrow [0, \infty)$ be continuous functions. Let $y : I \longrightarrow \mathbb{R}$ be a twice continuously differential function satisfying the differential inequality of (2.15) for all $x \in I$. It follows from the ordinary differential equations that $y_1(x) = \exp(\alpha x)\cos(\beta x)$. Then it follows from Theorem 2.1, Remark 2.3, and (2.15) that there exists a solution $y_0 : I \longrightarrow \mathbb{R}$ of (2.14) such that

$$\begin{aligned} y_0(x) &= \exp(\alpha x)\cos(\beta x)\cdot\left(x_0\cos^2(\beta a)\int_a^x \frac{\exp((2\alpha+b)(a-s))}{\cos^2(\beta s)}ds\right) \\ &\quad+\exp(\alpha x)\cos(\beta x)\cdot\left(\int_a^x \frac{\exp(-(2\alpha+b)s)}{\cos^2(\beta s)}\cdot\left(\int_a^s f(v)\cdot\exp v(\alpha+b)\right.\right. \\ &\quad\left.\left.\cdot\cos(\beta v)dv\cdot\exp v(\alpha+b)\right)ds+k\right) \end{aligned} \tag{2.18}$$

for all $x \in I$, where $k = y(a)/(\exp(\alpha a)\cos(\beta a))$ and $x_0 \in \mathbb{R}$ is unique and

$$|y(x)-y_0(x)| \le |\exp(\alpha x)\cos(\beta x)|\cdot\int_a^x\left(\frac{\exp(-(2\alpha+b)s)}{\cos^2(\beta s)}\cdot\left|\int_s^b \cos^2(\beta t)\cdot\exp((\alpha+b)t)\cdot\varphi(t)dt\right|\right)ds. \tag{2.19}$$

Example 2.6. Consider the equation

$$y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y = 6\left(1+x^2\right). \tag{2.20}$$

Let $I = (a,b)$ be an open interval, where $a,b \in [1,+\infty]$ are arbitrarily given with $a < b$, $f : I \longrightarrow \mathbb{R}$ and $\varphi : I \longrightarrow [0,\infty)$ are continuous functions. Assume that $y : I \longrightarrow \mathbb{R}$ is a twice continuously differential function satisfying the differential inequality

$$\left|y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y - 6\left(1+x^2\right)\right| \le \varphi(x) \tag{2.21}$$

for all $x \in I$. By the trial of $y_0(x) = x$, we see that it is a solution of corresponding homogeneous equation of (2.20). Then it follows from Theorem 2.1, Remark 2.3, and (2.21) that there exists a solution $y_0 : I \longrightarrow \mathbb{R}$ of (2.20) such that

$$y_0(x) = x\left(x_0 a\left(\frac{1-a^2}{1+a^2}\right) + k - 6a + 2a^3\right) + \left(x^2-1\right)\left(x_0\frac{a^2}{1+a^2} - 3a^2\right) + x^4 + 3x^2 \tag{2.22}$$

for all $x \in I$, where $k = y(a)/a$ and $x_0 \in \mathbb{R}$ is unique and

$$|y(x) - y_0(x)| \le x\cdot\int_a^x\left(\left(\frac{1+s^2}{s^2}\right)\cdot\left|\int_s^b \frac{t}{1+t^2}\cdot\varphi(t)dt\right|\right)ds. \tag{2.23}$$

Remark 2.7. We know that Eulars differential equation of second order has the general solution in ordinary differential equations, then we can use Theorem 2.1 and Remark 2.3 for the Hyers-Ulam-Rassias stability in this case.

Let p be a real constant, and $I = [-1, 1]$. We know that Legender's differential equation

$$\left(1 - x^2\right) y'' - 2xy' + p(p + 1)y = 0 \tag{2.24}$$

has the general solution

$$y = a_0 y_1(x) + a_1 y_2(x), \tag{2.25}$$

where

$$\begin{aligned} y_1(x) &= 1 - \frac{p(p+1)}{2}x^2 + \frac{(p-2)p(p+1)(p+3)}{4!}x^4 - \cdots, \\ y_2(x) &= x - \frac{(P-1)(p+2)}{3!}x^3 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!}x^5 - \cdots \end{aligned} \tag{2.26}$$

and a_0, a_1 are arbitrary constants. By Theorem 2.1 and Remark 2.3, Legender's differential equation has Hyers-Ulam-Rassias stability.

Hermite's differential equation

$$y'' - 2xy' + 2py = 0, \tag{2.27}$$

where p is a real constant, has the general solution

$$y = a_0 y_1(x) + a_1 y_2(x) \tag{2.28}$$

that

$$\begin{aligned} y_1(x) &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n p(p-2)\cdots(p-2n+2)}{(2n)!}x^{2n}, \\ y_2(x) &= x + \frac{(-1)^n 2^n (p-1)(p-3)\cdots(p-2n+1)}{(2n+1)!}x^{2n+1} \end{aligned} \tag{2.29}$$

for all $x \in \mathbb{R}$, and a_0, a_1 are arbitrary constants. Thus Hermites differential equation has generalized Hyers-Ulam stability.

It is well known from the ordinary differential equations that

$$y_1(x) = J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}, \tag{2.30}$$

for all $x \in \mathbb{R}$, is a solution of Bessel's differential equation

$$x^2 y'' + xy' + \left(x^2 - p^2\right) y = 0 \tag{2.31}$$

that $p \geq 0$.

Then Bessel's differential equation has Hyers-Ulam-Rassias stability.

We know from the ordinary differential equations that Laguerre, Chebyshev, and Gauss hypergeometric differential equations have the general solution. Then we can show that those have generalized Hyers-Ulam stability.

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