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Research Article

Hybrid Algorithm for Common Fixed Points of Uniformly Closed Countable Families of Hemirelatively Nonexpansive Mappings and Applications

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The authors have obtained the following results: (1) the definition of uniformly closed countable family of nonlinear mappings, (2) strong convergence theorem by the monotone hybrid algorithm for two countable families of hemirelatively nonexpansive mappings in a Banach space with new method of proof, (3) two examples of uniformly closed countable families of nonlinear mappings and applications, (4) an example which is hemirelatively nonexpansive mapping but not weak relatively nonexpansive mapping, and (5) an example which is weak relatively nonexpansive mapping but not relatively nonexpansive mapping. Therefore, the results of this paper improve and extend the results of Plubtieng and Ungchittrakool (2010) and many others.

1. Introduction and Preliminaries

Let E be a Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that the normalized duality J has the following properties: (1) if E is smooth, then J is single valued; (2) if E is strictly convex, then J is one-to-one (i.e., $Jx \cap Jy = \emptyset$ for all $x \neq y$); (3) if E is reflexive, then J is surjective; (4) if E has Frchet differentiable norm, then J is uniformly norm-to-norm continuous; (5) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E; (7) if E is a Hilbert space, then J is the identity operator.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces, and, consequently, it is not available in more general Banach spaces. In this connection, Alber [1] has recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Let E be a smooth Banach space. Consider the function defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x,y \in E. \tag{1.2}$$

Observe that, in a Hilbert space H, (1.2) reduces to $\phi(x,y) = \|x-y\|^2$, $x,y \in H$.

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y\in C} \phi(y,x), \tag{1.3}$$

existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see, e.g., [1, 2]). In Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y,x) \le (\|y\| + \|x\|)^2 \quad \forall x,y \in E. \tag{1.4}$$

If E is a reflexive strictly convex and smooth Banach space, then for $x,y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$, then $x = y$. From (1.4), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definitions of j, we have $Jx = Jy$. That is, $x = y$; see [3, 4] for more details.

Let C be a closed convex subset of E, and let T be a mapping from C into itself with nonempty set of fixed points. We denote by $F(T)$ the set of fixed points of T. T is called hemi-relatively nonexpansive if $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(Tx_n - x_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A hemi-relatively nonexpansive mapping T from C into itself is called relatively nonexpansive if $\widehat{F}(T) = F(T)$ (see, [5]).

A point p in C is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that the strong $\lim_{n\to\infty}(Tx_n - x_n) = 0$. The set of strong asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A hemi-relatively nonexpansive mapping T from C into itself is called weak relatively nonexpansive if $\widetilde{F}(T) = F(T)$ (see, [6]).

The following conclusions are obvious: (1) relatively nonexpansive mapping must be weak relatively nonexpansive mapping; (2) weak relatively nonexpansive mapping must be hemi-relatively nonexpansive mapping.

In this paper, we will give two examples to show that the inverses of above two conclusions are not hold.

In an infinite-dimensional Hilbert space, Mann's iterative algorithm has only weak convergence, in general, even for nonexpansive mappings. Hence in order to have strong convergence, in recent years, the hybrid iteration methods for approximating fixed points of nonlinear mappings has been introduced and studied by various authors.

In 2003, Nakajo and Takahashi [7] proposed the following modification of Mann iteration method for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{aligned} x_0 &\in C \text{ chosen only arbitrarily}, \\ y_n &= \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n &= \{ z \in C : \| y_n - z \| \le \| x_n - z \| \}, \\ Q_n &= \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0 \}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \end{aligned} \tag{1.5}$$

where C is a closed convex subset of H, and P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(T)}(x_0)$, where $F(T)$ denote the fixed points set of T.

The ideas to generalize the process (1.5) from Hilbert space to Banach space have recently been made. By using available properties on uniformly convex and uniformly smooth Banach space, Matsushita and Takahashi [8] presented their ideas as the following method for a single relatively nonexpansive mapping T in a Banach space E:

$$\begin{aligned} x_0 &\in C \text{ chosen only arbitrarily}, \\ y_n &= J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J T x_n), \\ C_n &= \{ z \in C : \phi(z, y_n) \le \phi(z, x_n) \}, \\ Q_n &= \{ z \in C : \langle x_n - z, J x_0 - J x_n \rangle \ge 0 \}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n}(x_0), \end{aligned} \tag{1.6}$$

where J is the duality mapping on E, and $\Pi_K(\cdot)$ is the generalized projection from E onto a nonempty closed convex subset K. They proved the following convergence theorem.

Theorem MT. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by (1.6), where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}(\cdot)$ is the generalized projection from C onto $F(T)$.*

Recently, Plubtieng and Ungchittrakool [9] here proposed the following hybrid iteration method for a countable family of relatively nonexpansive mappings in a Banach space and proved the convergence theorem.

Theorem PU. *Let E be a uniformly smooth and uniformly convex Banach space and let $\widehat{C}$ and C be two nonempty closed convex subsets of E such that $\widehat{C} \subset C$. Let $\{T_n\}$ be a sequence of relatively nonexpansive mappings from C into E such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{aligned} x_0 &\in \widehat{C}, \\ C_1 &= C, \\ x_1 &= \Pi_{C_1} x_0, \\ y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_n x_n), \\ C_{n+1} &= \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \quad n \geq 1, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \end{aligned} \tag{1.7}$$

where $\alpha_n \in [0,1]$ satisfies either

(a) $0 \leq \alpha_n < 1$ *for all* $n \geq 1$ *and* $\limsup_{n\to\infty} \alpha_n < 1$ *or*

(b) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$.

Suppose that for any bounded subset B of C there exists an increasing, continuous and convex function h_B from R^+ into R^+ such that $h_B(0) = 0$, and

$$\lim_{l,k\to\infty} \sup\{h_B(\|T_l z - T_k z\|) : z \in B\} = 0. \tag{1.8}$$

Let T be a mapping from C into E defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that

$$F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} \widehat{F}(T_n) = \widehat{F}(T). \tag{1.9}$$

Then $\{x_n\}$, $\{T_n x_n\}$, and $\{y_n\}$ converge strongly to $\Pi_{F(T)} x_0$.

In this paper, the authors have obtained the following results: (1) the definition of uniformly closed countable family of nonlinear mappings, (2) strong convergence theorem by the monotone hybrid algorithm for a countable family of hemi-relatively nonexpansive mappings in a Banach space with new method of proof, (3) two examples of uniformly closed countable families of nonlinear mappings and applications, (4) an example which is hemi-relatively nonexpansive mapping but not weak relatively nonexpansive mapping, and (5) an example which is weak relatively nonexpansive mapping but not relatively nonexpansive mapping. Therefore, the results of this paper improve and extend the results of Plubtieng and Ungchittrakool [9] and many others.

We need the following definitions and lemmas.

Lemma 1.1 (Kamimura and Takahashi [10]). *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

Lemma 1.2 (Alber [1]). *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0 \quad \text{for } y \in C. \tag{1.10}$$

Lemma 1.3 (Alber [1]). *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed convex subset of E, and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \quad \forall y \in C. \tag{1.11}$$

The following lemma is not hard to prove.

Lemma 1.4. *Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a hemi-relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

In this paper, we present the definition of uniformly closed for a sequence of mappings as follows.

Definition 1.5. Let E be a Banach space, C be a closed convex subset of E, let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C into E such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. We say that $\{T_n\}_{n=1}^{\infty}$ is uniformly closed if $p \in \bigcap_{n=1}^{\infty} F(T_n)$ whenever $\{x_n\} \subset C$ converges strongly to p and $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$.

Lemma 1.6 (see [11, 12]). *Let E be a p-uniformly convex Banach space with $p \geq 2$. Then, for all $x, y \in E$, $j(x) \in J_p(x)$, and $j(y) \in J_p(y)$,*

$$\langle x - y, j(x) - j(y) \rangle \geq \frac{c^p}{c^{p-2} p} \|x - y\|^p, \tag{1.12}$$

where J_p is the generalized duality mapping from E into E^*, and $1/c$ is the p-uniformly convexity constant of E.

2. Main Results

Definition 2.1. Let E be a Banach space, let C be a nonempty closed convex subset of E, and let $\{T_n\}_{n=1}^{\infty} : C \to E$, $\{S_n\}_{n=1}^{\infty} : C \to E$ be two sequences of mappings. If for any convergence sequence $\{x_n\} \subset C$, the following holds:

$$\lim_{n \to +\infty} \|T_n x_n - S_n x_n\| = 0. \tag{2.1}$$

Then $\{T_n\}$ and $\{S_n\}$ are said to satisfy the only *asymptotically condition*.

Theorem 2.2. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $\{T_n\}_{n=1}^{\infty} : C \to E$, $\{S_n\}_{n=1}^{\infty} : C \to E$ be two uniformly closed sequences of hemi-relatively nonexpansive mappings satisfying the asymptotically condition such that*

$F_1 = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, $F_2 = \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$, and $F = F_1 \bigcap F_2 \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\delta_n\}_{n=0}^{\infty}$ are four sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$, $\lim_{n\to\infty}\alpha_n = 0$ and $0 < \gamma \leq \gamma_n \leq 1$, $\delta_n \to 0$ for some constant $\gamma \in (0,1)$. Define a sequence $\{x_n\}$ in C by the following algorithm:

$$
\begin{aligned}
&x_0 \in C \text{ arbitrarily},\\
&y_n = J^{-1}\left(\alpha_n J x_0 + \beta_n J x_n + \gamma_n J T_n x_n + \delta_n J S_n x_n\right), \quad n \geq 1,\\
&C_n = \left\{z \in C_{n-1} : \phi\left(z, y_n\right) \leq (1-\alpha_n)\phi(z, x_n) + \alpha_n \phi(z, x_0)\right\}, \quad n \geq 1,\\
&C_0 = C,\\
&x_{n+1} = \Pi_{C_n} x_0, \quad n \geq 0.
\end{aligned}
\tag{2.2}
$$

Then $\{x_n\}$ converges strongly to $q = \Pi_F x_0$.

Proof. We first show that C_n is closed and convex for all $n \geq 0$. From the definitions of C_n, it is obvious that C_n is closed for all $n \geq 0$. Next, we prove that C_n is convex for all $n \geq 0$. Since

$$\phi(z, y_n) \leq (1-\alpha_n)\phi(z, x_n) + \alpha_n \phi(z, x_0) \tag{2.3}$$

is equivalent to

$$2\langle z, (1-\alpha_n) J x_n + \alpha_n J x_0 - J y_n \rangle \leq (1-\alpha_n)\|x_n\|^2 + \alpha_n \|x_0\|^2 - \left\|y_n\right\|^2, \tag{2.4}$$

it is easy to get that C_n is convex for all $n \geq 0$.

Next, we show that $F \subset C_n$ for all $n \geq 1$. Indeed, for each $p \in F$, we have

$$
\begin{aligned}
\phi(p, y_n) &= \phi\Big(p, J^{-1}\left(\alpha_n J x_0 + \beta_n J x_n + \gamma_n J T_n x_n + \delta_n J S_n x_n\right)\Big)\\
&= \|p\|^2 - 2\langle p, \left(\alpha_n J x_0 + \beta_n J x_n + \gamma_n J T_n x_n + \delta_n J S_n x_n\right)\rangle\\
&\quad + \left\|\alpha_n J x_0 + \beta_n J x_n + \gamma_n J T_n x_n + \delta_n J S_n x_n\right\|^2\\
&\leq \|p\|^2 - 2\alpha_n \langle p, J x_0\rangle - 2\beta_n \langle p, J x_n\rangle - 2\gamma_n \langle p, J T_n x_n\rangle - 2\delta_n \langle p, J S_n x_n\rangle\\
&\quad + \alpha_n \|x_0\|^2 + \beta_n \|x_n\| + \gamma_n \|T_n x_n\|^2 + \delta_n \|S_n x_n\|^2\\
&\leq \alpha_n \phi(p, x_0) + \beta_n \phi(p, x_n) + \gamma_n \phi(p, T_n x_n) + \delta_n \phi(p, S_n x_n)\\
&\leq \alpha_n \phi(p, x_0) + (1-\alpha_n)\phi(p, x_n).
\end{aligned}
\tag{2.5}
$$

So, $p \in C_n$, which implies that $F \subset C_n$ for all $n \geq 1$.

Since $x_{n+1} = \Pi_{C_n} x_0$ and $C_n \subset C_{n-1}$, then we get the following:

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0. \tag{2.6}$$

Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. On the other hand, by Lemma 1.3, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_{n-1}} x_0, x_0) \le \phi(p, x_0) - \phi(p, x_n) \le \phi(p, x_0), \tag{2.7}$$

for all $p \in F(T) \subset C_{n-1}$ and for all $n \ge 1$. Therefore, $\phi(x_n, x_0)$ is also bounded. This together with (3.1) implies that the limit of $\{\phi(x_n, x_0)\}$ exists. Put the following:

$$\lim_{n\to\infty} \phi(x_n, x_0) = d. \tag{2.8}$$

From Lemma 1.3, we have, for any positive integer m, that

$$\begin{aligned}\phi(x_{n+m}, x_{n+1}) &= \phi(x_{n+m}, \Pi_{C_n} x_0) \le \phi(x_{n+m}, x_0) - \phi(\Pi_{C_n} x_0, x_0)\\ &= \phi(x_{n+m}, x_0) - \phi(x_{n+1}, x_0),\end{aligned} \tag{2.9}$$

for all $n \ge 0$. This together with (3.6) implies that

$$\lim_{n\to\infty} \phi(x_{n+m}, x_{n+1}) = 0 \tag{2.10}$$

is, uniformly for all m, holds. By using Lemma 1.1, we get that

$$\lim_{n\to\infty} \|x_{n+m} - x_{n+1}\| = 0 \tag{2.11}$$

is, uniformly for all m, holds. Then $\{x_n\}$ is a Cauchy sequence; therefore, there exists a point $p \in C$ such that $x_n \to p$.

Since $x_{n+1} = \Pi_{C_n} x_0 \in C_n$, from the definition of C_n, we have the following:

$$\phi(x_{n+1}, y_n) \le (1-\alpha_n)\phi(x_{n+1}, x_n) + \alpha_n \phi(x_{n+1}, x_0). \tag{2.12}$$

This together with (2.10) and $\lim_{n\to\infty}\alpha_n = 0$ implies that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{2.13}$$

Therefore, by using Lemma 1.1, we obtain the following:

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{2.14}$$

Since J is uniformly norm-to-norm continuous on bounded sets, then we have the following:

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{2.15}$$

Noticing that

$$\begin{aligned}\|Jx_{n+1}-Jy_n\| &= \|Jx_{n+1}-(\alpha_n Jx_0+\beta_n Jx_n+\gamma_n JT_nx_n+\delta_n JS_nx_n)\| \\ &= \|\alpha_n(Jx_{n+1}-Jx_0)+\beta_n(Jx_{n+1}-Jx_n)+\gamma_n(Jx_{n+1}-JT_nx_n)+\delta_n(Jx_{n+1}-JT_nx_n)\| \\ &\geq \gamma_n\|Jx_{n+1}-JT_nx_n\|-\delta_n\|Jx_{n+1}-JS_nx_n\| \\ &\quad -\alpha_n\|Jx_{n+1}-Jx_0\|-\beta_n\|Jx_{n+1}-Jx_n\|,\end{aligned} \tag{2.16}$$

which leads to

$$\begin{aligned}\gamma_n\|Jx_{n+1}-JT_nx_n\| \leq \|Jx_{n+1}-Jy_n\|+\alpha_n\|Jx_0-Jx_{n+1}\| \\ +\beta_n\|Jx_{n+1}-Jx_n\|+\delta_n\|Jx_{n+1}-JS_nx_n\|.\end{aligned} \tag{2.17}$$

From (2.15) and $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\delta_n=0$, $0<\gamma\leq\gamma_n\leq 1$ we obtain that

$$\lim_{n\to\infty}\|Jx_{n+1}-JT_nx_n\|=0. \tag{2.18}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, then we obtain that

$$\lim_{n\to\infty}\|x_{n+1}-T_nx_n\|=0. \tag{2.19}$$

This together with (2.11) implies that

$$\lim_{n\to\infty}\|x_n-T_nx_n\|=0. \tag{2.20}$$

Sine $\{T_n\}$ and $\{S_n\}$ satisfy the asymptotically condition, we also have

$$\lim_{n\to\infty}\|x_n-S_nx_n\|=0. \tag{2.21}$$

Since $x_n \to p$ and $\{T_n\}_{n=1}^{\infty}$, $\{S_n\}_{n=1}^{\infty}$ are uniformly closed, we have

$$p\in F=\left(\bigcap_{n=1}^{\infty}F(T_n)\right)\bigcap\left(\bigcap_{n=1}^{\infty}F(S_n)\right). \tag{2.22}$$

Finally, we prove that $p=\Pi_F x_0$, from Lemma 1.3, we have

$$\phi(p,\Pi_F x_0)+\phi(\Pi_F x_0,x_0)\leq\phi(p,x_0). \tag{2.23}$$

On the other hand, $x_{n+1}=\Pi_{C_n}x_0$ and $F\subset C_n$, for all n. Also from Lemma 1.3, we have

$$\phi(\Pi_F x_0,x_{n+1})+\phi(x_{n+1},x_0)\leq\phi(\Pi_F x_0,x_0). \tag{2.24}$$

By the definition of $\phi(x,y)$, we know that

$$\lim_{n\to\infty}\phi(x_{n+1},x_0)=\phi(p,x_0). \tag{2.25}$$

Combining (2.24) and (2.25), we know that $\phi(p,x_0)=\phi(\Pi_F x_0,x_0)$. Therefore, it follows from the uniqueness of $\Pi_F x_0$ that $p=\Pi_F x_0$. This completes the proof. □

When $\alpha_n\equiv 0$, $\delta_n\equiv 0$ in Theorem 2.2, we obtain the following result.

Theorem 2.3. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, and let $\{T_n\}_{n=1}^{\infty}: C\to E$ be a uniformly closed sequence of hemi-relatively nonexpansive mappings such that $F=\bigcap_{n=1}^{\infty}F(T_n)\neq\emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequences in [0,1] such that $0\le\alpha_n\le\alpha<1$ for some constant $\alpha\in(0,1)$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned}
x_0&\in C \text{ arbitrarily},\\
y_n&=J^{-1}(\alpha_n Jx_n+(1-\alpha_n)JT_nx_n),\quad n\ge 1,\\
C_n&=\{z\in C_{n-1}:\phi(z,y_n)\le\phi(z,x_n)\},\quad n\ge 1,\\
C_0&=C,\\
x_{n+1}&=\Pi_{C_n}x_0,\quad n\ge 0.
\end{aligned} \tag{2.26}$$

Then $\{x_n\}$ converges strongly to $q=\Pi_F x_0$.

3. Applications for Equilibrium Problem

Let E be a real Banach space, and let E^* be the dual space of E. Let C be a closed convex subset of E. Let f be a bifunction from $C\times C$ to $R=(-\infty,+\infty)$. The equilibrium problem is to find $x\in C$ such that

$$f(x,y)\ge 0,\quad \forall y\in C. \tag{3.1}$$

The set of solutions of (1.2) is denoted by $\mathrm{EP}(f)$. Given a mapping $T: C\to E^*$ let $f(x,y)=\langle Tx,y-x\rangle$ for all $x,y\in C$. Then, $p\in\mathrm{EP}(f)$ if and only if $\langle Tp,y-p\rangle\ge 0$ for all $y\in C$, that is, p is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.2). Some methods have been proposed to solve the equilibrium problem in Hilbert spaces, see, for instance, [13–15].

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

(A1) $f(x,x)=0$, for all $x\in E$,

(A2) f is monotone, that is, $f(x,y)+f(y,x)\le 0$, for all $x,y\in E$,

(A3) for all $x,y,z\in E$, only $\limsup_{t\downarrow 0}f(tz+(1-t)x,y)\le f(x,y)$,

(A4) for all $x\in C$, $f(x,\cdot)$ is convex and lower semicontinuous.

Lemma 3.1 (Blum and Oettli [13]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C\times C$ to $R=(-\infty,+\infty)$ satisfying (A1)–(A4), and let $r>0$ and $x\in E$. Then, there exists $z\in C$ such that*

$$f(z,y)+\frac{1}{r}\langle y-z,\ Jz-Jx\rangle\geq 0,\qquad \forall y\in C. \tag{3.2}$$

Lemma 3.2 (Takahashi and Zembayashi [15]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let f be a bifunction from $C\times C$ to $R=(-\infty,+\infty)$ satisfying (A1)–(A4). For $r>0$, define a mapping $T_r: E\to C$ as follows:*

$$T_r(x)=\left\{z\in C: f(z,y)+\frac{1}{r}\langle y-z,Jz-Jx\rangle\geq 0,\forall y\in C\right\} \tag{3.3}$$

for all $x\in E$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is a firmly nonexpansive-type mapping, that is, for all $x,y\in E$,*

$$\langle T_rx-T_ry,JT_rx-JT_ry\rangle\leq\langle T_rx-T_ry,\ Jx-Jy\rangle, \tag{3.4}$$

(3) *$F(T_r)=\mathrm{EP}(f)$;*

(4) *$\mathrm{EP}(f)$ is closed and convex;*

(5) *T_r is also a relatively nonexpansive mapping.*

Lemma 3.3 (Takahashi and Zembayashi [15]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let f be a bifunction from $C\times C$ to $R=(-\infty,+\infty)$ satisfying (A1)–(A4), let $r>0$, and let $x\in E$, $q\in F(T_r)$, then the following holds:*

$$\phi(q,T_rx)+\phi(T_rx,x)\leq\phi(q,x). \tag{3.5}$$

Lemma 3.4. *Let E be a p-uniformly convex with $p\geq 2$ and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C\times C$ to $R=(-\infty,+\infty)$ satisfying (A1)–(A4). Let $\{r_n\}$ be a positive real sequence such that $\lim_{n\to\infty}r_n=r>0$. Then the sequence of mappings $\{T_{r_n}\}$ is uniformly closed.*

Proof. (1) Let $\{x_n\}$ be a convergent sequence in C. Let $z_n=T_{r_n}x_n$ for all n, then

$$f(z_n,y)+\frac{1}{r_n}\langle y-z_n,Jz_n-Jx_n\rangle\geq 0,\quad \forall y\in C, \tag{3.6}$$

$$f(z_{n+m},y)+\frac{1}{r_{n+m}}\langle y-z_{n+m},Jz_{n+m}-Jx_{n+m}\rangle\geq 0,\quad \forall y\in C. \tag{3.7}$$

Putting $y = z_{n+m}$ in (3.6) and $y = z_n$ in (3.7), we have

$$f(z_n, z_{n+m}) + \frac{1}{r_n}\langle z_{n+m} - z_n, Jz_n - Jx_n\rangle \geq 0, \quad \forall y \in C,$$
$$f(z_{n+m}, z_n) + \frac{1}{r_{n+m}}\langle z_n - z_{n+m}, Jz_{n+m} - Jx_{n+m}\rangle \geq 0, \quad \forall y \in C. \tag{3.8}$$

So, from (A2), we have

$$\left\langle z_{n+m} - z_n, \frac{Jz_n - Jx_n}{r_n} - \frac{Jz_{n+m} - Jx_{n+m}}{r_{n+m}}\right\rangle \geq 0, \tag{3.9}$$

and hence

$$\left\langle z_{n+m} - z_n, Jz_n - Jx_n - \frac{r_n}{r_{n+m}}(Jz_{n+m} - Jx_{n+m})\right\rangle \geq 0. \tag{3.10}$$

Thus, we have

$$\left\langle z_{n+m} - z_n, Jz_n - Jz_{n+m} + Jz_{n+m} - Jx_n - \frac{r_n}{r_{n+m}}(Jz_{n+m} - Jx_{n+m})\right\rangle \geq 0, \tag{3.11}$$

which implies that

$$\langle z_{n+m} - z_n, Jz_{n+m} - Jz_n\rangle \leq \left\langle z_{n+m} - z_n, Jz_{n+m} - Jx_n - \frac{r_n}{r_{n+m}}(Jz_{n+m} - Jx_{n+m})\right\rangle \geq 0. \tag{3.12}$$

By using Lemma 1.6, we obtain the following:

$$\begin{aligned}\frac{c^p}{c^{p-2}p}\|z_{n+m} - z_n\|^p &\leq \left\langle z_{n+m} - z_n, Jz_{n+m} - Jx_n - \frac{r_n}{r_{n+m}}(Jz_{n+m} - Jx_{n+m})\right\rangle \geq 0\\ &= \left\langle z_{n+m} - z_n, \left(1 - \frac{r_n}{r_{n+m}}\right)Jz_{n+m} + \frac{r_n}{r_{n+m}}(Jx_{n+m} - Jx_n)\right\rangle.\end{aligned} \tag{3.13}$$

Therefore, we get the following:

$$\frac{c^p}{c^{p-2}p}\|z_{n+m} - z_n\|^{p-1} \leq \left|1 - \frac{r_n}{r_{n+m}}\right|\|Jz_{n+m}\| + \left\|\frac{r_n}{r_{n+m}}Jx_{n+m} - Jx_n\right\|. \tag{3.14}$$

On the other hand, for any $p \in \mathrm{EP}(f)$, from $z_n = T_{r_n}x_n$, we have

$$\|z_n - p\| = \|T_{r_n}x_n - p\| \leq \|x_n - p\|, \tag{3.15}$$

so that $\{z_n\}$ is bounded. Since $\lim_{n\to\infty} r_n = r > 0$, this together with (3.14) implies that $\{z_n\}$ is a Cauchy sequence. Hence $T_{r_n} x_n = z_n$ is convergent.

(2) By using Lemma 3.2, we know that

$$\bigcap_{n=1}^{\infty} F(T_{r_n}) = \mathrm{EP}(f) \neq \emptyset. \tag{3.16}$$

(3) From (1) we know that, $\lim_{n\to\infty} T_{r_n} x$ exists for all $x \in C$. So, we can define a mapping T from C into itself by

$$Tx = \lim_{n\to\infty} T_{r_n} x, \quad \forall x \in C. \tag{3.17}$$

It is obvious that T is nonexpansive. It is easy to see that

$$\mathrm{EP}(f) = \bigcap_{n=1}^{\infty} F(T_{r_n}) \subset F(T). \tag{3.18}$$

On the other hand, let $w \in F(T)$ and $w_n = T_{r_n} w$, we have

$$f(w_n, y) + \frac{1}{r_n}\langle y - w_n, Jw_n - Jw\rangle \geq 0, \quad \forall y \in C. \tag{3.19}$$

By (A2) we know that

$$\frac{1}{r_n}\langle y - w_n, Jw_n - Jw\rangle \geq f(y, w_n), \quad \forall y \in C. \tag{3.20}$$

Since $w_n \to Tw = w$ and from (A4), we have $f(y, w) \leq 0$, for all $y \in C$. Then, for $t \in (0,1]$ and $y \in C$,

$$\begin{aligned} 0 &= f(ty + (1-t)w, ty + (1-t)w) \\ &\leq t f(ty + (1-t)w, y) + (1-t) f(ty + (1-t)w, w) \\ &\leq t f(ty + (1-t)w, y). \end{aligned} \tag{3.21}$$

Therefore, we have

$$f(ty + (1-t)w, y) \geq 0. \tag{3.22}$$

Letting $t \downarrow 0$ and using (A3), we get the following:

$$f(w, y) \geq 0, \quad \forall y \in C, \tag{3.23}$$

and hence $w \in \mathrm{EP}(f)$. From above two respects, we know that $F(T) = \bigcap_{n=0}^{\infty} F(T_{r_n})$.

Next we show $\{T_{r_n}\}$ is uniformly closed. Assume $x_n \to x$ and $\|x_n - T_{r_n}x_n\| \to 0$, from above results, we know that $Tx = \lim_{n\to\infty} T_{r_n}x$. On the other hand, from $\|x_n - T_{r_n}x_n\| \to 0$, we also get $\lim_{n\to\infty} T_{r_n}x = x$, so that $x \in F(T) = \bigcap_{n=1}^{\infty} F(T_{r_n})$. That is, the sequence of mappings $\{T_{r_n}\}$ is uniformly closed. This completes the proof. □

Theorem 3.5. *Let E be a p-uniformly convex with $p \geq 2$ and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let f and g be two bifunctions from $C \times C$ to $R = (-\infty, +\infty)$ satisfying (A1)–(A4). Assume that $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, and $\{\delta_n\}_{n=1}^{\infty}$ are four sequences in $[0,\ 1]$ such that $\alpha_n+\beta_n+\gamma_n+\delta_n = 1$, $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\delta_n = 0$, and $0 < \gamma \leq \gamma_n \leq 1$ for some constant $\gamma \in (0,1)$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
&x_0 \in C \text{ arbitrarily},\\
&y_n = J^{-1}\left(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JT_{r_n}x_n + \delta_n JS_{r_n}x_n\right), \quad n \geq 1,\\
&C_n = \left\{z \in C_{n-1} : \phi(z, y_n) \leq (1-\alpha_n)\phi(z, x_n) + \alpha_n\phi(z, x_0)\right\}, \quad n \geq 1,\\
&C_0 = C,\\
&x_{n+1} = \Pi_{C_n}x_0, \quad n \geq 0,
\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}
T_r(x) &= \left\{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C\right\}, \quad \forall x \in E,\\
S_r(x) &= \left\{z \in C : g(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C\right\}, \quad \forall x \in E,
\end{aligned} \tag{3.25}$$

and $\lim_{n\to\infty} r_n = r > 0$. Assume that the mappings T_n and S_n satisfy the asymptotically condition. Then $\{x_n\}$ converges strongly to $q = \Pi_{\mathrm{EP}(f)\cap\cap\mathrm{EP}(g)}x_0$.

Proof. By Lemma 3.4, $\{T_{r_n}\}_{n=1}^{\infty}$, $\{S_{r_n}\}_{n=1}^{\infty}$ are uniformly closed; therefore, by using Theorem 2.2 and Lemma 3.2, we can obtain the conclusion of Theorem 3.5. This completes the proof. □

When $\alpha_n \equiv 0$, $\delta_n \equiv 0$ in the Theorem 3.5, we obtain the following result.

Theorem 3.6. *Let E be a p–uniformly convex with $p \geq 2$ and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $R = (-\infty, +\infty)$ satisfying (A1)–(A4). Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequences in $[0,1]$ such that $0 \leq \alpha_n \leq \alpha < 1$ for some constant $\alpha \in (0,1)$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned}
&x_0 \in C \text{ arbitrarily},\\
&y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_{r_n}x_n), \quad n \geq 1,\\
&C_n = \left\{z \in C_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\right\}, \quad n \geq 1,\\
&C_0 = C,\\
&x_{n+1} = \Pi_{C_n}x_0, \quad n \geq 0,
\end{aligned} \tag{3.26}$$

where $\lim_{n\to\infty} r_n = r > 0$. Then $\{x_n\}$ converges strongly to $q = \Pi_{\mathrm{EP}(f)}x_0$.

4. Applications for Maximal Monotone Operators

In this section, we apply our above results to prove some strong convergence theorem concerning maximal monotone operators in a Banach space E.

Let A be a multivalued operator from E to E^* with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \{z \in E : z \in D(A)\}$. An operator A is said to be monotone if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0 \tag{4.1}$$

for each $x_1, x_2 \in D(A)$ and $y_1 \in Ax_1$, $y_2 \in Ax_2$. A monotone operator A is said to be maximal if it's graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $A^{-1}0$ is closed and convex. The following result is also wellknown.

Theorem 4.1. *Let E be a reflexive, strictly convex, and smooth Banach space, and let A be a monotone operator from E to E^*. Then A is maximal if and only if $R(J + rA) = E^*$. for all $r > 0$.*

Let E be a reflexive, strictly convex, and smooth Banach space, and let A be a maximal monotone operator from E to E^*. Using Theorem 4.1 and strict convexity of E, we obtain that for every $r > 0$ and $x \in E$, there exists a unique x_r such that

$$Jx \in Jx_r + rAx_r. \tag{4.2}$$

Then we can define a single-valued mapping $J_r : E \longrightarrow D(A)$ by $J_r = (J + rA)^{-1}J$ and such a J_r is called the resolvent of A, we know that $A^{-1}0 = F(J_r)$ for all $r > 0$.

Theorem 4.2. *Let E be a uniformly convex and uniformly smooth Banach space, let A be a maximal monotone operator from E to E^*, and let J_r be a resolvent of A for $r > 0$. Then for any sequence $\{r_n\}_{n=1}^{\infty}$ such that $\liminf_{n\to\infty} r_n > 0$, $\{J_{r_n}\}_{n=1}^{\infty}$ is a uniformly closed sequence of hemi-relatively nonespansive mappings.*

Proof. Firstly, we show that $\{J_{r_n}\}_{n=1}^{\infty}$ is uniformly closed. Let $\{z_n\} \subset E$ be a sequence such that $z_n \longrightarrow p$ and $\lim_{n\to\infty}\|z_n - J_{r_n}z_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we obtain that

$$\frac{1}{r_n}(Jz_n - JJ_{r_n}z_n) \longrightarrow 0. \tag{4.3}$$

It follows from

$$\frac{1}{r_n}(Jz_n - JJ_{r_n}z_n) \in AJ_{r_n}z_n \tag{4.4}$$

and the monotonicity of A that

$$\left\langle w - J_{r_n}z_n, w^* - \frac{1}{r_n}(Jz_n - JJ_{r_n}z_n) \right\rangle \geq 0 \tag{4.5}$$

for all $w \in D(A)$ and $w^* \in Aw$. Letting $n \to \infty$, we have $\langle w - p, w^* \rangle \geq 0$ for all $w \in D(A)$ and $w^* \in Aw$. Therefore, from the maximality of A, we obtain that $p \in A^{-1}0 = F(J_{r_n})$ for all $n \geq 1$, that is, $p \in \bigcap_{n=1}^{\infty} F(J_{r_n})$.

Next we show that J_{r_n} is a hemi-relatively nonexpansive mapping for all $n \geq 1$. For any $w \in E$ and $p \in F(J_{r_n}) = A^{-1}0$, from the monotonicity of A, we have

$$\begin{aligned}
\phi(p, J_{r_n}w) &= \|p\|^2 - 2\langle p, JJ_{r_n}w\rangle + \|J_{r_n}w\|^2 \\
&= \|p\|^2 + 2\langle p, Jw - JJ_{r_n}w - Jw\rangle + \|J_{r_n}w\|^2 \\
&= \|p\|^2 + 2\langle p, Jw - JJ_{r_n}w\rangle - 2\langle p, Jw\rangle + \|J_{r_n}w\|^2 \\
&= \|p\|^2 - 2\langle J_r w - p - J_{r_n}w, Jw - JJ_{r_n}w\rangle - 2\langle p, Jw\rangle + \|J_{r_n}w\|^2 \\
&= \|p\|^2 - 2\langle J_{r_n}w - p, Jw - JJ_{r_n}w\rangle \\
&\quad + 2\langle J_{r_n}w, Jw - JJ_{r_n}w\rangle - 2\langle p, Jw\rangle + \|J_{r_n}w\|^2 \\
&\leq \|p\|^2 + 2\langle J_{r_n}w, Jw - JJ_{r_n}w\rangle - 2\langle p, Jw\rangle + \|J_{r_n}w\|^2 \\
&= \|p\|^2 - 2\langle p, Jw\rangle + \|w\|^2 - \|J_{r_n}w\|^2 + 2\langle J_{r_n}w, Jw\rangle - \|w\|^2 \\
&= \phi(p, w) - \phi(J_{r_n}w, w) \\
&\leq \phi(p, w).
\end{aligned} \tag{4.6}$$

This implies that J_{r_n} is a hemi-relatively nonexpansive mapping for all $n \geq 1$. This completes the proof. □

Theorem 4.3. *Let E be a uniformly convex and uniformly smooth Banach space, let A and B be two maximal monotone operators from E to E^* with nonempty common zero point set $A^{-1}(0) \cap B^{-1}(0)$, let J_r^A be a resolvent of A and J_r^B a resolvent of B for $r > 0$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, and $\{\delta_n\}_{n=1}^{\infty}$ are four sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\delta_n = 0$, and $0 < \gamma \leq \gamma_n \leq 1$ for some constant $\gamma \in (0,1)$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
&x_0 \in C \text{ arbitrarily}, \\
&y_n = J^{-1}\left(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JJ_{r_n}^A x_n + \delta_n JJ_{r_n}^B x_n\right), \quad n \geq 1, \\
&C_n = \left\{z \in C_{n-1} : \phi(z, y_n) \leq (1-\alpha_n)\phi(z, x_n) + \alpha_n\phi(z, x_0)\right\}, \quad n \geq 1, \\
&C_0 = C, \\
&x_{n+1} = \Pi_{C_n}x_0, \quad n \geq 0,
\end{aligned} \tag{4.7}$$

where $\liminf_{n\to\infty} r_n > 0$. Assume that the mappings $J_{r_n}^A$ and $J_{r_n}^B$ satisfy the asymptotically condition. Then $\{x_n\}$ converges strongly to $q = \Pi_{A^{-1}(0)\cap B^{-1}(0)}x_0$.

Proof. From Theorem 4.2, $\{J_{r_n}\}_{n=1}^{\infty}$ is uniformly closed countable family of hemi-relatively nonexpansive mappings, on the other hand, $A^{-1}(0) = \bigcap_{n=1}^{\infty} F(J_{r_n})$, by using Theorem 2.2, we can obtain the conclusion of Theorem 4.3. □

When $\alpha_n \equiv 0, \delta_n \equiv 0$ in Theorem 4.3, we obtain the following result.

Theorem 4.4. *Let E be a uniformly convex and uniformly smooth Banach space, let A be a maximal monotone operator from E to E^* with nonempty zero point set $A^{-1}(0)$, and let J_r be a resolvent of A for $r > 0$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$ such that $0 \le \alpha_n \le \alpha < 1$ for some constant $\alpha \in (0,1)$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{aligned}
&x_0 \in C \text{ arbitrarily},\\
&y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J J_{r_n} x_n), \quad n \ge 1,\\
&C_n = \{z \in C_{n-1} : \phi(z, y_n) \le \phi(z, x_n)\}, \quad n \ge 1,\\
&C_0 = C,\\
&x_{n+1} = \Pi_{C_n} x_0, \quad n \ge 0,
\end{aligned} \tag{4.8}$$

where $\liminf_{n\to\infty} r_n > 0$. *Then $\{x_n\}$ converges strongly to $q = \Pi_{A^{-1}(0)} x_0$.*

5. Examples

Firstly, we give an example which is hemi-relatively nonexpansive mapping but not weak relatively nonexpansive mapping.

Example 5.1. Let $E = R^n$ and $x_0 \neq 0$ be a any element of E. We define a mapping $T : E \to E$ as follows:

$$T(x) = \begin{cases} \left(\dfrac{1}{2} + \dfrac{1}{2^{n+1}}\right) x_0 & \text{if } x = \left(\dfrac{1}{2} + \dfrac{1}{2^n}\right) x_0, \\ -x & \text{if } x \neq \left(\dfrac{1}{2} + \dfrac{1}{2^n}\right) x_0, \end{cases} \tag{5.1}$$

for $n = 1, 2, 3, \ldots$. Next we show that T is a hemi-relatively nonexpansive mapping but no weak relatively nonexpansive mapping. First, it is obvious that $F(T) = \{0\}$. In addition, it is easy to see that

$$\|Tx\| \le \|x\|, \quad \forall x \in E. \tag{5.2}$$

This implies that

$$\|Tx\|^2 - \|x\|^2 \le 2\langle 0, JTx - Jx\rangle = 2\langle p, JTx - Jx\rangle \tag{5.3}$$

for all $p \in F(T)$. It follows from above inequality that

$$\|p\|^2 - 2\langle p, JTx\rangle + \|Tx\|^2 \le \|p\|^2 - 2\langle p, Jx\rangle + \|x\|^2, \tag{5.4}$$

for all $p \in F(T)$ and $x \in E$. That is

$$\phi(p, Tx) \le \phi(p, x), \tag{5.5}$$

for all $p \in F(T)$ and $x \in E$; hence T is a hemi-relatively nonexpansive mapping. Finally, we show that T is not weak relatively nonexpansive mapping. In fact that, letting

$$x_n = \left(\frac{1}{2} + \frac{1}{2^n}\right)x_0, \quad n = 1, 2, 3, \ldots \tag{5.6}$$

from the definition of T, we have

$$Tx_n = \left(\frac{1}{2} + \frac{1}{2^{n+1}}\right)x_0, \quad n = 1, 2, 3, \ldots \tag{5.7}$$

which implies that $\|x_n - Tx_n\| \to 0$ and $x_n \to x_0$ $(x_n \rightharpoonup x_0)$ as $n \to \infty$. That is $x_0 \in \widetilde{F}(T)$ but $x_0 \overline{\in} F(T)$.

Next, we give an example which is weak relatively nonexpansive mapping but not relatively nonexpansive mapping.

Example 5.2. Let $E = l^2$, where

$$\begin{aligned} l^2 &= \left\{\xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) : \sum_{n=1}^{\infty}|\xi_n|^2 < \infty\right\}, \\ \|\xi\| &= \left(\sum_{n=1}^{\infty}|\xi_n|^2\right)^{1/2}, \quad \forall \xi \in l^2, \\ \langle \xi, \eta\rangle &= \sum_{n=1}^{\infty}\xi_n\eta_n, \quad \forall \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots), \eta = (\eta_1, \eta_2, \eta_3, \ldots, \eta_n, \ldots) \in l^2. \end{aligned} \tag{5.8}$$

It is well known that l^2 is a Hilbert space, so that $(l^2)^* = l^2$. Let $\{x_n\} \subset E$ be a sequence defined by

$$\begin{aligned}
x_0 &= (1, 0, 0, 0, \ldots), \\
x_1 &= (1, 1, 0, 0, \ldots), \\
x_2 &= (1, 0, 1, 0, 0, \ldots), \\
x_3 &= (1, 0, 0, 1, 0, 0, \ldots), \\
&\vdots \\
x_n &= (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \ldots, \xi_{n,k}, \ldots),
\end{aligned} \tag{5.9}$$

where

$$\xi_{n,k} = \begin{cases} 1 & \text{if } k = 1,\ n+1, \\ 0 & \text{if } k \neq 1,\ k \neq n+1, \end{cases} \tag{5.10}$$

for all $n \geq 1$. Define a mapping $T : E \longrightarrow E$ as follows:

$$T(x) = \begin{cases} \dfrac{n}{n+1} x_n & \text{if } x = x_n \ (\exists n \geq 1), \\ -x & \text{if } x \neq x_n \ (\forall n \geq 1). \end{cases} \tag{5.11}$$

Conclusion 1. $\{x_n\}$ converges weakly to x_0.

Proof. For any $f = (\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_k, \ldots) \in l^2 = (l^2)^*$, we have

$$f(x_n - x_0) = \langle f, x_n - x_0 \rangle = \sum_{k=2}^{\infty} \zeta_k \xi_{n,k} = \zeta_{n+1} \longrightarrow 0, \tag{5.12}$$

as $n \longrightarrow \infty$. That is, $\{x_n\}$ converges weakly to x_0. □

Conclusion 2. $\{x_n\}$ is not a Cauchy sequence, so that, it does not converge strongly to any element of l^2.

Proof. In fact, we have $\|x_n - x_m\| = \sqrt{2}$ for any $n \neq m$. Then $\{x_n\}$ is not a Cauchy sequence. □

Conclusion 3. T has a unique fixed point 0, that is, $F(T) = \{0\}$.

Proof . The conclusion is obvious. □

Conclusion 4. x_0 is an asymptotic fixed point of T.

Proof. Since $\{x_n\}$ converges weakly to x_0 and

$$\|Tx_n - x_n\| = \left\| \frac{n}{n+1}x_n - x_n \right\| = \frac{1}{n+1}\|x_n\| \longrightarrow 0 \tag{5.13}$$

as $n \to \infty$, so that, x_0 is an asymptotic fixed point of T. □

Conclusion 5. T has a unique strong asymptotic fixed point 0, so that $F(T) = \widetilde{F}(T)$.

Proof . In fact that, for any strong convergent sequence $\{z_n\} \subset E$ such that $z_n \to z_0$ and $\|z_n - Tz_n\| \to 0$ as $n \to \infty$, from Conclusion 2, there exists sufficiently large nature number N such that $z_n \neq x_m$, for any $n, m > N$. Then $Tz_n = -z_n$ for $n > N$, and it follows from $\|z_n - Tz_n\| \to 0$ that $2z_n \to 0$ and hence $z_n \to z_0 = 0$. □

Conclusion 6. T is a weak relatively nonexpansive mapping.

Proof. Since $E = l^2$ is a Hilbert space, we have

$$\phi(0, Tx) = \|0 - Tx\|^2 = \|Tx\|^2 \leq \|x\|^2 = \|x - 0\|^2 = \phi(0, x), \quad \forall x \in E. \tag{5.14}$$

From Conclusion 2, we have $F(T) = \widetilde{F}(T)$, then T is a weak relatively nonexpansive mapping. □

Conclusion 7. T is not a relatively nonexpansive mapping.

Proof. From Conclusions 3 and 4, we have $F(T) \neq \widehat{F}(T)$, so that T is not a relatively nonexpansive mapping. □

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