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Research Article

On the Composition and Neutrix Composition of the Delta Function with the Hyperbolic Tangent and Its Inverse Functions

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Let F be a distribution in $\mathfrak{D}'$ and let f be a locally summable function. The composition $F(f(x))$ of F and f is said to exist and be equal to the distribution $h(x)$ if the limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$ and $\{\delta_n(x)\}$ is a certain regular sequence converging to the Dirac delta function. It is proved that the neutrix composition $\delta^{(rs-1)}((\tanh x_+)^{1/r})$ exists and $\delta^{(rs-1)}((\tanh x_+)^{1/r}) = \sum_{k=0}^{s-1}\sum_{i=0}^{K_k}((-1)^k c_{s-2i-1,k}(rs)!/2sk!)\delta^{(k)}(x)$ for $r, s = 1, 2, \ldots$, where K_k is the integer part of $(s-k-1)/2$ and the constants $c_{j,k}$ are defined by the expansion $(\tanh^{-1}x)^k = \{\sum_{i=0}^{\infty}(x^{2i+1}/(2i+1))\}^k = \sum_{j=k}^{\infty} c_{j,k}x^j$, for $k = 0, 1, 2, \ldots$. Further results are also proved.

1. Introduction

In the following, we let $\mathfrak{D}$ be the space of infinitely differentiable functions φ with compact support and let $\mathfrak{D}[a,b]$ be the space of infinitely differentiable functions with support contained in the interval $[a,b]$. A *distribution* is a continuous linear functional defined on $\mathfrak{D}$. The set of all distributions defined on $\mathfrak{D}$ is denoted by $\mathfrak{D}'$ and the set of all distributions defined on $\mathfrak{D}[a,b]$ is denoted by $\mathfrak{D}'[a,b]$.

Now let $\rho(x)$ be a function in $\mathfrak{D}[-1,1]$ having the following properties:

(i) $\rho(x) \geq 0$,

(ii) $\rho(x) = \rho(-x)$,

(iii) $\int_{-1}^{1}\rho(x)dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is an arbitrary distribution in $\mathfrak{D}'$ and $F_n(x) = F(x) * \delta_n(x) = \langle F(x-t), \varphi(t)\rangle$, then $\{F_n(x)\}$ is a regular sequence converging to $F(x)$.

Since the theory of distributions is a linear theory, we can extend some operations which are valid for ordinary functions to space of distributions; such operations may be called regular operations, and among them are addition and multiplication by scalars, see [1]. Other operations can be defined only for particular class of distributions; these may be called irregular, and among them are multiplication of distributions, see [2], and composition [3, 4], convolution products, see [5–7], further in [8], where some singular integrals were defined as distributions. Note that it is a difficult task to give a meaning to the expression $F(f(x))$, if F and f are singular distributions.

Thus there have been several attempts recently to define distributions of the form $F(f(x))$ in $\mathfrak{D}'$, where F and f are distributions in $\mathfrak{D}'$, see for example [9–12]. In the following, we are going to consider an alternative approach. As a starting point, we look at the following definition which is a generalization of Gel'fand and Shilov's definition of the composition involving the delta function [13], and was given in [10].

Definition 1.1. Let F be a distribution in $\mathfrak{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b), with $-\infty < a < b < \infty$, if

$$N\underset{n\to\infty}{-\lim}\int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle, \tag{1.1}$$

for all φ in $\mathfrak{D}[a, b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$ and N is the neutrix, see [14], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^{\lambda}\ln^{r-1}n, \quad \ln^{r}\mathrm{n}: \ \lambda > 0, \ r = 1, 2, \ldots \tag{1.2}$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle, \tag{1.3}$$

for all φ in $\mathfrak{D}[a, b]$.

Note that taking the neutrix limit of a function $f(n)$ is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$. If f, g are two distributions then in the ordinary sense the composition $f(g)$ does not necessarily exist. Thus the definition of the neutrix composition of distributions was originally given in [10] but was then simply called the composition of distributions.

We also note that, Ng and van Dam applied the neutrix calculus, in conjunction with the Hadamard integral, developed by van der Corput, to the quantum field theories,

in particular, to obtain finite results for the coefficients in the perturbation series. They also applied neutrix calculus to quantum field theory, and obtained finite renormalization in the loop calculations, see [15, 16].

The following two theorems involving derivatives of the Dirac-delta function were proved in [17] and [12], respectively.

Theorem 1.2. *The neutrix composition* $\delta^{(s)}(\operatorname{sgn} x|x|^{\lambda})$ *exists and*

$$\delta^{(s)}\left(\operatorname{sgn} x|x|^{\lambda}\right) = 0, \tag{1.4}$$

for $s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda = 1, 3, \ldots$ *and*

$$\delta^{(s)}\left(\operatorname{sgn} x|x|^{\lambda}\right) = \frac{(-1)^{(s+1)(\lambda+1)} s!}{\lambda[(s+1)\lambda - 1]!}\delta^{((s+1)\lambda-1)}(x), \tag{1.5}$$

for $s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda = 2, 4, \ldots$.

Theorem 1.3. *The compositions* $\delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s})$ *and* $\delta^{(s-1)}(|x|^{1/s})$ *exist and*

$$\begin{aligned} \delta^{(2s-1)}\left(\operatorname{sgn} x|x|^{1/s}\right) &= \frac{1}{2}(2s)!\delta'(x), \\ \delta^{(s-1)}\left(|x|^{1/s}\right) &= (-1)^{s-1}\delta(x), \end{aligned} \tag{1.6}$$

for $s = 1, 2, \ldots$.

The following two theorems were also proved in [18].

Theorem 1.4. *The neutrix composition* $\delta^{(s)}(\ln^r(1+|x|))$ *exists and*

$$\delta^{(s)}(\ln^r(1+|x|)) = \sum_{k=0}^{sr+r-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{s-i}\left[1+(-1)^k\right]s!(i+1)^{rs+r-1}}{2r(rs+r-1)!k!}\delta^{(k)}(x), \tag{1.7}$$

for $s = 0, 1, 2, \ldots$ *and* $r = 1, 2, \ldots$.

In particular, the composition $\delta(\ln(1+|x|))$ *exists and*

$$\delta(\ln[1+|x|]) = \delta(x). \tag{1.8}$$

Theorem 1.5. *The neutrix composition* $\delta^{(s)}(\ln(1+|x|^{1/r}))$ *exists and*

$$\delta^{(s)}\left(\ln\left(1+|x|^{1/r}\right)\right) = \sum_{k=0}^{m-1}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{r+s+i-1}\left[1+(-1)^k\right]r(i+1)^s}{2k!}\delta^{(k)}(x), \tag{1.9}$$

for $s = 0, 1, 2, \ldots$ *and* $r = 2, 3, \ldots$, *where* m *is the smallest nonnegative integer greater than* $(s-r+1)r^{-1}$.

In particular, the composition $\delta^{(s)}(\ln(1+|x|^{1/r}))$ *exists and*

$$\delta^{(s)}\left(\ln\left(1+|x|^{1/r}\right)\right)=0, \tag{1.10}$$

for $s=0,1,2,\ldots,r-2$ *and* $r=2,3,\ldots$ *and*

$$\delta^{(r-1)}\left(\ln\left(1+|x|^{1/r}\right)\right)=(-1)^{r-1}r!\delta(x), \tag{1.11}$$

for $r=2,3,\ldots$.

The following two theorems were proved in [4].

Theorem 1.6. *The neutrix composition* $\delta^{(s)}((\sinh^{-1}x_+)^r)$ *exists and*

$$\delta^{(s)}\left(\left(\sinh^{-1}x_+\right)^r\right)=\sum_{k=0}^{sr+r-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^k r c_{s,k,i}}{2^{k+1}k!}\delta^{(k)}(x), \tag{1.12}$$

for $s=0,1,2,\ldots$ *and* $r=1,2,\ldots$, *where*

$$c_{r,s,k,i}=\frac{(-1)^s s!\left[(k-2i+1)^{rs+r-1}+(k-2i-1)^{rs+r-1}\right]}{2(rs+r-1)!}. \tag{1.13}$$

In particular, the neutrix composition $\delta(\sinh^{-1}x_+)$ *exists and*

$$\delta\left(\sinh^{-1}x_+\right)=\frac{1}{2}\delta(x). \tag{1.14}$$

Theorem 1.7. *The neutrix composition* $\delta^{(2s-1)}(\sinh^{-1}(\operatorname{sgn}x\cdot x^2))$ *exists and*

$$\delta^{(2s-1)}\left(\sinh^{-1}\left(\operatorname{sgn}x\cdot x^2\right)\right)=\sum_{k=0}^{2s-1}\sum_{i=0}^{i+k+1}\binom{k}{i}\frac{(-1)^k b_{s,k,i}}{2^{k+1}(2k+1)!}\delta^{(k)}(x), \tag{1.15}$$

for $s=1,2,\ldots$, *where*

$$b_{s,k,i}=(k-2i+1)^{2s-1}+(k-2i-1)^{2s-1}. \tag{1.16}$$

In particular

$$\delta'\left(\sinh^{-1}\left(\operatorname{sgn}x\cdot x^2\right)\right)=\frac{\delta'(x)}{4.3!}-2\delta(x). \tag{1.17}$$

2. Main Result

In the next theorem, the constants $c_{j,k}$ are defined by the expansion

$$\left(\tanh^{-1}x\right)^k = \left\{\sum_{i=0}^{\infty}\frac{x^{2i+1}}{2i+1}\right\}^k = \sum_{j=k}^{\infty}c_{j,k}x^j, \tag{2.1}$$

for $k = 0, 1, 2, \ldots$.

Theorem 2.1. *The neutrix composition* $\delta^{(rs-1)}((\tanh x_+)^{1/r})$ *exists and*

$$\delta^{(rs-1)}\left((\tanh x_+)^{1/r}\right) = \sum_{k=0}^{s-1}\sum_{i=0}^{K_k}\frac{(-1)^k c_{s-2i-1,k}(rs)!}{2sk!}\delta^{(k)}(x), \tag{2.2}$$

for $r, s = 1, 2, \ldots$, *where* K_k *is the integer part of* $(s-k-1)/2$.

In particular, the neutrix compositions $\delta^{(r-1)}((\tanh x_+)^{1/r})$ *and* $\delta^{(2r-1)}((\tanh x_+)^{1/r})$ *exist and*

$$\delta^{(r-1)}\left((\tanh x_+)^{1/r}\right) = \frac{(-1)^{r-1}r!}{2}\delta(x), \tag{2.3}$$

$$\delta^{(2r-1)}\left((\tanh x_+)^{1/r}\right) = \frac{(2r)!}{4}\left[\delta(x) - \delta'(x)\right], \tag{2.4}$$

for $r = 1, 2, \ldots$.

Proof. To prove (2.2), first of all we evaluate

$$\int_{-1}^{1}\delta_n^{(rs-1)}\left[(\tanh x_+)^{1/r}\right]x^k dx. \tag{2.5}$$

We have

$$\begin{aligned}\int_{-1}^{1}\delta_n^{(rs-1)}\left[(\tanh x_+)^{1/r}\right]x^k dx &= n^{rs}\int_{-1}^{1}\rho^{(rs-1)}\left[n(\tanh x_+)^{1/r}\right]x^k dx \\ &= n^{rs}\int_{0}^{1}\rho^{(rs-1)}\left[n(\tanh x)^{1/r}\right]x^k dx \\ &\quad + n^{rs}\int_{-1}^{0}\rho^{(rs-1)}(0)x^k dx \\ &= I_{1,k} + I_{2,k}.\end{aligned} \tag{2.6}$$

It is obvious that

$$N-\lim_{n\to\infty} I_{2,k} = N-\lim_{n\to\infty}\int_{-1}^{0}\delta_n^{(rs-1)}\left[(\tanh x_+)^{1/r}\right]x^k dx = 0, \tag{2.7}$$

for $k = 0, 1, 2, \ldots$.

Making the substitution $t = n(\tanh x)^{1/r}$, we have for large enough n

$$\begin{aligned} I_{1,k} &= rn^{rs-r}\int_0^1 t^{r-1}\left[\tanh^{-1}\left(\frac{t}{n}\right)^r\right]^k\left[1-\left(\frac{t}{n}\right)^{2r}\right]^{-1}\rho^{(rs-1)}(t)dt \\ &= r\sum_{i=0}^{\infty}\sum_{j=k}^{\infty}\int_0^1 \frac{c_{j,k}t^{rj+2ri+r-1}}{n^{rj+2ri+r-rs}}\rho^{(rs-1)}(t)dt. \end{aligned} \tag{2.8}$$

It follows that

$$\begin{aligned} N-\lim_{n\to\infty} I_{1,k} &= N-\lim_{n\to\infty} n^{rs}\int_0^1 \rho^{(rs-1)}\left[n(\tanh x)^{1/r}\right]x^k dx \\ &= r\sum_{i=0}^{k_0}\int_0^1 c_{s-2i-1,k}t^{rs-1}\rho^{(rs-1)}(t)dt \end{aligned} \tag{2.9}$$

$$= \sum_{i=0}^{k_0}\frac{(-1)^{rs-1}rc_{s-2i-1,k}(rs-1)!}{2}, \tag{2.10}$$

for $k = 0, 1, 2, \ldots, s-1$, where k_0 denotes the integer part of $(s-k-1)/2$ for $k = 0, 1, 2, \ldots$.

In particular, when $s = 1$, we have $k = 0 = k_0$. It follows from (2.9) that

$$I_{1,0} = r\int_0^1 c_{0,0}t^{2ri+r-1}\rho^{(r-1)}(t)dt, \tag{2.11}$$

and so ordinary limit exists

$$\lim_{n\to\infty} I_{1,0} = r\int_0^1 t^{r-1}\rho^{(r-1)}(t)dt = \frac{(-1)^{r-1}r!}{2}, \tag{2.12}$$

for $r = 1, 2, \ldots$, since $c_{0,0} = 1$.

Further, when $s = 2$, we have $k = 0, 1$ and $k_0 = 0$. It follows from (2.10) that

$$\begin{aligned} N-\lim_{n\to\infty} I_{1,0} &= \frac{rc_{1,0}(2r-1)!}{2} = \frac{(2r)!}{4}, \\ N-\lim_{n\to\infty} I_{1,1} &= \frac{rc_{1,1}(2r-1)!}{2} = \frac{(2r)!}{4}, \end{aligned} \tag{2.13}$$

for $r = 1, 2, \ldots$, since $c_{1,0} = 0$, and $c_{1,1} = 1$.

When $k = s$, we have

$$\begin{aligned}
|I_{1,s}| &= rn^{rs-r}\int_0^1 \left| t^{r-1}\left[\tanh^{-1}\left(\frac{t}{n}\right)^r\right]^s \left[1-\left(\frac{t}{n}\right)^{2r}\right]^{-1} \rho^{(rs-1)}(t)\right| dt \\
&\le rn^{rs-r}\int_0^1 \left| \left[\tanh^{-1}\left(\frac{1}{n}\right)^r\right]^s \left[1-\left(\frac{t}{n}\right)^{2r}\right]^{-1} \rho^{(rs-1)}(t)\right| dt \\
&= rn^{rs-r}\int_0^1 \left| \left[n^{-rs}+O\left(n^{-rs-2r}\right)\right]\rho^{(rs-1)}(t)\right| dt \\
&= O\left(n^{-r}\right).
\end{aligned} \tag{2.14}$$

Thus, if ψ is an arbitrary continuous function, then

$$\lim_{n\to\infty}\int_0^1 \delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right] x^s\psi(x)dx = 0. \tag{2.15}$$

We also have

$$\int_{-1}^0 \delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]\psi(x)dx = n^{rs}\int_{-1}^0 \rho^{(rs-1)}(0)\psi(x)dx, \tag{2.16}$$

and it follows that

$$N-\lim_{n\to\infty}\int_{-1}^0 \delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]\psi(x)dx = 0. \tag{2.17}$$

If now φ is an arbitrary function in $\mathfrak{D}[-1,1]$, then by Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{s-1}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^s}{s!}\varphi^{(s)}(\xi x), \tag{2.18}$$

where $0 < \xi < 1$, and so

$$\begin{aligned}
&N-\lim_{n\to\infty}\left\langle \delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right], \varphi(x)\right\rangle \\
&\quad = N-\lim_{n\to\infty}\sum_{k=0}^{s-1}\frac{\varphi^{(k)}(0)}{k!}\int_0^1 \delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]x^k dx \\
&\qquad + N-\lim_{n\to\infty}\sum_{k=0}^{s-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^0 \delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]x^k dx \\
&\qquad + \lim_{n\to\infty}\frac{1}{s!}\int_0^1 \delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]x^s\varphi^{(s)}(\xi x)dx
\end{aligned}$$

$$
\begin{aligned}
&+ N-\lim_{n\to\infty}\frac{1}{s!}\int_{-1}^{0}\delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]x^s\varphi^{(s)}(\xi x)dx\\
&=\sum_{k=0}^{s-1}\sum_{i=0}^{k_0}\frac{rc_{s-2i-1,k}(rs-1)!}{2}\frac{\varphi^{(k)}(0)}{k!}+0\\
&=\sum_{k=0}^{s-1}\sum_{i=0}^{k_0}\frac{c_{s-2i-1,k}(rs)!}{2s}\frac{(-1)^k}{k!}\left\langle\delta^{(k)}(x),\varphi(x)\right\rangle,
\end{aligned}
\tag{2.19}
$$

on using (2.6), (2.7), (2.10), (2.15), and (2.17). This proves (2.2) on the interval $(-1,1)$.

It is also clear that $\delta^{(rs-1)}((\tanh^{-1}x_+)^{1/r})=0$ for $x>0$ and so (2.2) holds for $x>-1$.

Now suppose that φ is an arbitrary function in $\mathfrak{D}[a,b]$, where $a<b<0$. Then

$$
\int_a^b\delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]\varphi(x)dx=n^{rs}\int_a^b\rho^{(rs-1)}(0)\varphi(x)dx,
\tag{2.20}
$$

and so

$$
N-\lim_{n\to\infty}\int_a^b\delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]\varphi(x)dx=0.
\tag{2.21}
$$

It follows that $\delta^{(rs-1)}((\tanh^{-1}x_+)^{1/r})=0$ on the interval (a,b). Since a and b are arbitrary, we see that (2.2) holds on the real line.

Equations (2.3) and (2.4) are just particular cases of (2.2). Equation (2.3) follows on using (2.12) and (2.4) follows on using (2.13). This completes the proof of the theorem. □

Corollary 2.2. *The neutrix composition* $\delta^{(rs-1)}((\tanh|x|)^{1/r})$ *exists and*

$$
\delta^{(rs-1)}\left((\tanh|x|)^{1/r}\right)=\sum_{k=0}^{s-1}\sum_{i=0}^{K_k}\frac{\left[1+(-1)^k\right]rc_{s-2i-1,k}(rs-1)!}{2k!}\delta^{(k)}(x),
\tag{2.22}
$$

for $r,s=1,2,\ldots$, *where* K_k *is the integer part of* $(s-k-1)/2$.

In particular, the composition $\delta^{(r-1)}((\tanh|x|)^{1/r})$ *exists and the neutrix composition* $\delta^{(2r-1)}((\tanh|x|)^{1/r})$ *exists and*

$$
\begin{aligned}
\delta^{(r-1)}\left((\tanh|x|)^{1/r}\right)&=(-1)^{r-1}r!\delta(x),\\
\delta^{(2r-1)}\left((\tanh|x|)^{1/r}\right)&=\frac{(2r)!}{2}\delta(x),
\end{aligned}
\tag{2.23}
$$

for $r=1,2,\ldots$.

Proof. To prove (2.22), we note that

$$\begin{aligned}\int_{-1}^{1}\delta_n^{(rs-1)}\left[\left(\tanh^{-1}|x|\right)^{1/r}\right]x^k dx &= n^{rs}\int_{-1}^{1}\rho^{(rs-1)}\left[\left(n\tanh^{-1}|x|\right)^{1/r}\right]x^k dx\\ &= n^{rs}\left[1+(-1)^k\right]\int_0^1\rho^{(s)}\left[n\left(\tanh^{-1}x\right)^{1/r}\right]x^k dx,\end{aligned} \tag{2.24}$$

and (2.22) now follow as above. Further, (2.23) are particular cases of (2.22) and so follows immediately. Note that in the particular case $s = 1$, the ordinary limit exists in (2.12) and so the composition $\delta^{(r-1)}((\tanh|x|)^{1/r})$ exists in this case. This completes the proof of the corollary. □

In the next theorem, the constants $b_{j,k}$ are defined by the following expansion

$$\tanh^k x = \sum_{j=k}^{\infty} b_{j,k}x^j, \tag{2.25}$$

for $k = 0, 1, 2, \ldots$.

Theorem 2.3. *The neutrix composition $\delta^{(s)}(\tanh^{-1}x_+^{1/r})$ exists and*

$$\delta^{(s)}\left(\tanh^{-1}x_+^{1/r}\right) = \sum_{k=0}^{K-1}\frac{(-1)^{s+k}b_{s+1,kr+r}(s+1)!}{2k!}\delta^{(k)}(x), \tag{2.26}$$

for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$, where K is the smallest integer for which $s < Kr + r - 1$.

Proof. To prove (2.26), first of all we evaluate

$$\int_{-1}^{1}\delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)x^k dx. \tag{2.27}$$

We have

$$\begin{aligned}\int_{-1}^{1}\delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)x^k dx &= n^{s+1}\int_{-1}^{1}\rho^{(s)}\left(n\tanh^{-1}x_+^{1/r}\right)x^k dx\\ &= n^{s+1}\int_0^1\rho^{(s)}\left(n\tanh^{-1}x^{1/r}\right)x^k dx + n^{s+1}\int_{-1}^{0}\rho^{(s)}(0)x^k dx\\ &= J_{1,k} + J_{2,k}.\end{aligned} \tag{2.28}$$

It is obvious that

$$N-\lim_{n\to\infty} J_{2,k} = N-\lim_{n\to\infty}\int_{-1}^{0}\delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)x^k dx = 0. \tag{2.29}$$

Making the substitution $t = n(\tanh^{-1} x^{1/r})$, we have for large enough n

$$\begin{aligned} J_{1,k} &= rn^s \int_0^1 \tanh^{kr+r-1}\left(\frac{t}{n}\right)\operatorname{sech}^2\left(\frac{t}{n}\right)\rho^{(s)}(t)dt \\ &= \frac{n^{s+1}}{k+1}\int_0^1 \rho^{(s)}(t)d\tanh^{kr+r}\left(\frac{t}{n}\right) \\ &= -\frac{n^{s+1}}{k+1}\int_0^1 \tanh^{kr+r}\left(\frac{t}{n}\right)\rho^{(s+1)}(t)dt \\ &= -\frac{1}{k+1}\sum_{j=kr+r}^{\infty} n^{s-j+1}\int_0^1 b_{j,kr+r}t^j\rho^{(s+1)}(t)dt, \end{aligned} \tag{2.30}$$

and it follows that

$$\begin{aligned} N-\lim_{n\to\infty} J_{1,k} &= N-\lim_{n\to\infty} rn^s \int_0^1 \tanh^{kr+r-1}\left(\frac{t}{n}\right)\operatorname{sech}^2\left(\frac{t}{n}\right)\rho^{(s)}(t)dt \\ &= -\frac{b_{s+1,kr+r}}{k+1}\int_0^1 t^{s+1}\rho^{(s+1)}(t)dt \\ &= \frac{(-1)^s b_{s+1,kr+r}(s+1)!}{2(k+1)}. \end{aligned} \tag{2.31}$$

In particular, when $s = 0$ and $r = 1$, we have $K = 1$ and then

$$\lim_{n\to\infty} J_{1,0} = \frac{b_{1,1}}{2} = \frac{1}{2}. \tag{2.32}$$

When $k = K$, we have

$$\begin{aligned} |J_{1,k}| &\le \frac{1}{K+1}\sum_{j=Kr+r}^{\infty} n^{s-j+1}\int_0^1 \left|b_{j,Kr+r}t^j\rho^{(s+1)}(t)dt\right| \\ &= O\left(n^{s-Kr-r+1}\right). \end{aligned} \tag{2.33}$$

Thus, if ψ is an arbitrary continuous function, then

$$\int_0^1 \delta_n^{(s)}\left(\tanh^{-1} x_+^{1/r}\right)x^K\psi(x)dx = O\left(n^{s-Kr-r+1}\right), \tag{2.34}$$

and so

$$\lim_{n\to\infty}\int_0^1 \delta_n^{(s)}\left(\tanh^{-1} x_+^{1/r}\right)\, x^K\psi(x)dx = \lim_{n\to\infty} rn^s\int_0^1 \tanh^{Kr+r-1}\left(\frac{t}{n}\right)\operatorname{sech}^2\left(\frac{t}{n}\right)\rho^{(s)}(t)\psi(t)dt = 0, \tag{2.35}$$

since $s - Kr - r + 1 < 0$.

We also have

$$\int_{-1}^{0} \delta_n^{(s)}\left(\tanh^{-1} x_+^{1/r}\right)\psi(x)dx = n^{s+1}\int_{-1}^{0} \rho^{(s)}(0)\psi(x)dx, \tag{2.36}$$

and it follows that

$$N-\lim_{n\to\infty}\int_{-1}^{0} \delta_n^{(s)}\left(\tanh^{-1} x_+^{1/r}\right)\psi(x)dx = 0. \tag{2.37}$$

If now φ is an arbitrary function in $\mathfrak{D}[-1,1]$, then by Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{K-1}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^K}{K!}\varphi^{(K)}(\xi x), \tag{2.38}$$

where $0 < \xi < 1$, and so

$$\begin{aligned}
&N-\lim_{n\to\infty}\left\langle \delta_n^{(s)}\left(\tanh^{-1} x_+^{1/r}\right), \varphi(x)\right\rangle \\
&\quad = N-\lim_{n\to\infty}\sum_{k=0}^{K-1}\frac{\varphi^{(k)}(0)}{k!}\int_0^1 \delta_n^{(s)}\left(\tanh^{-1} x_+^{1/r}\right)x^k dx \\
&\qquad + N-\lim_{n\to\infty}\sum_{k=0}^{K-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{0} \delta_n^{(s)}\left[\left(\tanh^{-1} x_+\right)^{1/r}\right] x^k dx \\
&\qquad + \lim_{n\to\infty}\frac{1}{K!}\int_0^1 \delta_n^{(s)}\left(\tanh^{-1} x_+^{1/r}\right)x^K\varphi^{(K)}(\xi x)dx \\
&\qquad + N-\lim_{n\to\infty}\frac{1}{K!}\int_{-1}^{0} \delta_n^{(s)}\left(\tanh^{-1} x_+^{1/r}\right)x^K\varphi^{(K)}(\xi x)dx \\
&\quad = \sum_{k=0}^{K-1}\frac{(-1)^s b_{s+1,kr+r}(s+1)!\varphi^{(k)}(0)}{2k!} + 0 \\
&\quad = \sum_{k=0}^{K-1}\frac{(-1)^{s+k} b_{s+1,kr+r}(s+1)!}{2k!}\left\langle \delta^{(k)}(x), \varphi(x)\right\rangle,
\end{aligned} \tag{2.39}$$

on using (2.9) and (2.12). This proves (2.26) on the interval $(-1,1)$. It is clear that $\delta^{(s)}(\tanh^{-1} x_+^{1/r}) = 0$ for $x > 0$ and so (2.2) holds for $x > -1$.

Now suppose that φ is an arbitrary function in $[a,b]$, where $a < b < 0$. Then

$$\int_a^b \delta_n^{(s)}\left(\tanh^{-1} x_+^{1/r}\right)\varphi(x)dx = n^{s+1}\int_a^b \rho^{(s)}(0)\varphi(x)dx, \tag{2.40}$$

and so

$$N-\lim_{n\to\infty}\int_a^b \delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)\varphi(x)dx=0. \tag{2.41}$$

It follows that $\delta^{(s)}(\tanh^{-1}x_+^{1/r})=0$ on the interval (a,b). Since a and b are arbitrary, we see that (2.26) holds on the real line. □

Corollary 2.4. *The neutrix composition $\delta^{(s)}(\tanh^{-1}|x|^{1/r})$ exists and*

$$\delta^{(s)}\left(\tanh^{-1}|x|^{1/r}\right)=\sum_{k=0}^{K-1}\frac{\left[(-1)^{s+k}+(-1)^s\right]b_{s+1,kr+r}(s+1)!}{2k!}\delta^{(k)}(x), \tag{2.42}$$

for $s=0,1,2,\ldots$ and $r=1,2,\ldots$, where K is the smallest integer for which $s<Kr+r-1$. In particular, the composition $\delta(\tanh^{-1}|x|)$ exists and

$$\delta\left(\tanh^{-1}|x|\right)=\delta(x). \tag{2.43}$$

Proof. To prove (2.42), we note that

$$\begin{aligned}\int_{-1}^{1}\delta_n^{(s)}\left[\left(\tanh^{-1}|x|\right)^{1/r}\right]x^k dx &= n^{s+1}\int_{-1}^{1}\rho^{(s)}\left[\left(n\tanh^{-1}|x|\right)^{1/r}\right]x^k dx\\ &= n^{s+1}\left[(-1)^{s+k}+(-1)^s\right]\int_0^1\rho^{(s)}\left[n\left(\tanh^{-1}x\right)^{1/r}\right]x^k dx,\end{aligned} \tag{2.44}$$

and (2.42) now follows as above. Equation (2.43) is a particular case of (2.42) and so follows immediately. Note that in the particular case $s=0$ and $r=1$, the ordinary limit exists in (2.12), and so the composition $\delta(\tanh|x|)$ exists in this case. For some related results on the neutrix composition of distributions, see [19–22]. □

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