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Research Article

Geodesic Effect Near an Elliptical Orbit

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Using a differential geometric treatment, we analytically derived the expression for De Sitter (geodesic) precession in the elliptical motion of the Earth through the gravitational field of the Sun with Schwarzschild's metric. The expression obtained in this paper in a simple way, using a classical approach, agrees with that given in B. M. Barker and R. F. O'Connell (1970, 1975) in a different setting, using the tools of Newtonian mechanics and the Euler-Lagrange equations.

1. Introduction

The geodesic effect, also named De Sitter precession or geodesic precession, represents the effect of the curvature of the space-time on a constant spin vector transported together with a body along an orbit through a gravitational field in Einstein's theory. De Sitter found that the Earth-Moon system would undergo a precession in the gravitational field of the Sun. De Sitter's work [1] was subsequently extended to rotating bodies, such as the Earth, by Schouten [2] and by Fokker [3]. Studying the Sun's gravitational field near a circular orbit with Schwarzschild's metric, it is emphasized the existence of a precessional motion along the Earth's orbit, as effect of the Sun's gravitational field [4].

The concept of geodesic precession has two slightly different meanings, as the body moving in orbit may have rotation or not. Nonrotating bodies move on geodesics, while the rotating bodies move in slightly different orbits. A geodesic is a curve which parallel transports a tangent vector. If a curve is not geodesic, then a vector tangent to it at some point does not remain tangent in parallel transport along this curve. In a Riemannian space, a vector parallel transported along a closed contour does not return, in general, to its original position.

The difference between De Sitter precession and Lense-Thirring precession is due to the rotation of the central mass. The total precession is calculated by combining De Sitter

precession with Lense-Thirring precession (see for more details [5, 6]). Barker and O'Connell [7] discussed the difference between De Sitter precession and geodesic precession, obtaining the correct expression for the precession in the case of the nearly circular orbits for binary systems with relatively massive components.

The geodesic precession is usually associated with the motion of the gyroscope orbiting in the static gravitational field of the source. This precession is obtained by parallel transport of a spin vector in curved space-time in the vicinity of the mass. The effect is presented even if the mass is not rotating. We also note that the geodesic precession has been recently studied in different settings in [8–10].

In this paper, using a classical treatment, we deduce the expression for the geodesic effect in the elliptical motion of the Earth through the gravitational field of the Sun with Schwarzschild's metric.

2. Elliptical Orbit in the Gravitational Field

The Schwarzschild metric

$$ds^2 = -\frac{dr^2}{1-2\mu/c^2 r} - r^2\left(d\theta^2 + \cos^2\theta d\varphi^2\right) + \left(c^2 - \frac{2\mu}{r}\right)dt^2 \tag{2.1}$$

is an exact, static, spherically symmetric solution of Einstein's equation of the general relativity, which represents the gravitational field of a cosmic object without rotation, which possesses mass [11, 12]. The quantity ds denotes the invariant space-time interval, an absolute measure of the distance between two events in space-time, c is the speed of light, t is the time coordinate measured by a stationary clock at infinity, r is the radial coordinate, while the variables θ and φ are the latitude and the longitude of mass M defined in the classical conception with respect to the equatorial plane and the prime meridian, passing through the center O of the spherical mass.

In the solar system, according to the formulas of the elliptical motion, the μ coefficient is equal to $4\pi^2 a^3/T^2$, $2a$ being the major axis of the orbit and T the period of revolution of a planet, expression that has significantly the same value for all the planets, according to Kepler's third law. $2\pi/T$ will be noted by n, where n is the mean angular velocity corresponding to the period T of the motion, called for simplicity mean motion. The coefficient μ is also equal to the product $f(S+M)$ of the universal gravitational constant and the mass of the planet-Sun system (where S is the mass of the Sun and M is the planet's mass).

Making the change of variable $\varphi = nt + \psi$, one obtains

$$d\varphi^2 = n^2 dt^2 + 2n d\psi dt + d\psi^2. \tag{2.2}$$

Thus, from the relations (2.1) and (2.2), we have

$$ds^2 = -\frac{dr^2}{1-2\mu/c^2 r} - r^2\left(d\theta^2 + \cos^2\theta d\psi^2\right) - 2nr^2\cos^2\theta d\psi dt + \left(c^2 - \frac{2\mu}{r} - n^2 r^2 \cos^2\theta\right)dt^2. \tag{2.3}$$

In the following, we will determine if the elliptical motion can occur in the plane of the equator. To this purpose, the existence of the geodesics of Schwarzschild's metric ds^2 along which the first three variables, namely r, θ, φ, have the constant values is studied.

From the equations of geodesics

$$\frac{d^2x_i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 0, \tag{2.4}$$

with $i \in \{1,2,3,4\}$ and $x_1 = r,\ x_2 = \theta,\ x_3 = \varphi,\ x_4 = t$, it remains for $\alpha = \beta = 4$

$$\frac{d^2x_i}{ds^2} + \Gamma^i_{44}\left(\frac{dx_4}{ds}\right)^2 = 0. \tag{2.5}$$

If in the relation (2.5) we take $i = 1, 2$ or 3, then it follows that

$$\Gamma^i_{44}\left(\frac{dx_4}{ds}\right)^2 = 0. \tag{2.6}$$

Because $dx_4/ds \neq 0$, one obtains $\Gamma^i_{44} = 0$, with $i \in \{1,2,3\}$. But

$$\Gamma^i_{44} = \frac{1}{2}g^{ik}\left(2\frac{\partial g_{4k}}{\partial x_4} - \frac{\partial g_{44}}{\partial x_k}\right), \tag{2.7}$$

and therefore it is found that

$$g^{ik}\left(2\frac{\partial g_{4k}}{\partial x_4} - \frac{\partial g_{44}}{\partial x_k}\right) = 0, \tag{2.8}$$

with Einstein summation convention. The coefficient

$$g_{44} = c^2 - \frac{2\mu}{r} - n^2r^2\cos^2\theta \tag{2.9}$$

does not depend on the variables φ and t, but only on r and θ. Then it follows that $\partial g_{44}/\partial\varphi = 0$ and $\partial g_{44}/\partial t = 0$. On the other hand, $g_{11} = -1/(1 - 2\mu/c^2r)$, $g_{22} = -r^2$, $g_{33} = -r^2\cos^2\theta$, $g_{34} = -nr^2\cos^2\theta$, and the remaining coefficients $g_{ij} = 0$, with $i \neq j$, $i, j \in \{1,2,3,4\}$.

Since $(g^{ij})_{i,j=\overline{1,4}} = \lfloor (g_{ij})_{i,j=\overline{1,4}} \rfloor^{-1}$, it is observed that $g^{22} \neq 0$. Consequently, for $i = 2$, we deduce from the relation (2.8) that $\partial g_{44}/\partial\theta = 0$. But

$$\frac{\partial g_{44}}{\partial\theta} = 2n^2r^2\cos\theta\sin\theta. \tag{2.10}$$

Hence,

$$2n^2r^2\cos\theta\sin\theta = 0, \tag{2.11}$$

and then one obtains $\theta = 0$; in other words the elliptical orbit is located in the plane of the equator.

In order to study the gravitational field near an elliptical orbit, a parallel transport of a vector whose origin describes the corresponding line of Universe is considered.

In the elliptical motion, at the perihelion passage r takes the minimum value, namely, $r_{\min} = a(1-e)$, and at the passage through aphelion r takes the maximum value $r_{\max} = a(1+e)$, where a represents the length of the semimajor axis and e is the eccentricity of the elliptical orbit.

In the following, we develop in the quadratic form ds^2 from (2.3), the coefficients of $d\psi^2$, $d\psi dt$, and dt^2 in the neighborhood of the system of values $1/r = 1/a(1-e^2)$ and $\theta = 0$, where for the inverse of the radius vector $1/r$ it has been taken its average value

$$\left(\frac{1}{r}\right)_{\text{av}} = \frac{1}{2}\left(\frac{1}{r_{\min}} + \frac{1}{r_{\max}}\right) = \frac{1}{a(1-e^2)}. \tag{2.12}$$

We obtain the following expression:

$$\begin{aligned} ds^2 = &-\frac{dr^2}{1-2\mu/c^2 r} - r^2 d\theta^2 - \left[r^2 + \frac{n^2 a^4 (1-e^2)^4}{c^2 - 3\mu/a}\right] d\psi^2 \\ &+ 4na^3\left(1-e^2\right)^3 \left[\frac{1}{r} - \frac{1}{a(1-e^2)}\right] d\psi dt \\ &+ \left(c^2 - \frac{3\mu}{a}\right)\left[dt - \frac{na^2(1-e^2)^2}{c^2 - 3\mu/a} d\psi\right]^2 + ds_1^2, \end{aligned} \tag{2.13}$$

where it has been noted by ds_1^2 a quadratic form that depends on the four differentials dr, $d\theta$, $d\psi$, and dt whose coefficients depend on the variables r and θ, which vanish for $1/r = 1/a(1-e^2)$ and $\theta = 0$. Therefore, ds_1^2 does not belong to the group of the Christoffel symbols which will be written in the following. Making the substitution

$$u = t - \frac{na^2(1-e^2)^2}{c^2 - 3\mu/a}\psi, \tag{2.14}$$

the expression of ds^2 can be rewritten as

$$\begin{aligned} ds^2 \quad = &-\frac{dr^2}{1-2\mu/c^2 r} - r^2 d\theta^2 + 4na^3\left(1-e^2\right)^3 \left[\frac{1}{r} - \frac{1}{a(1-e^2)}\right] d\psi du \\ &- \left\{r^2 + \frac{n^2 a^4 (1-e^2)^4 + 4n^2 a^5 (1-e^2)^5 \left[1/r - 1/a(1-e^2)\right]}{c^2 - 3\mu/a}\right\} d\psi^2 \\ &+ \left(c^2 - \frac{3\mu}{a}\right) du^2 + ds_1^2. \end{aligned} \tag{2.15}$$

3. Geodesic Effect in the Elliptical Motion

In the following, we study the phenomenon of precession in the elliptical motion. We will calculate the variations of the components of a contravariant vector (X^1, X^2, X^3, X^4) whose origin describes the line of Universe considered in the elliptical motion and whose components are proportional by definition with $dr, d\theta, d\psi, du$.

According to the general theory, these variations are given by the equation

$$dX^i = -\Gamma^i_{\alpha\beta} X^\alpha dx_\beta, \tag{3.1}$$

where

$$\Gamma^i_{\alpha\beta} = \frac{1}{2} g^{ik} \left(\frac{\partial g_{\alpha k}}{\partial x_\beta} + \frac{\partial g_{\beta k}}{\partial x_\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x_k} \right) \tag{3.2}$$

and $\beta = 4$, since in the considered movement $dr, d\theta, d\psi$ are null, and where the Christoffel symbols are calculated from the metric (2.15) for $1/r = 1/a(1-e^2)$ and $\theta = 0$. From (2.15) we have

$$(g_{ij})_{i,j=\overline{1,4}} = \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 \\ 0 & 0 & g_{33} & g_{34} \\ 0 & 0 & g_{34} & g_{44} \end{pmatrix}, \tag{3.3}$$

where

$$\begin{gathered} g_{11} = -\frac{1}{1 - 2\mu/c^2 r}, \qquad g_{22} = -r^2, \\ g_{33} = -r^2 - \frac{n^2 a^4 (1-e^2)^4 + 4n^2 a^5 (1-e^2)^5 \left[1/r - 1/a(1-e^2)\right]}{c^2 - 3\mu/a}, \\ g_{34} = 2na^3 \left(1-e^2\right)^3 \left[\frac{1}{r} - \frac{1}{a(1-e^2)}\right], \qquad g_{44} = c^2 - \frac{3\mu}{a}. \end{gathered} \tag{3.4}$$

On the other hand, we have

$$\left(g^{ij}\right)_{i,j=\overline{1,4}} = \frac{1}{g_{11} g_{22} \Delta} \begin{pmatrix} g_{22}\Delta & 0 & 0 & 0 \\ 0 & g_{11}\Delta & 0 & 0 \\ 0 & 0 & g_{11} g_{22} g_{44} & -g_{11} g_{22} g_{34} \\ 0 & 0 & -g_{11} g_{22} g_{34} & g_{11} g_{22} g_{33} \end{pmatrix}, \tag{3.5}$$

where $\Delta = g_{33} g_{44} - g_{34}^2$. Particularly, for $1/r = 1/a(1-e^2)$, we have $g_{34} = 0$, and so we obtain $g^{ii} = 1/g_{ii}$, for all $i = \overline{1,4}$.

By direct calculation, it is obtained that

$$\begin{gathered}\Gamma^1_{14} = \Gamma^1_{24} = \Gamma^1_{44} = 0, \qquad \Gamma^1_{34} = g^{11} na\left(1 - e^2\right), \qquad \Gamma^2_{14} = \Gamma^2_{24} = \Gamma^2_{34} = \Gamma^2_{44} = 0, \\ \Gamma^3_{14} = -g^{33} na\left(1 - e^2\right), \qquad \Gamma^3_{24} = \Gamma^3_{34} = 0, \qquad \Gamma^3_{44} = 0, \qquad \Gamma^4_{14} = \Gamma^4_{24} = \Gamma^4_{34} = \Gamma^4_{44} = 0,\end{gathered} \tag{3.6}$$

hence

$$\begin{gathered}dX^1 = -nag^{11}\left(1 - e^2\right)X^3 du, \\ dX^2 = 0, \\ dX^3 = nag^{33}\left(1 - e^2\right)X^1 du, \\ dX^4 = 0.\end{gathered} \tag{3.7}$$

Therefore, the components X^2 and X^4 have constant values, whatever be the initial vector. Particularly, we suppose the component X^4 of this initial vector to be null. Then the component X^4, proportional with du, will be constantly zero, and the vector of Universe will be projected on a vector from the 3D space having the origin in M. Furthermore, if the initial component X^2, proportional with $d\theta$, is also zero, then θ remains constantly null, because it is zero at the initial time. The projection MR of the vector of Universe in the 3D space is in the plane of the elliptical orbit at the initial time and remains in this plane when M describes this orbit. The variations dX^1 and dX^3 are given by the following expressions:

$$\begin{gathered}dX^1 = -nag^{11}\left(1 - e^2\right)X^3 du, \\ dX^3 = nag^{33}\left(1 - e^2\right)X^1 du.\end{gathered} \tag{3.8}$$

Taking the variable $1/r = 1/a(1 - e^2)$ and taking account that $g^{11} = 1/g_{11}$ and $g^{33} = 1/g_{33}$, then, using the relations (3.8), we derive

$$\begin{gathered}d\left(\sqrt{-g_{11}}X^1\right) = \frac{na\left(1 - e^2\right)}{\sqrt{g_{11}g_{33}}}\left(\sqrt{-g_{33}}X^3\right)du, \\ d\left(\sqrt{-g_{33}}X^3\right) = -\frac{na\left(1 - e^2\right)}{\sqrt{g_{11}g_{33}}}\left(\sqrt{-g_{11}}X^1\right)du.\end{gathered} \tag{3.9}$$

Making the change of variable

$$\begin{gathered}\sqrt{-g_{11}}X^1 = x, \\ \sqrt{-g_{33}}X^3 = y,\end{gathered} \tag{3.10}$$

one can define the direction of the vector MR in a reference system linked to the mass M in his motion by the cartesian coordinates x and y of the point R with respect to the tangent and the normal to the ellipse at the point M.

It is known that for all the planets the ratio μ/ac^2 is smaller than $1/(4\cdot 10^7)$, a being the arithmetic average of the extreme values of the distance r during the motion (see [4]). Therefore, if $1/r = 1/a(1-e^2)$, neglecting the second-order terms in μ/ac^2, the coefficient of $d\psi^2$ from (2.15) becomes $-a^2(1-e^2)^2(1+\mu/ac^2)$, and

$$\frac{1}{\sqrt{g_{11}g_{33}}} = \frac{1}{a(1-e^2)}\left[1-\frac{3\mu}{2ac^2(1-e^2)}\right]. \tag{3.11}$$

With the change of variable (3.10) and taking account of (3.11), the relations (3.9) can be rewritten in the following form:

$$\begin{aligned} \frac{dx}{dt} &= n\left[1-\frac{3\mu}{2ac^2(1-e^2)}\right]y, \\ \frac{dy}{dt} &= -n\left[1-\frac{3\mu}{2ac^2(1-e^2)}\right]x. \end{aligned} \tag{3.12}$$

Thus, it was obtained a system of linear differential equations with constant coefficients, having the solutions

$$\begin{aligned} x &= A\cos\left\{n\left[1-\frac{3\mu}{2ac^2(1-e^2)}\right](t-t_0)\right\}, \\ y &= -A\sin\left\{n\left[1-\frac{3\mu}{2ac^2(1-e^2)}\right](t-t_0)\right\}, \end{aligned} \tag{3.13}$$

where A and t_0 are constants.

It follows that, with respect to the mobile reference system formed by the tangent and the normal in M to ellipse, the vector MR has a retrograde rotational motion whose angular velocity is $[-n+(3\mu/2ac^2(1-e^2))n]$. During the revolution of the mass M, the vector MR moves in direct sense as well as the radius vector FM, but with the velocity $3\mu n/2ac^2(1-e^2)$.

The planet M, in its rotation with respect to the chosen reference system, seems to reach in the proper angular position of the end of its revolution before that this revolution to be effectively complete.

We note that the expression for the geodesic precession

$$p_{g_e} = \frac{3}{2}\cdot\left(\frac{na}{c}\right)^2\cdot\frac{n}{(1-e^2)}, \tag{3.14}$$

where n and a are the mean motion and semimajor axis of the orbit of the Earth-Moon system about the Sun, agrees with that given in Barker and O'Connell [5, 7], but in a different setting. They find the precession of the spin and the precession of the orbit for the two-body problem in general relativity with arbitrary masses, spins, and quadrupole moments, starting from a gravitational potential energy derived from Gupta's quantum theory of gravitation [13]. Their calculations were performed using the tools of Newtonian mechanics and the Euler-Lagrange equations. Our approach is clearly more simple, since it was used a purely classical treatment.

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