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Research Article

Some New Common Fixed Point Theorems under Strict Contractive Conditions in *G*-Metric Spaces

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We introduce some new types of pairs of mappings (f, g) on G-metric space called G-weakly commuting of type (A_f) and G-R-weakly commuting of type (A_f). We obtain also several common fixed point results for these mappings under certain contractive condition in G-metric space. Also some examples illustrated to support our results, and comparison between different types of pairs of mappings are studied.

1. Introduction and Preliminaries

The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of strong research activity and, being the area of the fixed point theory, has very important application in applied mathematics and sciences. In 1976 Jungck [1] proved a common fixed point theorem for commuting maps, but his results required the continuity of one of the maps.

Sessa [2] in 1982 first introduced a weaker version of commutativity for a pair of self-maps, and it is shown in Sessa [2] that weakly commuting pair of maps in metric pace is commuting, but the converse may not be true.

Later, Jungck [3] introduced the notion of compatible mappings in order to generalize the concepts of weak commutativity and showed that weak commuting map is compatible, but the reverse implication may not hold.

In 1996, Jungck [4] defined a pair of self-mappings to be weakly compatible if they commute at their coincidence points.

Therefore, we have one-way implication, namely, commuting maps $\Rightarrow$ weakly commuting maps $\Rightarrow$ compatible maps $\Rightarrow$ weakly Compatible maps. Recently various authors have introduced coincidence points results for various classes of mappings on metric spaces for more detail of coincidence point theory and related results see [5–7].

However, the study of common fixed point of noncompatible mappings has recently been initiated by Pant (see [8, 9]).

In 2002 Amari and El Moutawakil [10] defined a new property called E.A. property which generalizes the concept of noncompatible mappings, and they proved some common fixed point theorem.

Definition 1.1 (see [10]). Let S and T be two self-mappings of a metric space (X, d). We say that T and S satisfy the E.A. property if there exists a sequence (x_n) such that

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t, \quad \text{for some } t \in X. \tag{1.1}$$

In 2005 Zead Mustafa and Brailey Sims introduced the notion of G-metric spaces as generalization of the concept of ordinary metric spaces. Based on the notion of G-metric space Mustafa et al. [11–15] obtained some fixed point results for mapping satisfying different contractive conditions on complete G-metric space, while in [16] the completeness property was omitted and replaced by sufficient conditions, where these conditions do not imply the completeness property.

Chugh et al. [17] obtained some fixed point results for maps satisfying property P in G-metric spaces. Saadati et al. [18] studied fixed point of contractive mappings in partially ordered G-metric spaces. Shatanawi obtained fixed points of ϕ-maps in G-metric spaces [19] and a number of fixed point results for the two weakly increasing mappings with respect to partial ordering in G-metric spaces [20]. In [21, 22] authors established coupled fixed point theorems in a partially ordered G-metric spaces.

Abbas and rhoades [23] proved several common fixed points for noncommuting mappings without continuity in G-metric space, and they show that the results 2.3–2.6 generalize Theorems 2.1–2.4 of [11].

In [24] Abbas et al. proved several unique common fixed points for mappings satisfying E.A. property under generalized contraction condition and show that Corollary 3.1 extends the main result in [13] (Theorem 2.1) and Corollary 3.3 is G-version of Theorem 2 from [10] in the case of two self-mappings. Also this corollary is in relation with Theorem 2.5 of [23].

In [25] the authors proved some coupled coincidence and common coupled fixed point results for mappings defined on a set equipped with two G-metric spaces and these results do not rely on continuity of mappings involved therein as well as they show that Theorem 2.13 is an extension and generalization of (1) Theorem 2.2, Corollary 2.3, Theorem 2.6, Corollaries 2.7 and 2.8 in [26] and (2) Theorem 2.4 and Corollary 2.5 in [27].

Aydi et al. [28] established some common fixed point results for two mappings f and g on G-metric spaces with assumption that f is a generalized weakly G-contraction mappings of type A and B with respect to g.

In this paper, we define new types of self-maps f and g on G-metric space called G-weakly commuting of type A_f and G-R-weakly commuting of type A_f. Also we obtain several common fixed point results for these mappings under certain contractive condition in G-metric space, and some examples are illustrated to support our results, and a comparison between different types of pairs of mappings are stated.

The following definitions and results will be needed in the sequel.

Definition 1.2 (see [29]). A G-metric space is a pair (X, G), where X is a nonempty set, and G is a nonnegative real-valued function defined on $X \times X \times X$ such that for all $x, y, z, a \in X$ we have

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$; for all $x, y \in X$, *with* $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, *with* $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

The function G is called G-metric on X.

Every G-metric on X defines a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \quad \forall x, y \in X. \tag{1.2}$$

Example 1.3 (see [29]). Let (X, d) be a metric space, and define G_s and G_m on $X \times X \times X$ to $\mathbf{R}^+$ by

$$\begin{aligned} G_s(x, y, z) &= d(x, y) + d(y, z) + d(x, z), \\ G_m(x, y, z) &= \max\{d(x, y), d(y, z), d(x, z)\}, \end{aligned} \tag{1.3}$$

for all $x, y, z \in X$. Then (X, G_s) and (X, G_m) are G-metric spaces.

Example 1.4 (see [29]). Let $X = \mathbf{R}$, and define $G : X \times X \times X \to \mathbf{R}^+$, by

$$G(x, y, z) = \begin{cases} |x - y| + |y - z| + |x - z|, & \text{if all } x, y, \text{ and } z \text{ are strictly positive} \\ & \text{or they are all strictly negative} \\ & \text{or all } x, y, \text{and } z \text{ are zero,} \\ 1 + |x - y| + |y - z| + |x - z|, & \text{otherwise,} \end{cases} \tag{1.4}$$

then (X, G) is G-metric space.

Definition 1.5 (see [29]). A sequence (x_n) in a G-metric space X is said to converge if there exists $x \in X$ such that $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and one says that the sequence (x_n) is G-convergent to x. We call x the limit of the sequence (x_n) and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$ (through this paper we mean by $\mathbf{N}$ the set of all natural numbers).

Proposition 1.6 (see [29]). *Let X be G-metric space. Then the following statements are equivalent:*

(1) (x_n) *is G-convergent to x,*

(2) $G(x_n, x_n, x) \to 0$, *as* $n \to \infty$,

(3) $G(x_n, x, x) \to 0$, *as* $n \to \infty$,

(4) $G(x_m, x_n, x) \to 0$, *as* $m, n \to \infty$.

Definition 1.7 (see [29]). In a G-metric space X, a sequence (x_n) is said to be G-Cauchy if given $\varepsilon > 0$, there is $N \in \mathbf{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$. That is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 1.8 (see [29]). *In a G-metric space X, the following statements are equivalent:*

(1) *the sequence (x_n) is G-Cauchy;*

(2) *for every $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.*

Definition 1.9 (see [29]). A G-metric space (X, G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$ and called nonsymmetric if it is not symmetric.

Example 1.10. Let $X = \mathbf{N}$ be the set of all natural numbers, and define

$G : X \times X \times X \to \mathbf{R}$ such that for all $x, y, z \in X$:

$G(x, y, z) = 0$ if $x = y = z$,

$G(x, y, y) = x + y$, if $x < y$,

$G(x, y, y) = x + y + 1/2$, if $x > y$,

$G(x, y, z) = x + y + z$ if $x \neq y \neq z$ and symmetry in all three variables.

Then, (X, G) is G-metric space and nonsymmetric since if $x < y$, we have $G(x, y, y) = x + y \neq x + y + 1/2 = G(y, x, x)$.

Proposition 1.11 (see [29]). *Let X be a G-metric space; then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 1.12 (see [29]). A G-metric space X is said to be complete if every G-Cauchy sequence in X is G-convergent in X.

Definition 1.13 (see [23]). Let f and g be self-maps of a set X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Recall that a pair of self-mappings are called weakly compatible if they commute at their coincidence points.

Proposition 1.14 (see [23]). *Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

In 2001, Abbas et al. [30] introduce a new type of pairs of mappings (f, g) called R-weakly commuting and they proved a unique common fixed point of four R-weakly commuting, maps satisfying generalized contractive condition.

Definition 1.15 (see [30]). Let X be a G-metric space, and let f and g be two self-mappings of X; then f and g are called R-weakly commuting if there exists a positive real number R such that

$$G(f(g(x)), f(g(x)), g(f(x))) \leq RG(f(x), f(x), g(x)) \text{ hold for each } x \in X. \tag{1.5}$$

Very recently, Mustafa et al. [31] introduce some new types of pairs of mappings (f, g) on G-metric space called G-weakly commuting of type G_f and G-R-weakly commuting of type G_f, and they obtained several common fixed point results by using E.A. property.

Definition 1.16 (see [31]). A pair of self-mappings (f, g) of a G-metric space (X, G) is said to be G-weakly commuting of type G_f if

$$G(fgx, gfx, ffx) \leq G(fx, gx, fx), \quad \forall x \in X. \tag{1.6}$$

Definition 1.17 (see [31]). A pair of self-mappings (f, g) of a G-metric space (X, G) is said to be G-R-weakly commuting of type G_f if there exists some positive real number R such that

$$G(fgx, gfx, ffx) \leq R \;\; G(fx, gx, fx), \quad \forall x \in X. \tag{1.7}$$

Remark 1.18. The G-R-weakly commuting maps of type G_f are R-weakly commuting since $G(fgx, fgx, gfx) \leq G(fgx, gfx, ffx) \leq G(fx, gx, fx)$, but the converse need not be true.

2. Main Results

2.1. New Concepts and Some Properties

In this section we introduce the concept of G-weakly commuting of type A_f for pairs of mapping (f, g) and comparison between this concept and Definitions 1.15, 1.16, and 1.17 is studied as well as examples illustrated to show that these types of mappings are different.

First, we introduce the following concepts as follows.

Definition 2.1. A pair of self-mappings (f, g) of a G-metric space (X, G) is said to be G-weakly commuting of type A_f if

$$G(fgx, ggx, ffx) \leq G(fx, gx, fx), \quad \forall x \in X. \tag{2.1}$$

Definition 2.2. A pair of self-mappings (f,g) of a G-metric space (X, G) is said to be G-R-weakly commuting of type A_f if there exists some positive real number R such that

$$G(fgx, ggx, ffx) \leq R \;\; G(fx, gx, fx), \quad \forall x \in X. \tag{2.2}$$

Remark 2.3. The G-weakly commuting maps of type A_f are G-R-weakly commuting of type A_f. Reciprocally, if $R \leq 1$, then G-R-weakly commuting maps of type A_f are G-weakly commuting of type A_f.

If we interchange f and g in (2.1) and (2.2), then the pair of mappings (f, g) is called G-weakly commuting of type A_g and G-R-weakly commuting of type A_g, respectively.

Example 2.4. Let $X = [0, 3/4]$, with the G-metric $G(x, y, z) = |x - y| + |y - z| + |x - z|$, for all $x, y, z \in X$. Define $f, g : X \to X$ by, $f(x) = (1/4)x^2$, $g(x) = x^2$; then as an easy calculation one can show that $G(fgx, ggx, ffx) = (126/64)x^4 \leq G(fx, gx, fx) = (6/4)x^2$, for all $x \in X$. Then the pair (f, g) is G-Weakly commuting of type A_f and G-R-Weakly commuting of type A_f.

Example 2.5. Let $X = [2, \infty]$, with the G-metric $G(x, y, z) = |x - y| + |y - z| + |x - z|$, for all $x, y, z \in X$. Define $f, g : X \to X$ by, $f(x) = x + 1$, $g(x) = 2x + 1$, then for $x = 2$ we see that $G(fgx, ggx, ffx) = G(gfx, ggx, ffx) = 20$ and $G(fx, gx, fx) = G(gx, fx, gx) = 6$. Therefore the pair (f, g) is not G-weakly commuting of type A_f or A_g, but it is G-R-weakly commuting of type A_f (and A_g) for $R \geq 4$.

The following examples show a pair of mappings (f, g) that G-weakly commuting of type G_f need not be G-weakly commuting of type A_f.

Example 2.6. Let $X = [0, 89/100]$, with the G-metric $G(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\}$, for all $x, y, z \in X$. Define $f(x) = (1/4)x^2$, $g(x) = x^2$; then we see that $G(fgx, gfx, ffx) = (15/16)x^4$ and $G(fx, gx, fx) = (3/4)x^2$, while as an easy calculation one can show that for $x = 88/100$ we have $G(f(g(x), gg(x), f(f(x))) = (59/100) \nleq G(g(x), f(x), g(x)) = (58/100)$. Therefore the pair (f, g) is not G-weakly commuting of type A_f, but it is G-weakly commuting of type G_f.

The following example shows that

(1) a pair of mappings (f, g) that is G-weakly commuting of type A_f need not be G-weakly commuting of type A_g;

(2) a pair of mappings (f, g) that is G-weakly commuting of type A_f need not be G-weakly commuting of type G_f;

(3) a pair of mappings (f, g) that is G-R-weakly commuting of type A_f need not be R-weakly commuting;

Example 2.7. Let $X = [2, 9]$ and $G(x, y, z) = max\{|x - y|, |y - z|, |x - z|\}$ for all $x, y, z \in X$. Define the mappings $f, g : X \to X$ by

$$f(x) = \begin{cases} 2 & \text{if } x = 2, \\ 6 & \text{if } 2 < x \leq 5, \\ 5 & \text{if } x > 5, \end{cases} \qquad g(x) = \begin{cases} 2 & \text{if } x = 2, \\ 9 & \text{if } 2 < x \leq 5, \\ 6 & \text{if } x > 5. \end{cases} \tag{2.3}$$

Then,

$$\begin{aligned} f(g(x)) &= \begin{cases} 2 & \text{if } x = 2, \\ 5 & \text{if } x > 2, \end{cases} \qquad g(f(x)) = \begin{cases} 2 & \text{if } x = 2, \\ 6 & \text{if } 2 < x \leq 5, \\ 9 & \text{if } x > 5, \end{cases} \\ g(g(x)) &= \begin{cases} 2 & \text{if } x = 2, \\ 6 & \text{if } x > 2, \end{cases} \qquad f(f(x)) = \begin{cases} 2 & \text{if } x = 2, \\ 5 & \text{if } 2 < x \leq 5, \\ 6 & \text{if } x > 5. \end{cases} \end{aligned} \tag{2.4}$$

Moreover,

$$\left|f(x)-g(x)\right| = \begin{cases} 0 & \text{if } x = 2, \\ 3 & \text{if } 2 < x \leq 5, \\ 1 & \text{if } x > 5. \end{cases} \tag{2.5}$$

If $x = 2$, we have $G(f(g(x)), g(g(x)), f(f(x))) = 0 = G(f(x), g(x), f(x))$.
If $2 < x \leq 5$, we have

$$G\big(f\big(g(x)\big), f\big(f(x)\big), g\big(g(x)\big)\big) = \max\{0,1,1\} \leq 3 = G\big(f(x), g(x), f(x)\big). \tag{2.6}$$

If $x > 5$, then

$$G\big(f\big(g(x)\big), f\big(f(x)\big), g\big(g(x)\big)\big) = \max\{1,1,0\} \leq 1 = G\big(g(x), f(x), g(x)\big). \tag{2.7}$$

Thus, f and g are G weakly commuting of type A_f, but for $x = 6$, we have

$$G\big(g\big(f(6)\big), f\big(f(6)\big), g\big(g(6)\big)\big) = \max\{3,3,0\} \nleq 1 = G\big(g(6), f(6), g(6)\big). \tag{2.8}$$

Therefore, the pair (f, g) is not G-weakly commuting of type A_g, but it is G-weakly commuting of type A_f.

Also for $x = 7$, we have

$$G\big(f\big(g(7)\big), g\big(f(7)\big), f\big(f(7)\big)\big) = 4 \nleq 1 = G\big(g(6), f(6), g(6)\big). \tag{2.9}$$

Therefore, the pair (f, g) is not G-weakly commuting of type G_f.

AS an easy calculation one can see that (f, g) are G-R-weakly commuting of type A_g for $R = 3$; but for $x = 6$ we have $G(f(g(6)), f(g(6)), g(f(6))) = 4 \nleq 3G(f(6), f(6), g(6)) = 3$, hence (f, g) is NOT R-weakly commuting for $R = 3$.

Lemma 2.8. *If f and g are G-weakly commuting of type A_f or G-R-weakly commuting of type A_f, then f and g are weakly compatible.*

Proof. Let x be a coincidence point of f and g, that is, $f(x) = g(x)$; then if the pair (f, g) is G-weakly commuting of type A_f, we have

$$G\big(f\big(g(x)\big), g\big(f(x)\big), f\big(g(x)\big)\big) = G\big(f\big(g(x)\big), g\big(g(x)\big), f\big(f(x)\big)\big) \leq G\big(f(x), g(x), f(x)\big) = 0. \tag{2.10}$$

It follows $f(g(x)) = g(f(x))$; then they commute at their coincidence point.

Similarly, if the pair (f, g) is G-R-weakly commuting of type G_f, we have

$$G(f(g(x)), g(f(x)), f(g(x))) = G(f(g(x)), g(g(x)), f(f(x))) \leq RG(f(x), g(x), f(x)) = 0. \tag{2.11}$$

Thus $f(g(x)) = g(f(x))$; then the pair (f, g) is weakly compatible. □

The following example shows that

(1) the converse of Lemma 2.8 fails (for the case of G-weakly commutativity),

(2) a pair of mappings (f, g) that is R-weakly commuting need not be G-R-weakly commuting of type A_f,

(3) a pair of mappings (f, g) that is R-weakly commuting need not be G-R-weakly commuting of type G_f.

Example 2.9. Let $X = [1, +\infty)$ and $G(x, y, z) = \max\{|x-y|, |y-z|, |x-z|\}$. Define $f, g : X \to X$ by $f(x) = 2x - 1$ and $g(x) = x^2$. We see that $x = 1$ is the only coincidence point and $f(g(1)) = f(1) = 1$ and $g(f(1)) = g(1) = 1$, so f and g are weakly compatible.

But, by an easy calculation, one can see that for $x = 3$ we have,

$$G(f(g(x)), g(g(x)), f(f(x))) = 72 \nleq 4 = G(f(x), g(x), f(x)). \tag{2.12}$$

Therefore, f and g are not G-weakly commuting of type A_f.

Also, we see that $G(f(g(x)), f(g(x)), g(f(x))) = 2x^2 - 4x + 2 \leq 2G(f(x), f(x), g(x)) = 2(x^2 - 2x + 1)$; therefore the mappings (f, g) are R-weakly commuting for $R = 2$, but for $x = 4$ we have $G(f(g(4)), g(g(4)), f(f(4))) = 243 \nleq 2G(f(4), g(4), f(4)) = 18$; hence (f, g) are not G-R-weakly commuting of type A_f for $R = 2$ and $G(f(g(4)), g(f(4)), f(f(4))) = 49 \nleq 2G(f(4), g(4), f(4)) = 18$; hence (f, g) are not G-R-weakly commuting of type G_f for $R = 2$.

Now, we rewrite Definition 1.1 on G-metric spaces setting.

Definition 2.10. Let S and T be two self-mappings of a G-metric space (X, G). We say that T and S satisfy the E.A. property if there exists a sequence (x_n) such that (Tx_n) and (Sx_n) G-converge to t for some $t \in X$; that is, thanks to Proposition 1.6,

$$\lim_{n\to\infty} G(Tx_n, Tx_n, t) = \lim_{n\to\infty} G(Sx_n, Sx_n, t) = 0. \tag{2.13}$$

Remark 2.11. In view of (1.2) and Example 1.3, Definition 1.1 is equivalent to Definition 2.10.

In the following example, we show that if f and g satisfy the E.A. property, then the pair (f, g) need not be G-weakly commuting of type A_f.

Example 2.12. We return to Example 2.9. Let $x_n = 1 + (1/3n)$. We have $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}(1 + (2/3n)) = 1$, and $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty}(1 + (1/3n))^2 = 1$, therefore, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = 1 \in [1, \infty)$. Then f and g satisfy the E.A. property, but we know that the pair (f, g) is not G-weakly commuting of type A_f.

Following Matkowski (see [32]), let Φ be the set of all functions ϕ such that $\phi : [0,\infty) \to [0,\infty)$ be a nondecreasing function with $\lim_{n\to\infty}\phi^n(t) = 0$ for all $t \in (0,+\infty)$. If $\phi \in \Phi$, then ϕ is called Φ-map. If ϕ is Φ-map, then it is easy to show that

(1) $\phi(t) < t$ for all $t \in (0,+\infty)$,

(2) $\phi(0) = 0$.

2.2. Some Common Fixed Point Results

We start this section with the following theorem.

Theorem 2.13. *Let (X,G) be a G-metric space; suppose mappings $f,g : X \to X$ satisfy the following condition:*

(1) *f and g be G-weakly commuting of type A_f,*

(2) *$f(X) \subseteq g(X)$,*

(3) *$g(X)$ is G-complete subspace of X,*

(4) $G(f(x),f(y),f(z)) \le \phi(M(x,y,z)), \quad for \quad all \quad x,y,z \in X, where$

$$M(x,y,z) = \max\left\{\begin{array}{c} G(g(x),g(y),g(z)), G(g(x),f(x),f(x)), \frac{1}{2}G(g(x),f(y),f(y)), \\ \frac{1}{2}G(g(x),f(z),f(z)), G(g(y),f(y),f(y)), G(g(y),f(x),f(x)), \\ G(g(y),f(z),f(z)), G(g(z),f(z),f(z)), G(g(z),f(x),f(x)), \\ G(g(z),f(y),f(y)) \end{array}\right\}. \tag{2.14}$$

Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$, and then choose $x_1 \in X$ such that $f(x_0) = g(x_1)$ and $x_2 \in X$ where $f(x_1) = g(x_2)$; then by induction we can define a sequence $(y_n) \in X$ as follows:

$$y_n = f(x_n) = g(x_{n+1}), \quad n \in N \cup \{0\} \tag{2.15}$$

We will show that the sequence (y_n) is G-cauchy sequence:

$$G(y_n,y_{n+1},y_{n+1}) = G(f(x_n),f(x_{n+1}),f(x_{n+1})) \le \phi(M(x_n,x_{n+1},x_{n+1})), \tag{2.16}$$

where

$$\begin{aligned} M(x_n,x_{n+1},x_{n+1}) &= \max\left\{\begin{array}{c} G(g(x_n),g(x_{n+1}),g(x_{n+1})), G(g(x_n),f(x_n),f(x_n)), \\ \frac{1}{2}G(g(x_n),f(x_{n+1}),f(x_{n+1})), G(g(x_{n+1}),f(x_n),f(x_n)), \\ G(g(x_{n+1}),f(x_{n+1}),f(x_{n+1})), \end{array}\right\} \\ &= \max\left\{\begin{array}{c} G(y_{n-1},y_n,y_n), \frac{1}{2}G(y_{n-1},y_{n+1},y_{n+1}), \\ G(y_n,y_n,y_n), G(y_n,y_{n+1},y_{n+1}), \end{array}\right\}. \end{aligned} \tag{2.17}$$

We will have different cases.

Case (1): if $M(x_n, x_{n+1}, x_{n+1}) = G(y_n, y_{n+1}, y_{n+1})$, then $G(y_n, y_{n+1}, y_{n+1}) \leq \phi(G(y_n, y_{n+1}, y_{n+1})) < G(y_n, y_{n+1}, y_{n+1})$, which is contradiction.

Case (2): if $M(x_n, x_{n+1}, x_{n+1}) = (1/2)G(y_{n-1}, y_{n+1}, y_{n+1})$, then in this case we have $\max\{G(y_{n-1}, y_n, y_n), G(y_n, y_{n+1}, y_{n+1})\} < (1/2)G(y_{n-1}, y_{n+1}, y_{n+1})$, which implies that

$$G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1}) < G(y_{n-1}, y_{n+1}, y_{n+1}), \tag{2.18}$$

but from G-metric property (G5) we have

$$G(y_{n-1}, y_{n+1}, y_{n+1}) \leq G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1}). \tag{2.19}$$

Thus, from (2.18) and (2.19) we see that case (2) is impossible.
Then, we must have the case

$$M(x_n, x_{n+1}, x_{n+1}) = G(y_{n-1}, y_n, y_n). \tag{2.20}$$

Thus, for $n \in \mathbf{N} \cup \{0\}$ and from (2.16) we have,

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+1}) &\leq \phi(G(y_{n-1}, y_n, y_n)) \\ &\leq \phi^2(G(y_{n-2}, y_{n-1}, y_{n-1})) \\ &\vdots \\ &\leq \phi^n(G(y_0, y_1, y_1)) \ldots (1). \end{aligned} \tag{2.21}$$

Given $\epsilon > 0$, since $\lim_{n\longrightarrow\infty}\phi^n(G(y_0, y_1, y_1)) = 0$, and $\phi(\epsilon) < \epsilon$, there is an integer $n_o \in \mathbf{N}$, such that

$$\phi^n(G(y_0, y_1, y_1)) < \epsilon - \phi(\epsilon), \quad \forall n \geq n_0. \tag{2.22}$$

Hence, we have

$$G(y_n, y_{n+1}, y_{n+1}) \leq \phi^n(G(y_0, y_1, y_1)) < \epsilon - \phi(\epsilon). \tag{2.23}$$

Now for $m, n \in \mathbf{N}$; $m > n$, we claim that

$$G(y_n, y_m, y_m) < \epsilon, \quad \forall m \geq n \geq n_0. \tag{2.24}$$

We will prove (2.24) by induction on m.
Inequality (2.24) holds for $m = n + 1$, by using (2.23) and the fact that $\epsilon - \phi(\epsilon) < \epsilon$.

Assume (2.24) holds for $m = k$. For $m = k + 1$, we have

$$\begin{aligned} G(y_n, y_{k+1}, y_{k+1}) &\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{k+1}, y_{k+1}) \\ &< \epsilon - \phi(\epsilon) + \phi(G(y_n, y_k, y_k)) \\ &< \epsilon - \phi(\epsilon) + \phi(\epsilon) = \epsilon. \end{aligned} \tag{2.25}$$

By induction on m, we conclude that (2.24) holds for all $m \geq n \geq n_0$.

Hence, the sequence $(y_n) = g(x_{n+1})$ is G-cauchy sequence in $g(X)$; since $g(X)$ is G-complete, then there exists $t \in g(X)$ such that $\lim_{n\to\infty} g(x_n) = t = \lim_{n\to\infty} f(x_n)$.

Thus, there exists $p \in X$ such that $g(p) = t$, also $\lim_{n\to\infty} f(x_n) = g(p)$.

We will show that $f(p) = g(p)$. Supposing that $f(p) \neq g(p)$, then condition (4) implies that, $G(f(p), f(p), f(x_n)) \leq \phi(M(p, p, x_n))$, where

$$\begin{aligned} &M(p, p, x_n) \\ &= \max\left\{\begin{array}{c} G(g(p), g(p), g(x_n)), G(g(p), f(p), f(p)), \frac{1}{2}G(g(p), f(p), f(p)), \\ \frac{1}{2}G(g(p), f(x_n), f(x_n)), G(g(p), f(p), f(p)), G(g(p), f(p), f(p)), \\ G(g(p), f(x_n), f(x_n)), G(g(x_n), f(x_n), f(x_n)), G(g(x_n), f(p), f(p)), \\ G(g(x_n), f(p), f(p)) \end{array}\right\}. \end{aligned} \tag{2.26}$$

Taking the limit as $n \to \infty$ and using the fact that the function G is continuous we get

$$\begin{aligned} G(f(p), f(p), g(p)) &\leq \phi\left(\max\left\{G(g(p), f(p), f(p)), \frac{1}{2}G(g(p), f(p), f(p))\right\}\right) \\ &= \phi(G(g(p), f(p), f(p))). \end{aligned} \tag{2.27}$$

Therefore,

$$G(f(p), f(p), g(p)) \leq \phi(G(g(p), f(p), f(p))) < G(g(p), f(p), f(p)), \tag{2.28}$$

which is contradiction; hence $fp = gp$.

Since f and g are G-weakly commuting of type A_f, then $(G(f(g(p)), g(g(p)), f(f(p))) \leq G(f(p), g(p), f(p)) = 0)$.

Thus, $ff(p) = fg(p) = gf(p) = gg(p)$; it follows that $f(t) = fg(p) = gf(p) = g(t)$.

Finally, we will show that $t := f(p)$ is common fixed point of f and g.

Supposing that $ft \neq t$, then

$$G(f(t), t, t) = G(f(t), f(p), f(p)) \leq \phi(M(t, p, p)), \tag{2.29}$$

where

$$M(t,p,p) = \max\left\{\begin{array}{c} G(g(t),g(p),g(p)), G(g(t),f(t),f(t)), \frac{1}{2}G(g(t),f(p),f(p)), \\ \frac{1}{2}G(g(t),f(p),f(p)), G(g(p),f(p),f(p)), G(g(p),f(t),f(t)), \\ G(g(p),f(p),f(p)), G(g(p),f(p),f(p)), G(g(p),f(t),f(t)), \\ G(g(p),f(p),f(p)) \end{array}\right\}. \tag{2.30}$$

Since $g(t) = f(t)$, and $g(p) = f(p)$, therefore (2.30) implies that

$$M(t,p,p) = \max\left\{G(f(t),t,t), \frac{1}{2}G(f(t),t,t), G(t,f(t),f(t))\right\}. \tag{2.31}$$

Hence, (2.29) becomes

$$\begin{aligned} G(f(t),t,t) &\leq \phi(\max\{G(f(t),t,t), G(t,f(t),f(t))\}) \\ &= \phi(G(t,f(t),f(t))) < G(t,f(t),f(t)). \end{aligned} \tag{2.32}$$

Similarly we get,

$$G(t,f(t),f(t)) < G(f(t),t,t). \tag{2.33}$$

So,

$$G(f(t),t,t) < G(f(t),t,t), \tag{2.34}$$

a contradiction which implies that $t = ft = gt$. Then t is a common fixed point.

To prove uniqueness suppose we have u and v such that $u \neq v$, $fu = gu = u$ and $fv = gv = v$; then condition (4) implies that

$$G(u,v,v) \leq \phi(G(v,u,u)). \tag{2.35}$$

Therefore,

$$G(u,v,v) \leq \phi(G(v,u,u)) < G(v,u,u). \tag{2.36}$$

Similarly, $G(v,u,u) < G(u,v,v)$; thus $G(u,v,v) < G(u,v,v)$, a contradiction which implies that $u = v$. Then t is a unique common fixed point of f and g. □

Now we give an example to support ourresult.

Example 2.14. Let $X = [0, 4/3]$, and define $G : X \times X \times X \to [0, \infty)$ by $G(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\}$ and $f, g : X \to X$ by $f(x) = x^3/8$, $g(x) = x^3/2$ and $\phi(t) = (2/3)t$. Then,

(a) $g(X)$ is G-complete subspace of X,

(b) $f(X) \subset g(X)$,

(c) f and g are G-weakly commuting of type A_f,

(d) f and g satisfy condition (4) of Theorem 2.13.

It is clear that (a) and (b) are satisfied.

To show (c), as an easy calculation one can show that $\forall x \in X$; we have $G(f(g(x)), g(g(x)), f(f(x))) = \max\{(3/64)x^9, (63/4096)x^9, (255/4096)x^9\} \leq (3/8)x^3 = G(f(x), g(x), f(x))$. Then f and g are G-weakly commuting of type A_f.

To show (d), for $x, y, z \in X$ we have

$$\begin{aligned} G(f(x), f(y), f(z)) &= \frac{1}{8}\max\left\{\left|x^3 - y^3\right|, \left|y^3 - z^3\right|, \left|x^3 - z^3\right|\right\} \\ &\leq \frac{1}{3}\max\left\{\left|x^3 - y^3\right|, \left|y^3 - z^3\right|, \left|x^3 - z^3\right|\right\} \\ &= \frac{2}{3}\left(\frac{1}{2}\max\left\{\left|x^3 - y^3\right|, \left|y^3 - z^3\right|, \left|x^3 - z^3\right|\right\}\right) \\ &= \phi(g(x), g(y), g(z)) \leq \phi(M(x, y, z)). \end{aligned} \tag{2.37}$$

Therefore, all hypotheses of Theorem 2.13 are satisfied and $x = 0$ unique common fixed point of f and g.

Corollary 2.15. *Let (X, G) be a G-metric space, and suppose mappings $f, g : X \to X$ satisfy the following conditions:*

(1) *f and g be G -weakly commuting of type A_f,*

(2) *$f(X) \subseteq g(X)$,*

(3) *$g(X)$ is G-complete subspace of X,*

(4) *$G(f(x), f(y), f(z)) \leq kM(x, y, z)$, where*

$$M(x, y, z) = \max\left\{\begin{array}{c} G(g(x), g(y), g(z)), G(g(x), f(x), f(x)), \frac{1}{2}G(g(x), f(y), f(y)), \\ \frac{1}{2}G(g(x), f(z), f(z)), G(g(y), f(y), f(y))), G(g(y), f(x), f(x)), \\ G(g(y), f(z), f(z)), G(g(z), f(z), f(z)), G(g(z), f(x), f(x)), \\ G(g(z), f(y), f(y)) \end{array}\right\}, \tag{2.38}$$

for all $x, y, z \in X$, where $k \in [0, 1)$; then f and g have a unique common fixed point.

Proof. It suffices to take $\phi(t) = kt$ in Theorem 2.13. □

Theorem 2.16. *Let (X,G) be a G-metric space. Suppose the mappings $f,g : X \to X$ are G-weakly commuting of type A_g and satisfy the following condition:*

(1) *f and g satisfy E.A. property,*

(2) *$g(X)$ is closed subspace of X,*

(3) $G(f(x),f(y),f(z)) \leq kM(x,y,z)$, *where*

$$M(x,y,z) = \max\left\{\begin{array}{c}[G(g(x),f(x),f(x)) + G(g(y),f(y),f(y)) + G(g(z),f(z),f(z))],\\ [G(g(x),f(y),f(y)) + G(g(y),f(x),f(x)) + G(g(z),f(y),f(y))],\\ [G(g(x),f(z),f(z)) + G(g(y),f(z),f(z)) + G(g(z),f(x),f(x))]\end{array}\right\} \tag{2.39}$$

for all $x,y,z \in X$, where $k \in [0,1/3)$; then f and g have a unique common fixed point.

Proof. Since f and g satisfy E.A. property, there exists in X a sequence (x_n) satisfying $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$ for some $t \in X$.

Since $g(X)$ is closed subspace of X and $\lim_{n\to\infty} g(x_n) = t$, there exists $p \in X$ such that $g(p) = t$, also $\lim_{n\to\infty} f(x_n) = g(p)$.

We will show that $f(p) = g(p)$ supposing that $f(p) \neq g(p)$, then condition (3) implies that

$$G(f(p),f(p),f(x_n)) \leq kM(p,p,x_n), \tag{2.40}$$

where,

$$M(p,p,x_n) = \max\left\{\begin{array}{c}[G(g(p),f(p),f(p)) + G(g(p),f(p),f(p)) + G(g(x_n),f(x_n),f(x_n))],\\ [G(g(p),f(p),f(p)) + G(g(p),f(p),f(p)) + G(g(x_n),f(p),f(p))],\\ [G(g(p),f(x_n),f(x_n)) + G(g(p),f(x_n),f(x_n)) + G(g(x_n),f(p),f(p))]\end{array}\right\}. \tag{2.41}$$

Taking the limit as $n \to \infty$ and using the fact that the function G is continuous, we get

$$G(f(p),f(p),g(p)) \leq 3kG(g(p),f(p),f(p)), \tag{2.42}$$

which is contradiction since $k \in [0,1/3)$, so $fp = gp$. Since f and g are G-weakly commuting of type A_g, then

$$G(gf(p),gg(p),ff(p)) \leq G(g(p),f(p),g(p)) = 0. \tag{2.43}$$

Therefore, $f(g(p)) = ff(p) = gf(p) = gg(p)$; then

$$f(t) := fg(p) = gf(p) = g(t). \tag{2.44}$$

Finally, we will show that $t = f(p)$ is common fixed point of f and g.

Supposing that $ft \neq t$, then

$$G(f(t),t,t) = G(f(t),f(p),f(p)) \leq kM(t,p,p), \tag{2.45}$$

where

$$M(t,p,p) = \max\left\{\begin{array}{l}[G(g(t),f(t),f(t)) + G(g(p),f(p),f(p)) + G(g(p),f(p),f(p))],\\ [G(g(t),f(p),f(p)) + G(g(p),f(t),f(t)) + G(g(p),f(p),f(p))],\\ [G(g(t),f(p),f(p)) + G(g(p),f(p),f(p)) + G(g(p),f(t),f(t))]\end{array}\right\}. \tag{2.46}$$

But $f(t) = g(t)$ and $f(p) = g(p)$. Thus,

$$\begin{aligned} G(f(t),t,t) &\leq k\max\{G(t,f(t),f(t)),[G(f(t),t,t) + G(t,f(t),f(t))]\} \\ &< k\{G(f(t),t,t) + G(t,f(t),f(t))\}. \end{aligned} \tag{2.47}$$

Hence,

$$G(f(t),t,t) \leq \left(\frac{k}{1-k}\right) G(t,f(t),f(t)). \tag{2.48}$$

Adjusting similarly, we get

$$G(t,f(t),f(t)) \leq \left(\frac{k}{1-k}\right) G(ft,t,t). \tag{2.49}$$

Therefore,

$$G(ft,t,t) \leq \left(\frac{k}{1-k}\right)^2 G(ft,t,t), \tag{2.50}$$

a contradiction which implies that $ft = t = fp$, but $gt = ft = t$. Then t is a common fixed point of f and g.

To prove uniqueness, suppose we have u and v such that $u \neq v$, $fu = gu = u$ and $fv = gv = v$; then

$$G(u,v,v) = G(f(u),f(v),f(v)) \leq k\{G(u,v,v) + G(v,u,u)\}. \tag{2.51}$$

Hence, $G(u,v,v) \leq (k/(1-k))G(v,u,u)$.

Similarly, $G(v,u,u) \leq (k/(1-k))G(u,v,v)$.

Therefore, $G(u,v,v) \leq (k/(1-k))^2 G(u,v,v)$ a contradiction which implies that $u = v$. Then t is a unique common fixed point of f and g.

Now we give an example to support our result. □

Example 2.17. Let $X = [0,3/4]$, define $G : X \times X \times X \to [0,\infty)$ by $G(x,y,z) = \max\{|x-y|,|y-z|,|x-z|\}$ and let $f,g : X \to X$ by $f(x) = x^2/5$, $g(x) = x^2$. Then,

(a) $g(X)$ is closed subspace of X,

(b) f and g are G-weakly commuting of type A_g,

(c) f and g satisfy E.A. property.

(d) f and g satisfy condition (4) for $k = (1/4)$.

Proof. (a) is obvious.

To show (b), as an easy calculation one can show that for all $x \in X$; we have $G(g(f(x)),g(g(x)),f(f(x))) = \max\{(4/125)x^4,(24/25)x^4,(124/125)x^4\} \leq (4/5)x^2 = G(f(x),g(x),f(x))$. Then f and g are G-weakly commuting of type A_g.

To show (c), if we consider the sequence $\{x_n\} = \{1/2n\}$, then $fx_n \to 0$ and $gx_n \to 0$ as $n \to \infty$. Thus, f and g satisfy the E.A. property.

To show (d), for $x,y,z \in X$ we have

$$\begin{aligned}
\left|\frac{x^2}{5} - \frac{y^2}{5}\right| &\leq \frac{x^2}{5} + \frac{y^2}{5}, \\
\left|\frac{y^2}{5} - \frac{z^2}{5}\right| &\leq \frac{y^2}{5} + \frac{z^2}{5}, \\
\left|\frac{x^2}{5} - \frac{z^2}{5}\right| &\leq \frac{x^2}{5} + \frac{z^2}{5}.
\end{aligned} \tag{2.52}$$

Then

$$\begin{aligned}
G(f(x),f(y),f(z)) &= \max\left\{\left|\frac{x^2}{5} - \frac{y^2}{5}\right|, \left|\frac{y^2}{5} - \frac{z^2}{5}\right|, \left|\frac{x^2}{5} - \frac{z^2}{5}\right|\right\} \\
&\leq \frac{x^2}{5} + \frac{y^2}{5} + \frac{z^2}{5} = \frac{1}{4}\left(\frac{4x^2}{5} + \frac{4y^2}{5} + \frac{4z^2}{5}\right) \\
&= \frac{1}{4}\big(G(g(x),f(x),f(x)) + G(g(y),f(y),f(y)) \\
&\quad + G(g(z),f(z),f(z))\big) \leq kM(x,y,z).
\end{aligned} \tag{2.53}$$

Therefore, all hypotheses of Theorem 2.16 are satisfied for $k = 1/4$ and $x = 0$, a unique common fixed point of f and g. □

Theorem 2.18. *Let (X, G) be a G-metric space, and suppose mappings $f, g : X \to X$ be G-R-weakly commuting of type A_f. Suppose that there exists a mapping $\psi : X \to [0, \infty)$ such that*

(1) $f(X) \subset g(X)$,

(2) $g(X)$ *is G-complete subspace of* X,

(3) $G(gx, fx, fx) < \psi(g(x)) - \psi(f(x))$, *for all* $x \in X$,

$$G(f(x), f(y), f(z)) < \max\left\{\begin{matrix} G(g(x), g(y), g(z)), G(g(x), f(x), g(y)), \\ G(g(z), f(z), f(x)), G(g(y), f(y), f(z)) \end{matrix}\right\}, \tag{2.54}$$

for all $x, y, z \in X$; then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$, and then choose $x_1 \in X$ such that $f(x_0) = g(x_1)$ and $x_2 \in X$ where $f(x_1) = g(x_2)$; then by induction we can define a sequence $(y_n) \in X$ as follows:

$$y_n = f(x_n) = g(x_{n+1}), \quad n \in N \cup \{0\}. \tag{2.55}$$

We will show that the sequence (y_n) is G-cauchy sequence:

$$\begin{aligned} G(g(x_n), g(x_{n+1}), g(x_{n+1})) &= G(g(x_n), f(x_n), f(x_n)) \\ &< \psi(g(x_n)) - \psi(f(x_n)) \\ &= \psi(g(x_n)) - \psi(g(x_{n+1})). \end{aligned} \tag{2.56}$$

Consider $a_n = \psi(g(x_n))$, $n = 1, 2, 3, 4, \ldots$, then

$$0 \le G(g(x_n), g(x_{n+1}), g(x_{n+1})) < a_n - a_{n+1}. \tag{2.57}$$

Thus, the sequence (a_n) is nonincreasing and bounded below by 0; hence (a_n) is convergent sequence.

On the other hand we have, from (G5) and (2.57), that for $m, n \in \mathbf{N}; m > n$

$$\begin{aligned} G(g(x_n), g(x_{n+m}), g(x_{n+m})) &\le \sum_{j=n}^{n+m-1} G(g(x_j), g(x_{j+1}), g(x_{j+1})) \\ &< \sum_{j=n}^{n+m-1} a_j - a_{j+1} \quad \text{(Telescoping sum)} \\ &= a_n - a_{n+m}. \end{aligned} \tag{2.58}$$

Therefore, the sequence $(g(x_n))$ is G-cauchy sequence in $g(X)$.

Since $g(X)$ is G-complete subspace, then there exists $t \in g(X)$ such that $\lim_{n\to\infty} g(x_n) = t$; having $t \in g(X)$ there exists $p \in X$ such that $g(p) = t$, also $\lim_{n\to\infty} f(x_n) = g(p) = t$.

We will show that $f(p) = g(p)$; supposing that $f(p) \neq g(p)$, then condition (4) implies that

$$G(f(p), f(p), f(x_n)) < \max\left\{\begin{matrix} G(g(p), g(p), g(x_n)), G(g(p), f(p), g(p)), \\ G(g(p), f(p), f(x_n)), G(g(x_n), f(x_n), f(p)) \end{matrix}\right\}. \tag{2.59}$$

Taking the limit as $n \to \infty$, we get

$$G(f(p), f(p), g(p)) < \max\left\{\begin{matrix} G(g(p), g(p), g(p)), \\ G(g(p), f(p), g(p)), \\ G(g(p), g(p), f(p)) \end{matrix}\right\}, \tag{2.60}$$

hence,

$$G(f(p), f(p), g(p)) < G(g(p), g(p), f(p)). \tag{2.61}$$

Adjusting similarly, we get

$$G(g(p), g(p), f(p)) < G(f(p), f(p), g(p)) \tag{2.62}$$

Therefore,

$$G(f(p), f(p), g(p)) < G(g(p), f(p), g(p)) < G(f(p), f(p), g(p)). \tag{2.63}$$

Thus, a contradiction implies $fp = gp$.
Since f and g are G-weakly commuting of type A_f, then

$$G(f(g(p)), g(g(p)), f(f(p))) \leq G(f(p), g(p), f(p)) = 0. \tag{2.64}$$

Thus, $ff(p) = fg(p) = gf(p) = gg(p)$, then $f(t) = fg(p) = gf(p) = g(t)$.
Finally, we will show that $t = f(p)$ is common fixed point of f and g.
Suppose that $ft \neq t$, so

$$G(f(t), t, t) = G(f(t), f(p), f(p)) < \max\left\{\begin{matrix} G(g(t), g(p), g(p)), \\ G(g(t), f(t), g(p)), \\ G(g(p), f(p), f(p)) \end{matrix}\right\}. \tag{2.65}$$

Since $g(p) = f(p)$ and $g(t) = f(t)$, therefore (2.65) implies that

$$G(f(t), t, t) < G(f(t), f(t), t). \tag{2.66}$$

Similarly, we have $G(f(t), f(t), t) < G(f(t), t, t)$.

A contradiction implies that $ft = fp = t$. Then t is a common fixed point.

To prove uniqueness suppose we have u and v such that $u \neq v$ where $fu = gu = u$ and $fv = gv = v$; then as an easy calculation one can get

$$G(u,v,v) < G(v,u,u). \tag{2.67}$$

Similarly, $G(v,u,u) < G(u,v,v)$, a contradiction which implies that $u = v$. Then, t is a unique common fixed point of f and g. □

Now we give an example to support our result.

Example 2.19. Let $X = [1,\infty)$, $\psi : X \longrightarrow [0,\infty)$ such that $\psi(t) = 3t, t \in X$ and $G(x,y,z) = \max\{|x-y|,|y-z|,|z-x|\}$. Define $f,g : X \longrightarrow X$ by $f(x) = 2x-1$ and $g(x) = 3x-2$.

Then,

(a) $f(X) \subset g(X)$,

(b) $g(X)$ is G-complete subspace of X,

(c) $G(gx,fx,fx) < \psi(g(x)) - \psi(f(x))$, for all $x \in X$,

(d) f and g satisfy condition (4) of Theorem 2.18.

Then as an easy calculation one can see that $G(f(g(x)),g(g(x)),f(f(x))) = \max\{2x-2, 5x-5, 3x-3\} = 5x-5 \leq R(x-1) = RG(f(x),g(x),f(x))$, for $R \geq 5$, then f and g are G-R-weakly commuting of type A_f.

Also we see that $f(X) \subset g(X)$ and $g(X)$ is G-complete subspace of X.

To prove (c), for all $x \in X$ we see that

$$G(gx,fx,fx) = x-1 \leq 3x-3 = \psi(g(x)) - \psi(f(x)). \tag{2.68}$$

To prove (d), for all $x,y,z \in X$ we have

$$\begin{aligned} G(fx,fy,fz) &= 2\max\{|x-y|,|y-z|,|x-z|\} \\ &\leq 3\max\{|x-y|,|y-z|,|x-z|\} = G(gx,gy,gz) \\ &\leq \max\{G(g(x),g(y),g(z)), G(g(x),f(x),g(y)), \\ &\qquad G(g(z),f(z),f(x)), G(g(y),f(y),f(z))\}. \end{aligned} \tag{2.69}$$

Therefore, all hypotheses of the previous theorem are satisfied and $x = 1$ a unique common fixed point of f and g.

Note that the main result of Mustafa [33] is not applicable in this case. Indeed, for $y = z = 1$ and $x = 3$,

$$G(f(3),f(1),f(1)) = 4 > 2k = kG(3,1,1) \quad \forall k \in [0,1). \tag{2.70}$$

Also, the Banach principle [34] is not applicable. Indeed, for $d(x,y) = |x-y|$ for all $x, y \in X$ we have for $x \neq y$

$$d(f(x), f(y)) = 2|x-y| > k|x-y| \quad \forall k \in [0,1). \tag{2.71}$$

Corollary 2.20. *Theorems 2.13, 2.16, and 2.18 remain true if we replace, respectively, G-weakly commuting of type A_f, G-weakly commuting of type A_g, weakly compatible and G-R-weakly commuting of type A_f by any one of them (retaining the rest of hypothesis).*

Corollary 2.21. *Some corollaries could be derived from Theorems 2.13, 2.16, and 2.18 by taking $z = y$ or $g = Id_X$.*

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