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Research Article

The Convergent Behavior for Parametric Generalized Vector Equilibrium Problems

Yen-Cherng Lin,[1] Po-Jen Cheng,[2] and Su-Ling Lee[3, 4]

[1] *Department of Occupational Safety and Health, China Medical University, Taichung 404, Taiwan*
[2] *Department of Applied Mathematics, National Chiayi University, Chiayi 60004, Taiwan*
[3] *Szu Chen Junior High School, Taichung 43441, Taiwan*
[4] *Department of Applied Statistics, Chung Hua University, Hsinchu 300, Taiwan*

Correspondence should be addressed to Yen-Cherng Lin, yclin@mail.cmu.edu.tw

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We study some properties for parametric generalized vector equilibrium problems and the convergent behavior for the correspondent solution sets of this problem under some suitable conditions. Several existence results and the topological structures of the efficient solutions set are established. Some new results of existence for weak solutions and strong solutions are derived. Finally, we give some examples to illustrate our theory including the example studied by Fang (1992), who established the perturbed nonlinear program (P_μ) and described successfully that the optimal solution of (P_μ) will approach the optimal solution of linear program (P).

1. Introduction and Preliminaries

In recent years, the topological structures of the set of efficient solutions for vector equilibrium problems or generalized systems or variational inequality problems have been discussed in several aspects, as we show in [1–29]. More precisely, we divide this subject into several topics as following. First, the closedness of the set of efficient solutions are studied in [1, 4, 6, 13–16, 27]. Second, the lower semicontinuity of the set of efficient solutions are studied in [1, 9, 10, 19, 21, 23–26, 30]. Third, the upper semicontinuity of the set of efficient solutions are studied in [1, 4, 7, 8, 16, 21, 23–26, 30]. Fourth, the connectedness of the set of efficient solutions are studied in [2, 3, 17, 20, 27, 29]. Fifth, the existence of efficient solutions are studied in [5, 6, 8–12, 16–18, 22, 27, 29, 31].

Gong and Yao [19] establish the lower semicontinuity of the set of efficient solutions for parametric generalized systems with monotone bifunctions in real locally convex

Hausdorff topological vector spaces. They also discuss the connectedness of the efficient solutions for generalized systems, we refer to [20]. Luc [27, Chapter 6] investigates the structures of efficient point sets of linear, convex, and quasiconvex problems and also points out that the closedness and connectedness of the efficient solutions sets are important in mathematical programming. Huang et al. [8] discuss a class of parametric implicit vector equilibrium problems in Hausdorff topological vector spaces, where the mappings f and K are perturbed by parameters, say η and μ, respectively. They establish the upper semicontinuity and lower semicontinuity of the solution mapping for such problems and derive the closedness of the set of efficient solutions. Li et al. [1] discuss the generalized vector quasivariational inequality problem and obtain both upper semicontinuous and lower semicontinuous properties of the set of efficient solutions for parametric generalized vector quasivariational inequality problems. The closedness of the set of efficient solutions is also derived. Cheng [2] discusses the connectedness of the set of weakly efficient solutions for vector variational inequalities in $\mathbb{R}^n$. In 1992, Fang [32] established the perturbed nonlinear program (P_μ) and described successfully that the optimal solution of (P_μ) will approach the optimal solution of linear program (P). We will state the result in Example 3.7 below. We further point out that, in some suitable conditions, such convergent behavior will display continuity. Furthermore, the correspondent solution sets will preserve some kinds of topological properties under the convergent process. These results will show the convergent behavior about the sets of solutions by two kinds of parameters. As mentioned in [20], for the connectedness, "there are few papers which deal with this subject." But from above descriptions, we can understand and the topological structures of the sets of efficient solutions for some problems are more and more popular and interesting subjects. On the other hand, for our recent result [15], we study the generalized vector equilibrium problems in real Hausdorff topological vector space settings. The concepts of weak solutions and strong solutions are introduced. Several new results of existence for weak solutions and strong solutions of the generalized vector equilibrium problems are derived. These inspired us to discuss the parametric generalized vector equilibrium problems (PGVEPs). Let us introduce some notations as follows. We will use these notations through all this paper.

Let X, Y, and Z be arbitrary real Hausdorff topological vector spaces, where X and Z are finite dimensional. Let Δ_1, and Δ_2 be two parametric sets, $K : \Delta_2 \to 2^X$ be a mapping with nonempty values, $\mathcal{K} = \cup_{\eta\in\Delta_1} K(\eta)$, $C : \mathcal{K} \to 2^Y$ a set-valued mapping such that for each $x \in \mathcal{K}$, $C(x)$ is a proper closed convex and pointed cone with apex at the origin and $\operatorname{int} C(x) \neq \emptyset$. For each $x \in \mathcal{K}$, we can define relations "$\leq_{C(x)}$" and "$\nleq_{C(x)}$" as follows: (1) $z \leq_{C(x)} y \Leftrightarrow y - z \in C(x)$ and (2) $z \nleq_{C(x)} y \Leftrightarrow y - z \notin C(x)$. Furthermore, we use the following notations:

$$y \geq_{C(x)} z \Longleftrightarrow z \leq_{C(x)} y, \qquad y \ngeq_{C(x)} z \Longleftrightarrow z \nleq_{C(x)} y. \tag{1.1}$$

Similarly, we can define the relations "$\leq_{\operatorname{int} C(x)}$" and "$\nleq_{\operatorname{int} C(x)}$" if we replace the set $C(x)$ by $\operatorname{int} C(x)$. If the mapping $C(x)$ is constant, then we denote it by C. The mappings $f : \Delta_1 \times Z \times \mathcal{K} \times \mathcal{K} \to Y$ and $T : \mathcal{K} \to 2^Z$ are given. The parametric generalized vector equilibrium problem (PGVEP, for short) is as follows: For every $(\xi, \eta) \in \Delta_1 \times \Delta_2$, we will like to find an $\overline{x} \in K(\eta)$ such that

$$f(\xi, \overline{s}, \overline{x}, y) \notin -\operatorname{int} C(\overline{x}), \tag{1.2}$$

for all $y \in K(\eta)$ and for some $\bar{s} \in T(\bar{x})$. Such set of weak efficient solutions for (PGVEP) is denoted by $\Gamma_w(\xi,\eta)$. If we find $\bar{x} \in K(\eta)$ and some $\bar{s} \in T(\bar{x})$ such that

$$f(\xi,\bar{s},\bar{x},y) \notin -\operatorname{int} C(\bar{x}), \tag{1.3}$$

for all $y \in K(\eta)$. Such set of efficient solutions for (PGVEP) is denoted by $\Gamma(\xi,\eta)$. Our main purpose is to find some topological structures for these two sets, $\Gamma_w(\xi,\eta)$ and $\Gamma(\xi,\eta)$, of efficient solutions of the parametric generalized vector equilibrium problem. Furthermore, we try to find some sufficient conditions lead them to be nonempty or closed or connected or even compact sets.

2. Some Properties for $\Gamma_w(\xi,\eta)$

Theorem 2.1. *Let $X, Y, Z, C, K, \mathcal{K}, T$, and f be given as in Section 1, the parametric spaces Δ_1, Δ_2 be two Hausdorff topological vector spaces. Let the mapping $f : \Delta_1 \times Z \times \mathcal{K} \times \mathcal{K} \to Y$ be such that $(\xi,s,x,y) \to f(\xi,s,x,y)$ is continuous and $y \to f(\xi,s,x,y)$ is $C(x)$-convex for every $(\xi,s,x) \in \Delta_1 \times Z \times \mathcal{K}$, the mapping $T : \mathcal{K} \to 2^Z$ be an upper semicontinuous with nonempty compact values, and the mapping $K : \Delta_2 \to 2^X$ is continuous with nonempty compact and convex values. Suppose that the following conditions hold the following:*

(a) *for any $\xi \in \Delta_1$, $x \in \mathcal{K}$, there is an $s \in Tx$, such that $f(\xi,s,x,x) \notin (-\operatorname{int} C(x))$;*

(b) *the mapping $x \to Y \setminus (-\operatorname{int} C(x))$ is closed [33] on $\mathcal{K}$.*

Then, we have

(1) *for every $(\xi,\eta) \in \Delta_1 \times \Delta_2$, the weak efficient solutions for (PGVEP) exist, that is, the set $\Gamma_w(\xi,\eta)$ is nonempty, where $\Gamma_w(\xi,\eta) = \{\bar{x} \in K(\eta) : f(\xi,\bar{s},\bar{x},y) \notin -\operatorname{int} C(\bar{x})$ for some $\bar{s} \in T(\bar{x})$ for all $y \in K(\eta)\}$.*

(2) *$\Gamma_w : \Delta_1 \times \Delta_2 \to 2^X$ is upper semicontinuous on $\Delta_1 \times \Delta_2$ with nonempty compact values.*

Proof. (1) For any fixed $(\xi,\eta) \in \Delta_1 \times \Delta_2$, we can easy check that the mappings $(s,x) \to f(\xi,s,x,y)$, $y \to f(\xi,s,x,y)$ satisfy all conditions of Corollary 2.2 in [15] with $K = \mathcal{K}$ and $D = \operatorname{conv}(\mathcal{K})$. Hence, from this corollary, we know that $\Gamma_w(\xi,\eta)$ is nonempty.

(2) For any fixed $(\xi,\eta) \in \Delta_1 \times \Delta_2$, we first claim that $\Gamma_w(\xi,\eta)$ is closed in $K(\eta)$, hence it is compact. Indeed, let a net $\{x_\alpha\} \subset \Gamma_w(\xi,\eta)$ and $x_\alpha \to p$ for some $p \in X$. Then, $x_\alpha \in K(\eta)$ and $f(\xi,s_{\alpha y},x_{\alpha y},y) \notin -\operatorname{int} C(x_\alpha)$ for all $y \in K(\eta)$ and for some $s_{\alpha y} \in T(x_\alpha)$. Since $K(\eta)$ is compact, $p \in K(\eta)$. For each α and for each $y \in K(\eta)$, there exists an $s_{\alpha y} \in T(x_\alpha)$ such that $f(\xi,s_{\alpha y},x_\alpha,y) \in Y \setminus (-\operatorname{int} C(x_\alpha))$. Since T is upper semicontinuous with nonempty compact values, and the set $\{x_\alpha\} \cup \{p\}$ is compact, $T(\{x_\alpha\} \cup \{p\})$ is compact. Therefore, without loss of generality, we may assume that the net $\{s_{\alpha y}\}$ converges to some s_y. Then $s_y \in T(p)$. Since the mapping $(s,x) \to f(\xi,s,x,y)$ is continuous, we have

$$\lim_{\alpha} f(\xi,s_{\alpha y},x_\alpha,y) = f(\xi,s_y,p,y). \tag{2.1}$$

Since $f(\xi,s_{\alpha y},x_\alpha,y) \in Y \setminus (-\operatorname{int} C(x_\alpha))$, $x_\alpha \to p$ and the mapping $x \to Y \setminus (-\operatorname{int} C(x))$ is closed, we have

$$f(\xi,s_y,p,y) \in Y \setminus (-\operatorname{int} C(p)). \tag{2.2}$$

This proves that $p \in \Gamma_w(\xi,\eta)$, and hence $\Gamma_w(\xi,\eta)$ is closed. Since $K(\eta)$ is compact, so is $\Gamma_w(\xi,\eta)$.

We next prove that the mapping $\Gamma_w : \Delta_1\times\Delta_2 \to 2^{K(\eta)}$ is upper semicontinuous. That is, for any $(\xi,\eta)\in\Delta_1\times\Delta_2$, if there is a net $\{(\xi_\beta,\eta_\beta)\}$ converges to (ξ,η) and some $x_\beta\in\Gamma_w(\xi_\beta,\eta_\beta)$, we need to claim that there is a $p\in\Gamma_w(\xi,\eta)$ and a subnet $\{x_{\beta_\nu}\}$ of $\{x_\beta\}$ such that $x_{\beta_\nu} \to p$. Indeed, since $x_\beta \in K(\eta_\beta)$ and $K : \Delta_2 \to 2^X$ are upper semicontinuous with nonempty compact values, there is a $p\in K(\eta)$ and a subnet $\{x_{\beta_\nu}\}$ of $\{x_\beta\}$ such that $x_{\beta_\nu} \to p$.

If we can claim that $p\in\Gamma_w(\xi,\eta)$, then we can see that $\Gamma_w : \Delta_1\times\Delta_2 \to 2^X$ is upper semicontinuous on $\Delta_1\times\Delta_2$, and complete our proof. Indeed, if not, there is a $y\in K(\eta)$ such that for every $s\in T(p)$ we have

$$f(\xi,s,x,y)\in -\operatorname{int}C(p). \tag{2.3}$$

Since K is lower semicontinuous, there is a net $\{y_{\beta_\nu}\}$ with $y_{\beta_\nu}\in K(\eta_{\beta_\nu})$ and $y_{\beta_\nu} \to y$. Since $x_{\beta_\nu}\in\Gamma_w(\xi_{\beta_\nu},\eta_{\beta_\nu})$, we have $x_{\beta_\nu}\in K(\eta_{\beta_\nu})$ and, for each y_{β_ν},

$$f(\xi_{\beta_\nu},s_{\beta_\nu},x_{\beta_\nu},y_{\beta_\nu})\in Y\setminus\left(-\operatorname{int}C(x_{\beta_\nu})\right), \tag{2.4}$$

for some $s_{\beta_\nu}\in T(x_{\beta_\nu})$.

Since T is upper semicontinuous and the net $x_{\beta_\nu} \to x$, without loss of generality, we may assume that $s_{\beta_\nu} \to s$ for some $s\in T(x)$. Since the mapping $(\xi,s,x,y) \to f(\xi,s,x,y)$ is continuous, we have

$$\lim_{\beta_\nu} f(\xi_{\beta_\nu},s_{\beta_\nu},x_{\beta_\nu},y_{\beta_\nu}) = f(\xi,s,p,y). \tag{2.5}$$

From (2.4) and the closedness of the mapping $x \to Y\setminus(-\operatorname{int}C(x))$, we have

$$f(\xi,s,p,y)\in Y\setminus\left(-\operatorname{int}C(p)\right), \tag{2.6}$$

which contradicts (2.3). Hence, we have $p\in\Gamma_w(\xi,\eta)$. □

3. Some Properties for $\Gamma(\xi,\eta)$

In the section, we discuss the set $\Gamma(\xi,\eta)$ of the efficient solutions for (PGVEP), where $\Gamma(\xi,\eta) = \{\overline{x}\in K(\eta) : \text{there is an } \overline{s}\in T(\overline{x}), \text{ such that } f(\xi,\overline{s},\overline{x},y)\notin -\operatorname{int}C(\overline{x}) \text{ for all } y\in K(\eta)\}$. The sets of minimal points, maximum points, weak minimal points, and weak maximum points for some set A with respect to the cone $C(\overline{x})$ are denoted by $\operatorname{Min}^{C(\overline{x})}A$, $\operatorname{Max}^{C(\overline{x})}A$, $\operatorname{Min}_w^{C(\overline{x})}A$, and $\operatorname{Max}_w^{C(\overline{x})}A$, respectively. For more detail, we refer the reader to Definition 1.2 of [28].

Theorem 3.1. *Under the framework of Theorem 2.1, for each $(\xi,\eta)\in\Delta_1\times\Delta_2$, there is an $\overline{x}\in\Gamma_w(\xi,\eta)$ with $\overline{s}\in T(\overline{x})$. In addition, if $T(\overline{x})$ is convex, the mapping $s \to -f(\xi,s,\overline{x},y)$ is properly quasi $C(\overline{x})$-convex (Definition 1.1 of [28]) on $T(\overline{x})$ for each $(\xi,y)\in\Delta_1\times K(\eta)$. Assume that the mapping $(s,y) \to f(\xi,s,\overline{x},y)$ satisfies the following conditions:*

(i)

$$\mathrm{Max}^{C(\overline{x})} \bigcup_{s\in T(\overline{x})} \mathrm{Min}_w^{C(\overline{x})} \bigcup_{y\in K(\eta)} \{f(\xi, s, \overline{x}, y)\} \subset \mathrm{Min}_w^{C(\overline{x})} \bigcup_{y\in K(\eta)} \{f(\xi, s, \overline{x}, y)\} + C(\overline{x}) \tag{3.1}$$

for every $s \in T(\overline{x})$;

(ii) *for any fixed $x \in K(\eta)$, if $\delta \in \mathrm{Max}^{C(\overline{x})} \bigcup_{s\in T(\overline{x})}\{f(\xi, s, \overline{x}, y)\}$ and δ cannot be comparable with $f(\xi, \overline{s}, \overline{x}, y)$ which does not equal to δ, then $\delta \nleq_{\mathrm{int}\, C(\overline{x})} 0$;*

(iii) *if $\mathrm{Max}^{C(\overline{x})} \bigcup_{s\in T(\overline{x})} \mathrm{Min}_w^{C(\overline{x})} \bigcup_{y\in K(\eta)}\{f(\xi, s, \overline{x}, y)\} \subset Y \setminus (-\mathrm{int}\, C(\overline{x}))$, there exists an $s \in T(\overline{x})$ such that $\mathrm{Min}_w^{C(\overline{x})} \bigcup_{y\in K(\eta)}\{f(\xi, s, \overline{x}, y)\} \subset Y \setminus (-\mathrm{int}\, C(\overline{x}))$.*

Then, we have

(a) *for every $(\xi, \eta) \in \Delta_1 \times \Delta_2$, the efficient solutions exists, that is, the set $\Gamma(\xi, \eta)$ is nonempty, furthermore, it is compact;*

(b) *the mapping $\Gamma : \Delta_1 \times \Delta_2 \to 2^X$ is upper semicontinuous on $\Delta_1 \times \Delta_2$ with nonempty compact values;*

(c) *for each $(\xi, \eta) \in \Delta_1 \times \Delta_2$, the set $\Gamma(\xi, \eta)$ is connected if $C : K(\eta) \to 2^Y$ is constant, and for any $(\xi, \eta) \in \Delta_1 \times \Delta_2$, $x \in K(\eta)$ and $s \in T(K(\eta))$, $f(\xi, s, x, K(\eta)) + C$ is convex.*

Proof. (a) Fixed any $(\xi, \eta) \in \Delta_1 \times \Delta_2$, we can easy see that all conditions of Theorem 2.3 of [15] hold, hence from Theorem 2.3 of [15], we know that $\Gamma(\xi, \eta)$ is nonempty and compact.

(b) Let $\{(\xi_\alpha, \eta_\alpha)\} \subset \Delta_1 \times \Delta_2$ be a net such that $(\xi_\alpha, \eta_\alpha) \to (\xi, \eta)$ and $\{x_\alpha\}$ be a net with $x_\alpha \in \Gamma(\xi_\alpha, \eta_\alpha)$. Since $x_\alpha \in K(\eta_\alpha)$ and $K : \Delta_2 \to 2^X$ are upper semicontinuous with nonempty compact values, there are an $x \in K(\eta)$ and a subnet $\{x_{\alpha_\iota}\}$ of $\{x_\alpha\}$ such that $x_{\alpha_\iota} \to x$. Since $T : K \to 2^Z$ is upper semicontinuous with nonempty compact values, $T(\{x_{\alpha_\iota}\} \cup \{x\})$ is compact. Since $s_{\alpha_\iota} \in T(x_{\alpha_\iota})$, there is an $s \in T(x)$ such that a subnet of $\{s_{\alpha_\iota}\}$ converges to s. Without loss of generality, we still denote the subnet by $\{s_{\alpha_\iota}\}$, and hence $s_{\alpha_\iota} \to s$.

If $x \notin \Gamma(\xi, \eta)$, then there is a $y \in K(\eta)$ such that

$$f(\xi, s, x, y) \in -\mathrm{int}\, C(x). \tag{3.2}$$

Since $K(\eta)$ is compact, there is a net, say $\{y_{\alpha_\iota}\}$, in $K(\eta)$ converges to y. Since the mapping $(\xi, s, x, y) \to f(\xi, s, x, y)$ is continuous, and the mapping $x \to Y \setminus (-\mathrm{int}\, C(x))$ is closed, we have

$$f(\xi, s, x, y) = \lim_{\alpha_\iota} f(\xi_{\alpha_\iota}, s_{\alpha_\iota}, x_{\alpha_\iota}, y_{\alpha_\iota}) \in Y \setminus (-\mathrm{int}\, C(x)), \tag{3.3}$$

which contracts (3.2). Thus, $x \in \Gamma(\xi, \eta)$.

In order to prove (c), we introduce Lemmas 3.2–3.4 as follows.

Let $Y^\star$ be the topological dual space of Y. For each $x \in \mathcal{K}$,

$$C^\star(x) = \{g \in Y^\star : g(y) \geq 0\ \forall y \in C(x)\}. \tag{3.4}$$

Let $C^\star = \cap_{x\in\mathcal{K}} C^\star(x)$, then $C^\star$ is nonempty and connected. If $C^\star : \mathcal{K} \to 2^{Y^\star}$ is a constant mapping, then $C^\star(x) = C^\star$ for all $x \in \mathcal{K}$. In the sequel, we suppose that $C^\star$ is not a singleton. That is, $C^\star \setminus \{0\} \neq \emptyset$, and hence it is connected. For each $g \in C^\star \setminus \{0\}$, let us denote the set of g-efficient solutions to (PGVEP) by

$$S^{\xi,\eta}(g) = \left\{ x \in K(\eta) : \sup_{s\in T(x)} g(f(\xi, s, x, y)) \geq 0 \text{ for every } y \in K(\eta) \right\}. \tag{3.5}$$

Lemma 3.2. *Under the framework of Theorem 3.1,*

$$S^{\xi,\eta}(g) \neq \emptyset, \tag{3.6}$$

for every $g \in C^\star \setminus \{0\}$.

Proof. From (a) of Theorem 3.1, we know that, for each $(\xi, \eta) \in \Delta_1 \times \Delta_2$, there is an $\overline{x} \in K(\eta)$ with $\overline{s} \in T(\overline{x})$ such that

$$g(f(\xi, \overline{s}, \overline{x}, y)) \geq 0, \tag{3.7}$$

for all $y \in K(\eta)$ and for all $g \in C^\star \setminus \{0\}$. Thus, $\overline{x} \in S^{\xi,\eta}(g)$ for every $g \in C^\star \setminus \{0\}$. Hence, $S^{\xi,\eta}(g) \neq \emptyset$ for every $g \in C^\star \setminus \{0\}$.

Lemma 3.3. *Suppose that for any $(\xi, \eta) \in \Delta_1 \times \Delta_2$ and $y \in K(\eta)$, $f(\xi, T(K(\eta)), K(\eta), y)$ are bounded. Then, the mapping $S^{\xi,\eta} : C^\star \setminus \{0\} \to 2^{K(\eta)}$ is upper semicontinuous with compact values.*

Proof. Fixed any $(\xi, \eta) \in \Delta_1 \times \Delta_2$. We first claim that the mapping $S^{\xi,\eta} : C^\star \setminus \{0\} \to 2^{K(\eta)}$ is closed. Let $x_\nu \in S^{\xi,\eta}(g_\nu)$, $x_\nu \to x$ and $g_\nu \to g$ with respect to the strong topology $\sigma(Y^\star, Y)$ in $Y^\star$.

Since $x_\nu \in S^{\xi,\eta}(g_\nu)$, there is an $s_\nu \in T(x_\nu)$ such that $g(f(\xi, s_\nu, x_\nu, y)) \geq 0$ for all $y \in K(\eta)$. Since T is upper semicontinuous with nonempty compact values, by a similar argument in the proof of Theorem 3.1(b), there is an $s \in T(x)$ such that a subnet of $\{s_\nu\}$ converges to s. Without loss of generality, we still denote the subnet by $\{s_\nu\}$.

For each $y \in K(\eta)$, we define $P_{f(\xi,T(K(\eta)),K(\eta),y)}(g) = \sup_{z\in f(\xi,T(K(\eta)),K(\eta),y)} |g(z)|$ for all $g \in Y^\star$. We note that the set $f(\xi, T(K(\eta)), K(\eta), y)$ is bounded by assumption, hence $P_{f(\xi,T(K(\eta)),K(\eta),y)}(g)$ is well defined and is a seminorm of $Y^\star$. For any $\varepsilon > 0$, let $\mathcal{U}_\varepsilon = \{g \in Y^\star : P_{f(\xi,T(K(\eta)),K(\eta),y)}(g) < \varepsilon\}$ be a neighborhood of 0 with respect to $\sigma(Y^\star, Y)$. Since $g_\nu \to g$, there is a $\alpha_0 \in \Lambda$ such that $g_\nu - g \in \mathcal{U}_\varepsilon$ for every $\nu \geq \nu_0$. That is, $P_{f(\xi,T(K(\eta)),K(\eta),y)}(g_\nu - g) = \sup_{z\in f(\xi,T(K(\eta)),K(\eta),y)} |(g_\nu - g)(z)| < \varepsilon/2$ for every $\nu \geq \nu_0$. This implies that

$$\left|(g_\nu - g)(f(\xi, s_\nu, x_\nu, y))\right| < \frac{\varepsilon}{2}, \tag{3.8}$$

for all $\nu \geq \nu_0$. Since the mapping $(s, x) \to f(\xi, s, x, y)$ is continuous and $(s_\nu, x_\nu) \to (s, x)$, we have

$$f(\xi, s_\nu, x_\nu, y) \to f(\xi, s, x, y). \tag{3.9}$$

By the continuity of g, we have

$$\left|g(f(\xi,s_\nu,x_\nu,y))-g(f(\xi,s,x,y))\right|<\frac{\varepsilon}{2}, \tag{3.10}$$

for some ν_1 and all $\nu \geq \nu_1$. Let us choose $\nu_2 = \max\{\nu_0,\nu_1\}$. Combining (3.8) and (3.10), we know that, for all $\nu \geq \nu_2$,

$$\begin{aligned}
&\left|g_\nu(f(\xi,s_\nu,x_\nu,y))-g(f(\xi,s,x,y))\right| \\
&\quad\leq \left|g_\nu(f(\xi,s_\nu,x_\nu,y))-g(f(\xi,s_\nu,x_\nu,y))\right| \\
&\qquad+\left|g(f(\xi,s_\nu,x_\nu,y))-g(f(\xi,s,x,y))\right| \\
&\quad<\frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\
&\quad=\varepsilon.
\end{aligned} \tag{3.11}$$

That is $g_\nu(f(\xi,s_\nu,x_\nu,y)) \to g(f(\xi,s,x,y))$. Since $g_\nu(f(\xi,s_\nu,x_\nu,y)) \geq 0$, there is an $s \in T(x)$ such that $g(f(\xi,s,x,y)) \geq 0$, which proves that $x \in S^{\xi,\eta}(g)$. Therefore, the mapping $S^{\xi,\eta} : C^\star \setminus \{0\} \to 2^{K(\eta)}$ is closed. By the compactness and Corollary 7 in [33, page 112], the mapping $S^{\xi,\eta}$ is upper semicontinuous with compact values.

Lemma 3.4. *Suppose that for any $(\xi,\eta) \in \Delta_1 \times \Delta_2$, $x \in K(\eta)$ and $s \in T(K(\eta))$, $f(\xi,s,x,K(\eta)) + C(x)$ is convex. Then*

$$\Gamma(\xi,\eta) \supseteq \bigcup_{g\in C^\star\setminus\{0\}} S^{\xi,\eta}(g). \tag{3.12}$$

Furthermore, if $C : K(\eta) \to 2^Y$ is constant, then we have

$$\Gamma(\xi,\eta) = \bigcup_{g\in C^\star\setminus\{0\}} S^{\xi,\eta}(g). \tag{3.13}$$

Proof. We first claim that $\Gamma(\xi,\eta) \supseteq \cup_{g\in C^\star\setminus\{0\}} S^{\xi,\eta}(g)$.

If $x \in \cup_{g\in C^\star\setminus\{0\}} S^{\xi,\eta}(g)$, there is a $g \in C^\star \setminus \{0\}$ such that $x \in S^{\xi,\eta}(g)$. Then, there is a $g \in C^\star \setminus \{0\}$ such that

$$g(f(\xi,s,x,y)) \geq 0 \tag{3.14}$$

for all $y \in K(\eta)$. This implies that $f(\xi,s,x,y) \notin -\operatorname{int} C(x)$ for all $y \in K(\eta)$. Indeed, if there is a $\overline{y} \in K(\eta)$ such that $f(\xi,s,x,\overline{y}) \in -\operatorname{int} C(x)$. Since $g \in C^\star \setminus \{0\}$, we have $g(f(\xi,s,x,y)) < 0$ which contracts (3.14). Thus, $x \in \Gamma(\xi,\eta)$. This proves (3.12) holds.

Second, if $C : K(\eta) \to 2^Y$ is constant, we claim that $\Gamma(\xi,\eta) \subseteq \cup_{g\in C^\star\setminus\{0\}} S^{\xi,\eta}(g)$.

If $x \in \Gamma(\xi, \eta)$, then $x \in K(\eta)$ with $s \in T(x)$ and $f(\xi, s, x, y) \notin -\operatorname{int} C$ for all $y \in K(\eta)$, that is, $f(\xi, s, x, K(\eta)) \cap (-\operatorname{int} C) = \emptyset$. Hence,

$$(f(\xi, s, x, K(\eta)) + C) \cap (-\operatorname{int} C) = \emptyset. \tag{3.15}$$

Since $f(\xi, s, x, K(\eta)) + C$ is convex, by Eidelheit separation theorem, there is a $g \in Y^{\star} \setminus \{0\}$ and $\rho \in \mathbb{R}$ such that

$$g(w') < \rho \leq g(f(\xi, s, x, y) + w), \tag{3.16}$$

for all $y \in K(\eta)$, $w \in C$, $w' \in -\operatorname{int} C$. Then,

$$(g - \rho)(w') < 0 \leq (g - \rho)(f(\xi, s, x, y) + w), \tag{3.17}$$

for all $y \in K(\eta)$, $w \in C$, $w\prime \in -\operatorname{int} C$.

Without loss of generality, we denote $g - \rho$ by g, then

$$g(w') < 0 \leq g(f(\xi, s, x, y) + w), \tag{3.18}$$

for all $y \in K(\eta)$, $w \in C$, $w' \in -\operatorname{int} C$. By the left-hand side inequality of (3.18) and the linearity of g, we have $g(m) > 0$ for all $m \in \operatorname{int} C$. Since C is closed, for any m in the boundary of C, there is a net $\{m_{\nu}\} \subset \operatorname{int} C$ such that $m_{\nu} \to m$. By the continuity of g, $g(m) = g(\lim_{\nu} m_{\nu}) = \lim_{\nu} g(m_{\nu}) \geq 0$. Hence, for all $w \in C$, $g(w) \geq 0$, that is $g \in C^{\star} \setminus \{0\}$.

By the right-hand side inequality of (3.18), for all $w \in C$, there is an $s \in T(x)$ such that $g(f(\xi, s, x, y) + w) \geq 0$ for all $y \in K(\eta)$. This implies that $g(f(\xi, s, x, y)) \geq 0$ for all $y \in K(\eta)$ if we choose $w = 0$. Hence, $\sup_{s \in T(x)} g(f(\xi, s, x, y)) \geq 0$ for all $y \in K(\eta)$. Thus, $x \in S^{\xi,\eta}(g)$. Therefore, $x \in \cup_{g \in C^{\star} \setminus \{0\}} S^{\xi,\eta}(g)$, and hence

$$\Gamma(\xi, \eta) \subseteq \bigcup_{g \in C^{\star} \setminus \{0\}} S^{\xi,\eta}(g). \tag{3.19}$$

Combining this with (3.12), we have

$$\Gamma(\xi, \eta) = \bigcup_{g \in C^{\star} \setminus \{0\}} S^{\xi,\eta}(g). \tag{3.20}$$

□

Now, we go back to prove Theorem 3.1(c).

Proof of Theorem 3.1(c). From Lemmas 3.2 and 3.3, the mapping $S^{\xi,\eta} : C^{\star} \setminus \{0\} \to 2^{K(\eta)}$ is upper semicontinuous with nonempty compact values. From Lemma 3.4 and Theorem 3.1 [29], we know that for each $(\xi, \eta) \in \Delta_1 \times \Delta_2$, the set $\Gamma(\xi, \eta)$ is connected. □

Modifying the Example 3.1 [8], we give the following examples to illustrate Theorems 2.1 and 3.1 as follows.

Example 3.5. Let $\Delta_1 = \Delta_2 = X = Y = Z = \mathbb{R}$, $K(\eta) = [0,1]$ for all $\eta \in \Delta_2$, $\mathcal{K} = \cup_{\eta\in\Delta_2} K(\eta) = [0,1]$, $C(x) = [0,\infty)$ for all $x \in \mathcal{K}$. Choose $T : \mathcal{K} \to 2^Z$ by $T(x) = \{x, x/2\}$ for all $x \in \mathcal{K}$. Define $f(\xi, s, x, y) = s - y + \xi^2$ for all $(\xi, x, y) \in \Delta_1 \times X \times Y$. Then, all the conditions of Theorem 2.1 hold, and $\Gamma_w(\xi,\eta) = [1-\xi^2, 1] \cap [0,1]$ for all $(\xi,\eta) \in \Delta_1 \times \Delta_2$. Indeed, since there are two choices for s, one is x, and the other is $x/2$. If the nonnegative number ξ^2 is less than 1, for any y in $[0,1]$, and we always choose $s = x/2$, then for this case, the set $\Gamma_w(\xi,\eta)$ will contain all elements of the set $[2(1-\xi^2), 1]$. Furthermore, if we always choose $s = x$, then the set $\Gamma_w(\xi,\eta)$ will contain all elements of the set $[(1-\xi^2), 1]$. If the nonnegative number ξ^2 is greater than or equal to 1, then the set $\Gamma_w(\xi,\eta)$ will contain all elements of the set $[0,1]$. Hence,

$$\begin{aligned}\Gamma_w(\xi,\eta) &= \left(\left[2\left(1-\xi^2\right),1\right] \cap [0,1]\right) \bigcup \left(\left[\left(1-\xi^2\right),1\right] \cap [0,1]\right) \bigcup [0,1] \\ &= \left[\left(1-\xi^2\right),1\right] \cap [0,1].\end{aligned} \tag{3.21}$$

Here, we note that we cannot apply Theorem 3.1 since $T(x)$ is not convex.

Example 3.6. Following Example 3.5, let $T(x) = [x/2, x]$ for all $x \in \mathcal{K} = [0,1]$. By Theorem 2.1, the set $\Gamma_w(\xi,\eta) \neq \emptyset$. We choose any $\overline{x} \in \Gamma_w(\xi,\eta)$, and we can see the mapping $s \to -f(\xi, s, \overline{x}, y)$ is properly quasi $C(\overline{x})$-convex on $T(\overline{x})$ for any $(\xi, y) \in \Delta_1 \times K(\eta)$. Since $\mathrm{Max}^{C(\overline{x})} \cup_{s\in[\overline{x}/2,\overline{x}]} \mathrm{Min}_w^{C(\overline{x})} \cup_{y\in[0,1]} \{s - y + \xi^2\} = \{\overline{x} - 1 + \xi^2\} \subset \{s - 1 + \xi^2\} + [0,\infty) = \mathrm{Min}_w^{C(\overline{x})} \cup_{y\in[0,1]} \{s - y + \xi^2\} + C(\overline{x})$ for all $s \in [\overline{x}/2, \overline{x}] = T(\overline{x})$. So, condition (i) of Theorem 3.1 holds. Obviously, the condition (ii) also holds, since no such δ exists in this example. Now, we can see condition (iii) holds. Indeed, from the facts

$$\begin{aligned}\mathrm{Min}_w^{C(\overline{x})} \bigcup_{y\in[0,1]} \left\{s - y + \xi^2\right\} &= \left\{s - 1 + \xi^2\right\}, \\ \mathrm{Max}^{C(\overline{x})} \bigcup_{s\in T(\overline{x})} \mathrm{Min}_w^{C(\overline{x})} \bigcup_{y\in[0,1]} \left\{s - y + \xi^2\right\} &= \left\{\overline{x} - 1 + \xi^2\right\},\end{aligned} \tag{3.22}$$

we know that if $\overline{x} - 1 + \xi^2 \geq 0$, then we can choose $s = \overline{x} \in T(\overline{x})$ such that $s - 1 + \xi^2 \geq 0$. Hence, we can apply Theorem 3.1, and we know that $\Gamma(\xi,\eta)$ is nonempty compact and connected. Let us compute the set $\Gamma(\xi,\eta)$ for any $(\xi,\eta) \in \Delta_1 \times \Delta_2$. If we choose any $\overline{s} = t\overline{x}$ for some $t \in [1/2, 1]$, we can see that all the points in the set $[(1-\xi^2)/t, 1]$ are efficient solutions for (PGVEP). Hence,

$$\begin{aligned}\Gamma(\xi,\eta) &= \bigcup_{t\in[1/2,1]} \left[\left(1-\xi^2\right)/t, 1\right] \cap [0,1] \\ &= \left[\left(1-\xi^2\right),1\right] \cap [0,1],\end{aligned} \tag{3.23}$$

for any $(\xi,\eta) \in \Delta_1 \times \Delta_2$.

Example 3.7 (see [32]). The perturbed nonlinear program (P_μ) described successfully that the optimal solutions set $\Gamma(\mu)$ of (P_μ) will approach the optimal solutions set Γ of linear program (P), where (P) and (P_μ) are as follows

$$\begin{aligned} \text{Minimise} \quad & c^T x, \\ \text{subject to} \quad & Ax = 0, \\ & e^T x = 1, \\ & x \geq 0, \end{aligned} \tag{P}$$

$$\begin{aligned} \text{minimise} \quad & \mathbf{c}^T \mathbf{x} + \mu \sum_{j=1}^{n} x_j \log x_j, \\ \text{subject to} \quad & Ax = 0, \\ & e^T x = 1, \\ & x \geq 0, \end{aligned} \tag{P_μ}$$

where $\mu > 0$.

We further note that, such convergent behavior will be described by upper semicontinuity by Theorems 2.1 and 3.1. That is,

$$\Gamma(\mu) \xrightarrow{\text{u.s.c.}} \Gamma \quad \text{as } \mu \to 0^+. \tag{3.24}$$

Furthermore, the correspondent solution sets will preserve some kinds of topological properties, such as compactness and connectedness, under the convergent process.

We would like to point out an open question that naturally raises from Theorems 2.1 and 3.1. Under what conditions the mappings Γ_w and Γ will be lower semicontinuous?

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