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Research Article

Convergence of an Iterative Algorithm for Common Solutions for Zeros of Maximal Accretive Operator with Applications

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The aim of this paper is to introduce an iterative algorithm for finding a common solution of the sets $(A + M_2)^{-1}(0)$ and $(B + M_1)^{-1}(0)$, where M is a maximal accretive operator in a Banach space and, by using the proposed algorithm, to establish some strong convergence theorems for common solutions of the two sets above in a uniformly convex and 2-uniformly smooth Banach space. The results obtained in this paper extend and improve the corresponding results of Qin et al. 2011 from Hilbert spaces to Banach spaces and Petrot et al. 2011. Moreover, we also apply our results to some applications for solving convex feasibility problems.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ with the dual space E^* and let $\langle\cdot,\cdot\rangle$ denote the pairing between E and E^*. Let C be a nonempty closed convex subset of E. We define the *generalized duality mapping* $J_q : E \to 2^{E^*}$ by

$$J_q(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1} \right\}, \quad \forall x \in E, \tag{1.1}$$

for all $q > 1$. In the special case, for $q = 2$, we called the mapping J_2 as the *normalized duality mapping* and as usual we write $J_2 = J$. The following is the well-known properties of the generalized duality mapping J_q:

(1) $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \in E$ with $x \neq 0$;

(2) $J_q(tx) = t^{q-1} J_q(x)$ for all $x \in E$ and $t \in [0,\infty)$;

(3) $J_q(-x) = -J_q(x)$ for all $x \in E$.

It is well known that if X is smooth, then J is single valued, which is denoted by j. Recall that the duality mapping j is said to be *weakly sequentially continuous* if, for each sequence $\{x_n\}$ with $x_n \to x$ weakly, we have $j(x_n) \to j(x)$ weakly*. We know that, if X admits a weakly sequentially continuous duality mapping, then X is smooth. For the details, see [1–3].

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be

(1) *uniformly convex* if there exists $\delta > 0$ such that, for any $x, y \in U$ and, for any $\epsilon \in (0,2]$, $\|x - y\| \geq \epsilon$ implies $\|(x+y)/2\| \leq 1 - \delta$.

We can see that every uniformly convex Banach space is also reflexive and strictly convex.

(2) *Smooth* if $\lim_{t\to 0}(\|x + ty\| - \|x\|)/t$ exists for all $x, y \in U$.

(3) *Uniformly smooth* if the limit is attained uniformly for $x, y \in U$. The *modulus of smoothness* of E is defined by

$$\rho(\tau) = \sup\left\{\frac{1}{2}\left(\|x+y\| + \|x-y\|\right) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\right\}, \tag{1.2}$$

where $\rho : [0,\infty) \to [0,\infty)$ is a function. In the other way, E is uniformly smooth if and only if $\lim_{\tau\to 0}\rho(\tau)/\tau = 0$.

(4) *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$ where q is a fixed real number with $1 < q \leq 2$. (see, for instance, [1, 4]).

We note that E is a uniformly smooth Banach space if and only if J_q is single valued and uniformly continuous on any bounded subset of E. Examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for any $p > 1$. Note also that no Banach space is q-uniformly smooth for $q > 2$ (see [1, 5] for more details).

Let $A : C \to E$ be a nonlinear mapping. The mapping A is said to be

(1) *accretive* if

$$\langle Ax - Ay, J(x-y)\rangle \geq 0, \quad \forall x, y \in C, \tag{1.3}$$

(2) *λ-strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, J(x-y)\rangle \geq \lambda\|x - y\|^2, \quad \forall x, y \in C, \tag{1.4}$$

(3) *λ-inverse-strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, J(x-y)\rangle \geq \lambda\|Ax - Ay\|^2, \quad \forall x, y \in C, \tag{1.5}$$

Definition 1.1. Let $M : E \to 2^E$ be a multivalued maximal accretive mapping. The single-valued mapping $J_{M,\rho} : E \to E$ defined by

$$J_{M,\rho}(u) = \left(I + \rho M\right)^{-1}(u), \quad \forall u \in E, \tag{1.6}$$

is called the *resolvent operator* associated with M, where ρ is any positive number and I is the identity mapping.

Let T be a mapping from E into itself. We use $F(T)$ to denote the set of fixed points of the mapping T. Recall that the mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in E. \tag{1.7}$$

A mapping $f : C \to C$ is said to be contractive if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in C. \tag{1.8}$$

Recently, Aoyama et al. [4] considered the following generalized variational inequality problem in a smooth Banach space: Find a point $x \in C$ such that

$$\langle Ax, j(y - x) \rangle \geq 0, \quad \forall y \in C, \tag{1.9}$$

where A is an accretive operator of C into E. This problem is related to the fixed point problem for nonlinear mappings, the problem of finding a zero point of an accretive operator, and so on. For the problem of finding a zero point of an accretive operator by the proximal point algorithm, see Agarwal et al. [6], Cho et al. [7, 8], Kamimura and Takahashi [9, 10], Qin et al. [11], Song et al. [12], and Wei and Cho [13]. In order to find a solution of the variational inequality (1.9), Aoyama et al. [4] studied the weak convergence theorem for accretive operators in Banach spaces, which is a generalization of the result by Iiduka et al. [14] from the class of Hilbert spaces.

Theorem AIT (see [4], Aoyama et al. Theorem 3.1). *Let E be a uniformly convex and 2-uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\alpha > 0$ and A be an α-inverse strongly accretive operator of C into E with $S(C, A) \neq \emptyset$, where*

$$S(C, A) = \left\{x^* \in C : \left\langle Ax^*, j(x - x^*)\right\rangle \geq 0, \forall x \in C\right\}. \tag{1.10}$$

If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen such that $\lambda_n \in [a, \alpha/K^2]$ for some $a > 0$ and $\alpha_n \in [b, c]$ for some b, c with $0 < b < c < 1$, then the sequence $\{x_n\}$ defined by the following manners: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n), \quad \forall n \geq 1, \tag{1.11}$$

converges weakly to an element z of $S(C, A)$, where K is the 2-uniformly smoothness constant of E and Q_C is a sunny nonexpansive retraction.

In 2011, Katchang and Kumam [15] presented an iterative algorithm for finding a common solution of fixed point problems and a general system of variational inequality problems for two accretive operators as shown in the following: for all $n \geq 0$,

$$\begin{aligned} x_0 &= u \in C, \\ y_n &= Q_C(x_n - \mu B x_n), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n S Q_C(y_n - \lambda A y_n). \end{aligned} \tag{1.12}$$

They proved that the sequence $\{x_n\}$ generated by the above algorithm converges strongly to a point $\overline{x} = Q_{\mathcal{F}} f(\overline{x})$. Moreover, they apply their theorem to find zeros of accretive operators and the class of k-strictly pseudocontractive mappings.

Recently, Petrot et al. [16] considered the problem so-called *quasivariational inclusion problem*, that is, determine an element $u \in H$ such that

$$0 \in A(u) + M(u), \tag{1.13}$$

where $A : H \to H$ is a single-valued nonlinear mapping and $M : H \to 2^H$ is a multivalued mapping. The set of solutions of the above problem is denoted by $\mathrm{VI}(H, A, M)$. Therefore, they presented a new iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of variational inclusion problem with a multivalued maximal monotone mapping and an α-inverse-strongly monotone mapping by using the iterative sequence $\{x_n\}$ defined as follows:

$$\begin{aligned} x_0 &\in H, \ \text{chosen arbitrarily}, \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n S z_n, \\ z_n &= J_M^{\lambda}(y_n - \lambda A y_n), \\ y_n &= J_M^{\lambda}(x_n - \lambda A x_n), \quad \forall n \geq 0, \end{aligned} \tag{1.14}$$

and, under appropriated conditions, they proved the the sequence $\{x_n\}$ generated by (1.14) converges strongly to a point $z_0 \in H$, which is the unique solution in $F(S) \cap \mathrm{VI}(H, A, M)$ to the following variational inequality:

$$\langle (f - I) z_0, z_0 - z \rangle \leq 0, \quad \forall z \in F(S) \cap \mathrm{VI}(H, A, M). \tag{1.15}$$

Very recently, Qin et al. [17] introduced an iterative scheme for a general variational inequality (VI) and proved the strong convergence theorems of common solutions of two variational inequalities in a uniformly convex and 2-uniformly smooth Banach space by using the following iterative sequence $\{x_n\}$:

$$\begin{aligned} x_0 &= u \in C, \\ y_n &= \delta_n Q_C(x_n - \rho B x_n) + (1-\delta_n) Q_C(x_n - \lambda A x_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n y_n, \quad \forall n \geq 0. \end{aligned} \tag{1.16}$$

They proved that the sequence $\{x_n\}$ generated by the above algorithm converges strongly to a point $q = Q_{\mathrm{VI}} u$, where Q_{VI} is the unique sunny nonexpansive retraction from C onto VI.

Motivated and inspired by the above recent works, in this paper, we introduce an iterative scheme for finding zeros of maximal accretive operators. Furthermore, we prove some strong convergence theorems and also propose applications for solving the convex feasibility problems. Our results improve and extend the corresponding results of Qin et al. [17] and Katchang and Kumam [15], Petrot et al. [16], and many others.

2. Preliminaries

Note that, if C and D are nonempty subsets of a Banach space E such that D is a subset of a closed convex subset C and $Q : C \to D$. Then Q is said to be *sunny* if

$$Q(Qx + t(x - Qx)) = Qx, \tag{2.1}$$

whenever $Qx + t(x - Qx) \in C$ for any $x \in C$ and $t \geq 0$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction Q of C onto D. A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z is in the range of Q (see [4, 18] for more details).

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 2.1 (see [19]). *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*

(1) *Q is sunny and nonexpansive;*

(2) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle$ *for all $x, y \in E$;*

(3) $\langle x - Qx, J(y - Qx)\rangle \leq 0$ *for all $x \in E$ and $y \in C$.*

Proposition 2.2 (see [20]). *Let C be a nonempty closed convex subset of a uniformly convex and let uniformly smooth Banach space E and T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

We need the following lemmas in order to prove our main results.

Lemma 2.3 (see [5]). *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E. \tag{2.2}$$

Lemma 2.4 (see [21]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1. \tag{2.3}$$

Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{2.4}$$

Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.5 (see [22]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \ \forall n \geq 0, \tag{2.5}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty}\delta_n/\alpha_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 (see [23]). *Let C be a closed convex subset of a strictly convex Banach space E. Let $T_m : C \to C$ be a nonexpansive mappings for each $1 \leq m \leq r$, where r is some integer. Suppose that $\cap_{m=1}^{r} F(T_m)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{m=1}^{r}\lambda_n = 1$. Then the mapping $S : C \to C$ defined by*

$$Sx = \sum_{m=1}^{r} \lambda_m T_m x, \quad \forall x \in C, \tag{2.6}$$

is well defined, nonexpansive, and $F(S) = \cap_{m=1}^{r} F(T_m)$ holds.

Lemma 2.7 (see [24]). *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and T be nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence in C such that $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then x is a fixed point of T.*

Lemma 2.8 (see [3, 4]). *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E. Let a mapping $A : C \to E$ be λ-inverse-strongly accretive. Then one has*

$$\|(I-\rho_2 A)x - (I-\rho_2 A)y\|^2 \leq \|x-y\|^2 + 2\rho_2\left(\rho_2 K^2 - \lambda\right)\|Ax - Ay\|^2. \tag{2.7}$$

If $\lambda \geq \rho_2 K^2$, then $I - \rho_2 A$ is nonexpansive.

Proof. For any $x, y \in C$, it follows from Lemma 2.3 that

$$\begin{aligned}\|(I-\rho_2 A)x-(I-\rho_2 A)y\|^2 &= \|(x-y)-\rho_2(Ax-Ay)\|^2 \\ &\le \|x-y\|^2 - 2\rho_2\langle Ax-Ay, j(x-y)\rangle + 2\rho_2^2K^2\|Ax-Ay\|^2 \\ &\le \|x-y\|^2 - 2\rho_2\lambda\|Ax-Ay\|^2 + 2\rho_2^2K^2\|Ax-Ay\|^2 \\ &= \|x-y\|^2 + 2\rho_2\left(\rho_2K^2-\lambda\right)\|Ax-Ay\|^2.\end{aligned} \tag{2.8}$$

If $\lambda \ge \rho_2 K^2$, then $I-\rho_2 A$ is nonexpansive. This completes the proof. □

Lemma 2.9. *Let C be a nonempty subset of a Banach space E. Let A be a mapping of C into E, M be a maximal accretive operator on E and $J_{M,\rho} = (I+\rho M)^{-1}$ be the resolvent of M for any $\rho > 0$. Then $F(J_{M,\rho}(I-\rho A)) = (A+M)^{-1}(0)$ for all $\rho > 0$.*

Proof. Let $\rho > 0$ be fixed. Then we have

$$\begin{aligned} u \in F(J_{M,\rho}(I-\rho A))u &\Longleftrightarrow u = J_{M,\rho}(I-\rho A)u = (I+\rho M)^{-1}(I-\rho A)u \\ &\Longleftrightarrow (I+\rho M)u \ni (I-\rho A)u \\ &\Longleftrightarrow \rho Mu \ni -\rho Au \\ &\Longleftrightarrow Mu \ni -Au \\ &\Longleftrightarrow (A+M)u \ni 0 \\ &\Longleftrightarrow u \in (A+M)^{-1}(0).\end{aligned} \tag{2.9}$$

This completes the proof. □

Lemma 2.10. *Let E be a Banach space. Then for all $x, y \in E$,*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle. \tag{2.10}$$

3. Main Results

In this section, we prove strong convergence theorems for a λ-inverse-strongly accretive mapping $A : C \to E$ and a β-inverse-strongly accretive $B : C \to E$ in a real 2-uniformly smooth Banach space E.

In order to prove our main results, we need the following lemma.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E with the best smooth constant K. Let J_{M_1,ρ_1}, J_{M_2,ρ_2} be a resolvent operator associated with M_1, M_2,* *where $M_1, M_2 : E \to 2^E$ is a multivalued maximal accretive mapping. Let the mappings $A, B : C \to E$ be λ-inverse-strongly accretive and β-inverse-strongly accretive, respectively. Let $G : C \to C$ be a mapping defined by*

$$Gx = \delta J_{M_1,\rho_1}(x - \rho_1 Bx) + (1-\delta) J_{M_2,\rho_2}(x - \rho_2 Ax), \quad \forall x \in C. \tag{3.1}$$

If $\lambda \geq \rho_2 K^2$ and $\beta \geq \rho_1 K^2$, then G is nonexpansive.

Proof. Since J_{M_1,ρ_1} and J_{M_2,ρ_2} are nonexpansive, for any $x, y \in C$, it follows from Lemma 2.8 that

$$\begin{aligned}
\|G(x) - G(y)\| &= \|\delta J_{M_1,\rho_1}(x - \rho_1 Bx) + (1-\delta) J_{M_2,\rho_2}(x - \rho_2 Ax) \\
&\qquad - \delta J_{M_1,\rho_1}(y - \rho_1 By) - (1-\delta) J_{M_2,\rho_2}(y - \rho_2 Ay)\| \\
&\leq \delta \| J_{M_1,\rho_1}(x - \rho_1 Bx) - J_{M_1,\rho_1}(y - \rho_1 By)\| \\
&\quad + (1-\delta)\| J_{M_2,\rho_2}(x - \rho_2 Ax) - J_{M_2,\rho_2}(y - \rho_2 Ay)\| \\
&\leq \delta \| J_{M_1,\rho_1}(I - \rho_1 B)x - J_{M_1,\rho_1}(I - \rho_1 B)y\| \\
&\quad + (1-\delta)\| J_{M_2,\rho_2}(I - \rho_2 A)x - J_{M_2,\rho_2}(I - \rho_2 A)y\| \\
&\leq \|x - y\|.
\end{aligned} \tag{3.2}$$

Therefore, G is nonexpansive. This completes the proof. □

Next, we state the main result of this work.

Theorem 3.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and C be a nonempty closed convex subset of E. Let $A, B : C \to E$ be λ-inverse-strongly accretive and β-inverse-strongly accretive, respectively, and K be the best smooth constant. Let f be a contraction of E into itself with coefficient $\alpha \in [0,1)$. Suppose that $\Omega := (A + M_2)^{-1}(0) \cap (B + M_1)^{-1}(0) \neq \emptyset$ and G is a mapping defined by Lemma 3.1. Let ρ_1, ρ_2 be any positive real numbers such that $\rho_1 \leq \beta/K^2$ and $\rho_2 \leq \lambda/K^2$. For arbitrary $x_0 = x \in C$, define the iterative sequence $\{x_n\}$ as follows:*

$$\begin{aligned}
x_0 &= u \in C, \\
y_n &= \delta_n J_{M_1,\rho_1}(x_n - \rho_1 Bx_n) + (1-\delta_n) J_{M_2,\rho_2}(x_n - \rho_2 Ax_n), \\
x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n,
\end{aligned} \tag{3.3}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ in $(0,1)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$;

(C2) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C4) $\lim_{n\to\infty}\delta_n = \delta \in (0,1)$.

Then the sequence $\{x_n\}$ *generated by* (3.3) *converges strongly to a point* $q = Q_\Omega f(q)$, *where* Q_Ω *is a sunny nonexpansive retraction on* Ω.

Proof. First, we prove that $J_{M_1,\rho_1}(I - \rho_1 B)$ and $J_{M_2,\rho_2}(I - \rho_2 A)$ are nonexpansive mappings. Consider the following:

$$\begin{aligned}\left\|J_{M_1,\rho_1}(I-\rho_1 B)x - J_{M_1,\rho_1}(I-\rho_1 B)y\right\|^2 &\le \left\|(x-y) - \rho_1(Bx - By)\right\|^2 \\ &\le \|x-y\|^2 - 2\rho_1\langle Bx - By, J(x-y)\rangle \\ &\quad + 2K^2\rho_1^2\|Bx - By\|^2 \\ &\le \|x-y\|^2 - 2\rho_1\beta\|Bx-By\|^2 + 2K^2\rho_1^2\|Bx-By\|^2 \\ &= \|x-y\|^2 + 2\rho_1\left(\rho_1 K^2 - \beta\right)\|Bx-By\|^2 \\ &\le \|x-y\|^2.\end{aligned} \tag{3.4}$$

Thus, it follows that $J_{M_1,\rho_1}(I - \rho_1 B)$ is nonexpansive and so is $J_{M_2,\rho_2}(I - \rho_2 A)$.

Step 1. We show that $\{x_n\}$ is bounded. For any $p \in \Omega$, we have

$$\begin{aligned}\|y_n - p\| &= \left\|\delta_n\left[J_{M_1,\rho_1}(x_{\mathrm{n}} - \rho_1 Bx_n) - p\right] + (1-\delta_n)\left[J_{M_2,\rho_2}(x_n - \rho_2 Ax_n) - p\right]\right\| \\ &\le \delta_n\left\|J_{M_1,\rho_1}(x_n - \rho_1 Bx_n) - p\right\| + (1-\delta_n)\left\|J_{M_2,\rho_2}(x_n - \rho_2 Ax_n) - p\right\| \\ &\le \delta_n\|x_n - p\| + (1-\delta_n)\|x_n - p\| \\ &\le \|x_n - p\|.\end{aligned} \tag{3.5}$$

It follows by induction that

$$\begin{aligned}\|x_{n+1} - p\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - p\| \\ &\le \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\ &\le \alpha\alpha_n\|x_n - p\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| + \gamma_n\|x_n - p\| \\ &\le \alpha\alpha_n\|x_n - p\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= (1 - \alpha_n + \alpha\alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &= (1 - \alpha_n(1-\alpha))\|x_n - p\| + \alpha_n(1-\alpha)\frac{\|f(p) - p\|}{1-\alpha}\end{aligned}$$

$$
\begin{aligned}
&\le \max\left\{ \|x_n - p\|, \frac{\|f(p) - p\|}{1-\alpha} \right\} \\
&\cdots \\
&\le \max\left\{ \|x_1 - p\|, \frac{\|f(p) - p\|}{1-\alpha} \right\}.
\end{aligned}
\tag{3.6}
$$

Thus the sequence $\{x_n\}$ is bounded and so is $\{y_n\}$.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Let $u_n = J_{M_1,\rho_1}(I - \rho_1 B)x_n$ and $v_n = J_{M_2,\rho_2}(I - \rho_2 A)x_n$ for each $n \ge 0$. Then we have

$$
\begin{aligned}
y_{n+1} - y_n &= (\delta_{n+1}u_{n+1} + (1-\delta_{n+1})v_{n+1}) - (\delta_n u_n + (1-\delta_n)v_n) \\
&= \delta_{n+1}(u_{n+1} - u_n) + (\delta_{n+1} - \delta_n)(u_n - v_n) + (1-\delta_{n+1})(v_{n+1} - v_n),
\end{aligned}
\tag{3.7}
$$

and so

$$
\begin{aligned}
\|y_{n+1} - y_n\| &\le \delta_{n+1}\|u_{n+1} - u_n\| + |\delta_{n+1} - \delta_n|\|u_n - v_n\| + (1-\delta_{n+1})\|v_{n+1} - v_n\| \\
&\le \delta_{n+1}\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|M_1 + (1-\delta_{n+1})\|x_{n+1} - x_n\| \\
&= \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|M_1,
\end{aligned}
\tag{3.8}
$$

where M_1 ia an appropriate constant such that $M_1 \ge \sup_{n\ge 0}\{\|u_n - v_n\|\}$.

Next, let $z_n = (x_{n+1} - \beta_n x_n)/(1-\beta_n)$ for all $n \ge 0$. Then we have $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ for all $n \ge 0$. Now, we compute

$$
\begin{aligned}
z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n y_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}f(x_{n+1}) + (1-\alpha_{n+1}-\beta_{n+1})y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + (1-\alpha_n-\beta_n)y_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}f(x_{n+1}) - \alpha_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) - \alpha_n y_n}{1-\beta_n} + y_{n+1} - y_n,
\end{aligned}
\tag{3.9}
$$

and so

$$
\|z_{n+1} - z_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - y_n\| + \|y_{n+1} - y_n\|. \tag{3.10}
$$

Substituting (3.8) into (3.10), we get

$$\begin{aligned}\|z_{n+1}-z_n\| \le{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left\|f(x_{n+1})-y_{n+1}\right\| + \frac{\alpha_n}{1-\beta_n}\left\|f(x_n)-y_n\right\| \\ &+ \|x_{n+1}-x_n\| + |\delta_{n+1}-\delta_n| M_1,\end{aligned} \tag{3.11}$$

that is,

$$\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left\|f(x_{n+1})-y_{n+1}\right\| + \frac{\alpha_n}{1-\beta_n}\left\|f(x_n)-y_n\right\| + |\delta_{n+1}-\delta_n| M_1. \tag{3.12}$$

From the conditions (C2) and (C3), it follows that

$$\limsup_{n\to\infty}\left(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|\right) \le 0. \tag{3.13}$$

Thus, from Lemma 2.4, it follows that

$$\lim_{n\to\infty}\|z_n-x_n\| = 0. \tag{3.14}$$

From the definition of x_{n+1} in this step, we observe that $x_{n+1}-x_n = (1-\beta_n)(z_n-x_n)$. Then we have

$$\begin{aligned}\lim_{n\to\infty}\|x_{n+1}-x_n\| &= \lim_{n\to\infty}\left\|(1-\beta_n)(z_n-x_n)\right\| \\ &= 0.\end{aligned} \tag{3.15}$$

Step 3. We show that $\limsup_{n\to\infty}\langle (f-I)q, J(x_n-q)\rangle \le 0$, where $q = Q_\Omega f(q)$. Define a mapping $G : C \to C$ by Lemma 3.1 Then, it follows that G is a nonexpansive mapping such that

$$F(G) = F\left(J_{M_1,\rho_1}(\mathrm{I}-\rho_1 B)\right) \cap F\left(J_{M_2,\rho_2}(I-\rho_2 A)\right) = (B+M)^{-1}(0) \cap (A+M)^{-1}(0) = \Omega. \tag{3.16}$$

Consider the following:

$$\begin{aligned}y_n - Gx_n &= \delta_n v_n + (1-\delta_n)u_n - (\delta v_n + (1-\delta)u_n) \\ &= (\delta_n-\delta)(v_n-u_n).\end{aligned} \tag{3.17}$$

From the condition (C4), we have

$$\lim_{n\to\infty}\left\|y_n - Gx_n\right\| = 0. \tag{3.18}$$

Next, we consider

$$\begin{aligned}\|x_n - Gx_n\| &= \left\|x_n - x_{n+1} + x_{n+1} - y_n + y_n - Gx_n\right\| \\ &\le \|x_n - x_{n+1}\| + \left\|x_{n+1} - y_n\right\| + \left\|y_n - Gx_n\right\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\left\|f(x_n) - y_n\right\| + \beta_n\left\|x_n - y_n\right\| + \left\|y_n - Gx_n\right\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\left\|f(x_n) - y_n\right\| + \beta_n\|x_n - Gx_n\| + \beta_n\left\|Gx_n - y_n\right\| + \left\|y_n - Gx_n\right\|.\end{aligned} \tag{3.19}$$

Therefore, we have

$$(1-\beta_n)\|x_n - Gx_n\| \le \|x_n - x_{n+1}\| + \alpha_n\left\|f(x_n) - y_n\right\| + (\beta_n + 1)\left\|Gx_n - y_n\right\|. \tag{3.20}$$

From the conditions (C2), (C3), (3.15), (3.18), and the inequality above, we obtain

$$\lim_{n\to\infty}\|x_n - Gx_n\| = 0. \tag{3.21}$$

Thus, since $Q_\Omega f(q)$ is a contraction, there exists a unique fixed point. We denote that q is the unique fixed point to the mapping $Q_\Omega f(q)$ which means that $q = Q_\Omega f(q)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$, it follows from (3.21) that

$$\lim_{n\to\infty}\|x_{n_i} - Gx_{n_i}\| = 0. \tag{3.22}$$

Since G is nonexpansive, it follows from Lemma 2.7 that $p = Gp$ we obtain that $p \in F(G)$. By (3.22), we have $p \in \Omega$.

Furthermore, with the reason that $\{x_n\}$ is bounded, we can choose the sequence $\{x_{n_i}\}$ of $\{x_n\}$ which $\{x_{n_i}\} \rightharpoonup p$ such that

$$\limsup_{n\to\infty}\left\langle (f-I)q, J(x_n - q)\right\rangle = \lim_{i\to\infty}\left\langle (f-I)q, J(x_{n_i} - q)\right\rangle. \tag{3.23}$$

Now, from (3.23) and Proposition 2.1(3) and the weakly sequential continuity of the duality mapping J, we have

$$\begin{aligned}\limsup_{n\to\infty}\left\langle (f-I)q, J(x_n - q)\right\rangle &= \lim_{i\to\infty}\left\langle (f-I)q, J(x_{n_i} - q)\right\rangle \\ &= \left\langle (f-I)q, J(p-q)\right\rangle \le 0.\end{aligned} \tag{3.24}$$

From (3.15), it follows that

$$\limsup_{n\to\infty}\left\langle (f-I)q, J(x_{n+1} - q)\right\rangle \le 0. \tag{3.25}$$

Step 4. We show that $\{x_n\}$ converges strongly to a point $q = Q_\Omega f(q)$. In fact, observe that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \langle x_{n+1}-q, J(x_{n+1}-q)\rangle \\
&= \langle \alpha_n(f(x_n)-q)+\beta_n(x_n-q)+\gamma_n(y_n-q), J(x_{n+1}-q)\rangle \\
&\le \alpha\alpha_n\langle x_n-q, J(x_{n+1}-q)\rangle + \alpha_n\langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\quad + \beta_n\langle x_n-q, J(x_{n+1}-q)\rangle + \gamma_n\langle y_n-q, J(x_{n+1}-q)\rangle \\
&\le \alpha\alpha_n\|x_n-q\|\|x_{n+1}-q\| + \alpha_n\langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\quad + \beta_n\|x_n-q\|\|x_{n+1}-q\| + \gamma_n\|y_n-q\|\|x_{n+1}-q\| \\
&\le (1-\alpha_n(1-\alpha))\|x_n-q\|\|x_{n+1}-q\| + \alpha_n\langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\le \frac{1-\alpha_n(1-\alpha)}{2}\left(\|x_n-q\|^2+\|x_{n+1}-q\|^2\right) + \alpha_n\langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\le \frac{1-\alpha_n(1-\alpha)}{2}\|x_n-q\|^2 + \frac{1}{2}\|x_{n+1}-q\|^2 + \alpha_n\langle f(q)-q, J(x_{n+1}-q)\rangle.
\end{aligned} \tag{3.26}$$

Thus it follows that

$$\|x_{n+1}-q\|^2 \le (1-\alpha_n(1-\alpha))\|x_n-q\|^2 + 2\alpha_n\langle f(q)-q, J(x_{n+1}-q)\rangle. \tag{3.27}$$

Therefore, from Condition (C2), (3.25), and Lemma 2.5, we get $\|x_n - q\| \to 0$ as $n \to \infty$. This completes the proof. □

Corollary 3.3. *Let E be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and let C be a nonempty closed convex subset of E. Let $A, B : C \to E$ be λ-inverse-strongly accretive and β-inverse-strongly accretive, respectively, and K be the best smooth constant. Suppose that $\Omega := (A+M_2)^{-1}(0) \cap (B+M_1)^{-1}(0) \neq \emptyset$, where G is a mapping defined by Lemma 3.1. Let ρ_1, ρ_2 be any positive real numbers such that $\rho_1 \le \beta/K^2$ and $\rho_2 \le \lambda/K^2$. For arbitrary $x_0 = x \in C$, define the iterative sequence $\{x_n\}$ by*

$$\begin{aligned}
x_0 &= u \in C, \\
y_n &= \delta_n J_{M_1,\rho_1}(x_n - \rho_1 B x_n) + (1-\delta_n) J_{M_2,\rho_2}(x_n - \rho_2 A x_n), \\
x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n y_n, \quad \forall n \ge 0,
\end{aligned} \tag{3.28}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ in $(0,1)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$;

(C2) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(C4) $\lim_{n\to\infty}\delta_n = \delta \in (0,1)$.

Then the sequence $\{x_n\}$ generated by (3.28) converges strongly to a point $q = Q_\Omega f(q)$, where Q_Ω is a sunny nonexpansive retraction on Ω.

Proof. Take $f(x_n) = u$ for all $n \geq 1$ for any fixed $u \in C$ in (3.3). Then, by Theorem 3.2, we can conclude the desired conclusion easily. □

4. Applications

4.1. Application to Convex Feasibility Problems

In this part, we consider the following convex feasibility problem (CFP): find $x \in \cap_{j=1}^{N} C_j$, where $j \in \{1, 2, \ldots, N\}$ and C_j denotes the set of zeros of a maximal accretive operator.

The following result can be obtained from Theorem 3.2.

Theorem 4.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly sequentially continuous duality mapping and let C be a nonempty closed convex subset of E. Let $\{A_i\}_i = 1^N : C \to E$ be an λ_i-inverse-strongly accretive and K be the best smooth constant. Let f be a contraction of E into itself with coefficient $\alpha \in [0, 1)$. Suppose that $\Omega := \bigcap_{i=1}^{N} (A_i + M_i)^{-1}(0) \neq \emptyset$, where G is a mapping defined by Lemma 3.1. Let ρ_i be any positive real numbers such that $\rho_i \leq \lambda_i / K^2$, $i = 1, 2, 3, \ldots, N$. For arbitrary $x_0 = x \in C$, define the iterative sequence $\{x_n\}$ by*

$$\begin{aligned} x_0 &= u \in C, \\ y_n &= \sum_{i=1}^{N} \delta_{i,n} J_{M_i, \rho_i}(x_n - \rho_i A_i x_n), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \quad \forall n \geq 0, \end{aligned} \tag{4.1}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ in $(0, 1)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$ *and* $\sum_{i=1}^{N} \delta_{i,n} = 1$;

(C2) $\lim_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$;

(C4) $\lim_{n \to \infty} \delta_{i,n} = \delta \in (0, 1)$.

Then the sequence $\{x_n\}$ generated by (4.1) converges strongly to a point $q = Q_\Omega$, where Q_Ω is a sunny nonexpansive retraction on Ω.

4.2. Application to Hilbert Spaces

Assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $A : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a multivalued mapping. The problem of finding $u \in H$ such that

$$\theta \in A(u) + M(u) \tag{4.2}$$

is called the *quasivariational inclusion problem*, and we denote the set of solutions of the above variational inclusion by $\mathrm{VI}(H, A, M)$.

If $M = \partial\delta_C$, where C is a nonempty closed convex subset of H and $\delta_C : H \to [0,\infty]$ is the indicator function of C, that is,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{4.3}$$

Then the variational inclusion problem (4.2) is equivalent to the problem of finding $u \in C$ such that

$$\langle A(u), v - u \rangle \geq 0, \quad \forall v \in C, \tag{4.4}$$

which is the well-known *Hartman-Stampacchia variational inequality problem* [25].

Theorem 4.2. *Let C be a closed convex subset of a real Hilbert space H. Let $A, B : C \to H$ be λ-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let f be a contraction of E into itself with coefficient $\alpha \in [0,1)$. Suppose that $\Omega := \mathrm{VI}(C,A) \cap \mathrm{VI}(C,B) \neq \emptyset$, where $\mathrm{VI}(C,A)$ and $\mathrm{VI}(C,B)$ are the sets of solutions of variational inequality (4.4). For arbitrary $x_0 = x \in C$, define the iterative sequence $\{x_n\}$ by*

$$\begin{aligned} x_0 &= u \in C, \\ y_n &= \delta_n P_C(x_n - \rho_1 B x_n) + (1-\delta_n) P_C(x_n - \rho_2 A x_n), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \quad \forall n \geq 0, \end{aligned} \tag{4.5}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ in $(0,1)$ satisfy the following conditions:

(C1) $\alpha_n + \beta_n + \gamma_n = 1$;

(C2) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(C3) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C4) $\lim_{n\to\infty}\delta_n = \delta \in (0,1)$.

Then the sequence $\{x_n\}$ generated by (4.5) converges strongly to a point $P_\Omega x_0$.

Proof. Take $M = \partial\delta_C : H \to 2^H$, where $\delta_C : H \to [0,\infty]$ is the indicator function of C. Let $J(M,\rho) = I$. Then we get

$$\begin{aligned} P_C(x_n - \rho_1 A x_n) &= J_{M,\rho_1} P_C(x_n - \rho_1 A x_n), \\ P_C(x_n - \rho_2 B x_n) &= J_{M,\rho_2} P_C(x_n - \rho_2 B x_n). \end{aligned} \tag{4.6}$$

Thus the conclusion can be obtained from Theorem 3.2 immediately. □

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