Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 438023, 12 pages
doi:10.1155/2012/438023

Research Article

Strong Convergence of the Viscosity Approximation Process for the Split Common Fixed-Point Problem of Quasi-Nonexpansive Mappings

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Received 14 December 2011; Accepted 11 January 2012

Academic Editor: Yonghong Yao

Very recently, Moudafi (2011) introduced an algorithm with weak convergence for the split common fixed-point problem. In this paper, we will continue to consider the split common fixed-point problem. We discuss the strong convergence of the viscosity approximation method for solving the split common fixed-point problem for the class of quasi-nonexpansive mappings in Hilbert spaces. Our results improve and extend the corresponding results announced by many others.

1. Introduction and Preliminary

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let I denote the identity operator on H. Let C and Q be nonempty closed convex subset of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is to find a point

$$x \in C \quad \text{such that } Ax \in Q, \tag{1.1}$$

where $A : H_1 \rightarrow H_2$ is a bounded linear operator. The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. The SFP attracts many authors' attention due to its application in signal processing. Various algorithms have been invented to solve it (see [3–9] and references therein).

Note that the split feasibility problem (1.1) can be formulated as a fixed-point equation by using the fact

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*; \tag{1.2}$$

that is, x^* solves the SFP (1.1) if and only if x^* solves the fixed point equation (1.2) (see [10] for the details). This implies that we can use fixed-point algorithms (see [11–13]) to solve SFP. A popular algorithm that solves the SFP (1.1) is due to Byrne's CQ algorithm [2] which is found to be a gradient-projection method (GPM) in convex minimization. Subsequently, Byrne [3] applied KM iteration to the CQ algorithm, and Zhao and Yang [14] applied KM iteration to the perturbed CQ algorithm to solve the SFP. It is well known that the CQ algorithm and the KM algorithm for a split feasibility problem do not necessarily converge strongly in the infinite-dimensional Hilbert spaces.

The split common fixed-point problem (SCFP) is a generalization of the split feasibility problem (SFP) and the convex feasibility problem (CFP); see [15]. In this paper, we introduce and study the convergence properties of a viscosity approximation algorithm for solving the SCFP for the class of quasi-nonexpansive operators S such that $I - S$ is demiclosed at the origin.

Now let us first recall the definition of quasi-nonexpansive operators which appear naturally when using subgradient projection operator techniques in solving some feasibility problems, and also some definitions of classes of operators often used in fixed-point theory and which are commonly encountered in the literature.

Let $T : H \to H$ be a mapping. A point $x \in H$ is said to be a fixed point of T provided that $Tx = x$. In this paper, we use $F(T)$ to denote the fixed-point set and use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. We use $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ stand for the weak ω-limit set of $\{x_k\}$.

(i) A mapping $T : H \to H$ belongs to the general class Φ_Q of (possibly discontinuous) quasi-nonexpansive mappings if

$$\|Tx - q\| \le \|x - q\|, \quad \forall (x, q) \in H \times F(T). \tag{1.3}$$

(ii) A mapping $T : H \to H$ belongs to the set Φ_N of nonexpansive mappings if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall (x, y) \in H \times H. \tag{1.4}$$

(iii) A mapping $T : H \to H$ belongs to the set Φ_{FN} of firmly nonexpansive mappings if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2, \quad \forall (x, y) \in H \times H. \tag{1.5}$$

(iv) A mapping $T : H \to H$ belongs to the set Φ_{FQ} of firmly quasi-nonexpansive mappings if

$$\|Tx - q\|^2 \le \|x - q\|^2 - \|x - Tx\|^2, \quad \forall (x, q) \in H \times F(T). \tag{1.6}$$

It is easily observed that $\Phi_{\mathrm{FN}} \subset \Phi_N \subset \Phi_Q$ and that $\Phi_{\mathrm{FN}} \subset \Phi_{\mathrm{FQ}} \subset \Phi_Q$. Furthermore, Φ_{FN} is well known to include resolvents and projection operators, while Φ_{FQ} contains subgradient projection operators (see, e.g., [16] and the reference therein).

A mapping $T : H \to H$ is called demiclosed at the origin if any sequence $\{x_n\}$ weakly converges to x, and if the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx = 0$. A mapping $f : H \to H$ is called a contraction of modulus $\rho \in [0,1)$ if

$$\|fx - fy\| \le \rho\|x - y\|, \quad \forall(x,y) \in H \times H. \tag{1.7}$$

In what follows, we will focus our attention on the following general two-operator split common fixed-point problem:

$$\text{find } x^* \in C \quad \text{such that } Ax^* \in Q, \tag{1.8}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $U : H_1 \to H_1$ and $S : H_2 \to H_2$ are two quasi-nonexpansive operators with nonempty fixed-point sets $F(U) = C$ and $F(S) = Q$, and denote the solution set of the two-operator SCFP by

$$\Gamma = \{y \in C; Ay \in Q\}. \tag{1.9}$$

Recall that $F(U)$ and $F(S)$ are nonempty closed convex subsets of H_1 and H_2, respectively. If $\Gamma \neq \emptyset$, we have Γ which is close convex subset of H_1. To solve (1.8), Censor and Segal [15] proposed and proved, in infinite-dimensional spaces, the convergence of the following algorithm:

$$x_{k+1} = U\left(x_k + \gamma A^t(S - I)Ax_k\right), \quad k \in N, \tag{1.10}$$

where $\gamma \in (0, 2/\lambda)$, with λ being the largest eigenvalue of the matrix A^tA (A^t stands for matrix transposition). Very recently, Moudafi [17] introduced the following relaxed algorithm:

$$x_{k+1} = (1 - \alpha_k)u_k + \alpha_k U(u_k), \quad k \in N, \tag{1.11}$$

where $u_k = x_k + \gamma\beta A^*(S - I)Ax_k$, $\beta \in (0,1)$, $\alpha_k \in (0,1)$, and $\gamma \in (0, 1/\lambda\beta)$, with λ being the spectral radius of the operator A^*A. Moudafi proved weak convergence result of the algorithm in Hilbert spaces.

Inspired by their work, we introduce the following viscosity approximation algorithm.

Algorithm 1. Initialization: Let $x_0 \in H$ be arbitrary.

Iterative step: Set $T = U(I + \gamma A^*(S - I)A)$. For $k \in N$, let

$$x_{k+1} = \alpha_k f(x_k) + (1 - \alpha_k)((1 - \omega_k)x_k + \omega_k Tx_k), \tag{1.12}$$

where $f : H \to H$ is a contraction of modulus ρ, $\omega_k \in (0, 1/2)$, $\gamma \in (0, 1/\lambda)$ with λ being the spectral radius of the operator A^*A, and $\alpha_k \in (0,1)$.

This paper establishes the strong convergence of the sequence given by (1.12) to the unique solution of the variational inequality problem $\mathrm{VIP}(I-f,\Gamma)$:

$$\text{find } x^* \in \Gamma \quad \text{such that } \langle (I-f)x^*, v-x^* \rangle \geq 0, \quad \forall v \in \Gamma. \tag{1.13}$$

Now we give a series of preliminary results needed for the convergence analysis of algorithm (1.12).

Lemma 1.1. *Let H be a real Hilbert space and $T : H \to H$ a quasi-nonexpansive mapping. Then, the following properties are reached:*

(i) $\langle x,y\rangle = -\frac{1}{2}\|x-y\|^2 + \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2, \quad \forall (x,y) \in H \times H;$

(ii) $\langle x-Tx, x-q\rangle \geq \frac{1}{2}\|x-Tx\|^2$ *and* $\langle x-Tx, q-Tx\rangle \leq \frac{1}{2}\|x-Tx\|^2, \quad \forall (x,q) \in H \times F(T).$

Remark 1.2. Let $F := I - f$, *where* f is the contraction defined in (1.7). It is a simple matter to see that the operator F is $(1-\rho)$ strongly monotone over H; that is,

$$\langle Fx - Fy, x-y\rangle \geq (1-\rho)\|x-y\|^2, \quad \forall (x,y) \in H \times H. \tag{1.14}$$

The next result is of fundamental importance for the techniques of analysis used in this paper. It was established in [18], and its proof is given for the sake of completeness.

Lemma 1.3 (see [18, Lemma 1.3]). *Let $\{\delta_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\delta_{n_j}\}_{j\geq 0}$ of $\{\delta_n\}$ which satisfies $\delta_{n_j} < \delta_{n_j+1}$ for all $j \geq 0$. Also consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by*

$$\tau(n) = \max\{k \leq n \mid \delta_k < \delta_{k+1}\}. \tag{1.15}$$

Then $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty}\tau(n) = \infty$, and, for all $n \geq n_0$, it holds that $\delta_{\tau(n)} \leq \delta_{\tau(n)+1}$ and one has

$$\delta_n \leq \delta_{\tau(n)+1}. \tag{1.16}$$

Proof. Clearly, we can see that $\{\tau(n)\}$ is a well-defined sequence, and the fact that it is nondecreasing is obvious as well as $\lim_{n\to\infty}\tau(n) = \infty$ and $\delta_{\tau(n)} \leq \delta_{\tau(n)+1}$. Let us prove (1.16). It is easily observed that $\tau(n) \leq n$. Consequently, we prove (1.16) by distinguishing the three cases: (c1) $\tau(n) = n$; (c2) $\tau(n) = n-1$; (c3) $\tau(n) < n-1$. In the first case (i.e., $\tau(n) = n$), (1.16) is immediately given by $\delta_{\tau(n)} \leq \delta_{\tau(n)+1}$. In the second case (i.e., $\tau(n) = n-1$), (1.16) becomes obvious. In the third case (i.e., $\tau(n) \leq n-2$), by (1.15) and for any integer $n \geq n_0$, we easily observe that $\delta_j \geq \delta_{j+1}$ for $\tau(n)+1 \leq j \leq n-1$; namely,

$$\delta_{\tau(n)+1} \geq \delta_{\tau(n)+2} \geq \cdots \geq \delta_{n-1} \geq \delta_n, \tag{1.17}$$

which entails the desired result. □

2. Main Results

Theorem 2.1. *Given a bounded linear operator* $A : H_1 \to H_2$, *let* $U : H_1 \to H_1$ *and* $S : H_2 \to H_2$ *be quasi-nonexpansive mappings with nonempty fixed-point set* $F(U) = C$ *and* $F(S) = Q$. *Assume that* $U - I$ *and* $S - I$ *are demiclosed at origin. Let* $\{x_k\}$ *be the sequence given by* (1.12) *with* $\gamma \in (0, 1/\lambda)$, $\omega_k \in (0, 1/2)$ *such that* $0 < \liminf_{k\to\infty}\omega_k \le \limsup_{k\to\infty}\omega_k < 1/2$ *and* $\{\alpha_k\} \subset (0,1)$ *such that* $\lim_{k\to\infty}\alpha_k = 0$ *and* $\sum_k \alpha_k = \infty$. *If* $\Gamma \neq \emptyset$, *then the sequence* $\{x_k\}$ *strongly converges to a split common fixed-point* $x^* \in \Gamma$, *verifying* $x^* = P_\Gamma f(x^*)$ *which equivalently solves the following variational inequality problem:*

$$x^* \in \Gamma, \quad \langle (I - f)x^*, v - x^* \rangle \ge 0, \quad \forall v \in \Gamma. \tag{2.1}$$

Proof. Set $T_{\omega_k} = (1 - \omega_k)I + \omega_k T$. Then $x_{k+1} = \alpha_k f(x_k) + (1 - \alpha_k)T_{\omega_k}x_k$.

Firstly, we prove that $\{x_k\}$ is bounded. Taking $y \in \Gamma$, that is, $y \in F(U)$, $Ay \in F(S)$. We have

$$\begin{aligned}\|x_{k+1} - y\| &= \|\alpha_k(f(x_k) - f(y)) + \alpha_k(f(y) - y) + (1 - \alpha_k)(T_{\omega_k}x_k - y)\| \\ &\le \alpha_k\|f(x_k) - f(y)\| + \alpha_k\|f(y) - y\| + (1 - \alpha_k)\|T_{\omega_k}x_k - y\| \\ &\le \alpha_k\rho\|x_k - y\| + \alpha_k\|f(y) - y\| + (1 - \alpha_k)\|T_{\omega_k}x_k - y\|.\end{aligned} \tag{2.2}$$

From the definition of T_{ω_k}, we get

$$\begin{aligned}\|T_{\omega_k}x_k - y\|^2 &= \|(1 - \omega_k)x_k + \omega_k Tx_k - y\|^2 \\ &= \|x_k - y + \omega_k(Tx_k - x_k)\|^2 \\ &= \|x_k - y\|^2 - 2\omega_k\langle x_k - y, x_k - Tx_k\rangle + \omega_k^2\|Tx_k - x_k\|^2.\end{aligned} \tag{2.3}$$

On the other hand, we have

$$\begin{aligned}\|Tx_k - y\|^2 &= \|U(I + \gamma A^*(S - I)A)x_k - y\|^2 \\ &\le \|(I + \gamma A^*(S - I)A)x_k - y\|^2 \\ &= \|x_k - y\|^2 + \gamma^2\|A^*(S - I)Ax_k\|^2 + 2\gamma\langle x_k - y, A^*(S - I)Ax_k\rangle \\ &= \|x_k - y\|^2 + \gamma^2\langle (S - I)Ax_k, AA^*(S - I)Ax_k\rangle + 2\gamma\langle x_k - y, A^*(S - I)Ax_k\rangle.\end{aligned} \tag{2.4}$$

From the definition of λ, it follows that

$$\begin{aligned}\gamma^2\langle (S - I)Ax_k, AA^*(S - I)Ax_k\rangle &\le \lambda\gamma^2\langle (S - I)Ax_k, (S - I)Ax_k\rangle \\ &= \lambda\gamma^2\|(S - I)Ax_k\|^2.\end{aligned} \tag{2.5}$$

Now, by using property (ii) of Lemma 1.1, we obtain

$$\begin{aligned}
2\gamma\langle x_k - y, A^*(S-I)Ax_k\rangle &= 2\gamma\langle A(x_k - y), (S-I)Ax_k\rangle \\
&= 2\gamma\langle A(x_k - y) + (S-I)Ax_k - (S-I)Ax_k, (S-I)Ax_k\rangle \\
&= 2\gamma\left(\langle S(Ax_k) - Ay, (S-I)Ax_k\rangle - \|(S-I)Ax_k\|^2\right) \\
&\le 2\gamma\left(\frac{1}{2}\|(S-I)Ax_k\|^2 - \|(S-I)Ax_k\|^2\right) \\
&= -\gamma\|(S-I)Ax_k\|^2.
\end{aligned} \tag{2.6}$$

Combining (2.4)–(2.6), we have

$$\begin{aligned}
\|Tx_k - y\|^2 &\le \|x_k - y\|^2 + \lambda\gamma^2\|(S-I)Ax_k\|^2 - \gamma\|(S-I)Ax_k\|^2 \\
&= \|x_k - y\|^2 - \gamma(1-\lambda\gamma)\|(S-I)Ax_k\|^2 \\
&\le \|x_k - y\|^2.
\end{aligned} \tag{2.7}$$

From property (i) of Lemma 1.1, we have

$$\begin{aligned}
\langle x_k - y, x_k - Tx_k\rangle &= -\frac{1}{2}\|Tx_k - y\|^2 + \frac{1}{2}\|x_k - y\|^2 + \frac{1}{2}\|x_k - Tx_k\|^2 \\
&\ge \frac{1}{2}\|x_k - Tx_k\|^2.
\end{aligned} \tag{2.8}$$

From (2.3) and (2.8), we have

$$\begin{aligned}
\|T_{\omega_k}x_k - y\|^2 &\le \|x_k - y\|^2 - \omega_k\|x_k - Tx_k\|^2 + \omega_k^2\|x_k - Tx_k\|^2 \\
&= \|x_k - y\|^2 - \omega_k(1-\omega_k)\|x_k - Tx_k\|^2 \\
&\le \|x_k - y\|^2,
\end{aligned} \tag{2.9}$$

Combining (2.2), (2.3), and (2.9), it follows that

$$\begin{aligned}
\|x_{k+1} - y\| &\le \alpha_k\rho\|x_k - y\| + \alpha_k\|f(y) - y\| + (1-\alpha_k)\|x_k - y\| \\
&= [1 - \alpha_k(1-\rho)]\|x_k - y\| + \alpha_k\|f(y) - y\| \\
&\le \max\left\{\|x_k - y\|, \frac{1}{1-\rho}\|f(y) - y\|\right\}.
\end{aligned} \tag{2.10}$$

It is obviously that

$$\|x_k - y\| \le \max\left\{ \|x_0 - y\|, \frac{1}{1-\rho}\|f(y) - y\| \right\}, \tag{2.11}$$

and hence $\{x_k\}$ is bounded. Let $x^* = P_\Gamma f(x^*)$. We have

$$x_{k+1} - x_k + \alpha_k\big(x_k - f(x_k)\big) = (1-\alpha_k)(T_{\omega_k}x_k - x_k), \tag{2.12}$$

and hence

$$\big\langle x_{k+1} - x_k + \alpha_k(I-f)x_k, x_k - x^* \big\rangle = -(1-\alpha_k)\langle x_k - T_{\omega_k}x_k, x_k - x^*\rangle. \tag{2.13}$$

By (2.9) we obtain that

$$\begin{aligned}\langle x_k - T_{\omega_k}x_k, x_k - x^*\rangle &= \frac{1}{2}\|x_k - T_{\omega_k}x_k\|^2 + \frac{1}{2}\|x_k - x^*\|^2 - \frac{1}{2}\|T_{\omega_k}x_k - x^*\|^2 \\ &\ge \frac{\omega_k^2}{2}\|x_k - Tx_k\|^2 + \frac{1}{2}\|x_k - x^*\|^2 - \frac{1}{2}\|x_k - x^*\|^2 + \frac{\omega_k}{2}(1-\omega_k)\|x_k - Tx_k\|^2 \\ &= \frac{\omega_k}{2}\|x_k - Tx_k\|^2.\end{aligned} \tag{2.14}$$

It follows from (2.13) that

$$\big\langle x_{k+1} - x_k + \alpha_k(I-f)x_k, x_k - x^* \big\rangle \le -\frac{\omega_k}{2}(1-\alpha_k)\|x_k - Tx_k\|^2, \tag{2.15}$$

and hence

$$-\langle x_k - x_{k+1}, x_k - x^*\rangle \le -\alpha_k\big\langle (I-f)x_k, x_k - x^*\big\rangle - \frac{\omega_k}{2}(1-\alpha_k)\|x_k - Tx_k\|^2. \tag{2.16}$$

Setting $\delta_k = \frac{1}{2}\|x_k - x^*\|^2$, we have

$$\begin{aligned}\langle x_k - x_{k+1}, x_k - x^*\rangle &= -\frac{1}{2}\|x_{k+1} - x^*\|^2 + \frac{1}{2}\|x_k - x^*\|^2 + \frac{1}{2}\|x_k - x_{k+1}\|^2 \\ &= -\delta_{k+1} + \delta_k + \frac{1}{2}\|x_k - x_{k+1}\|^2,\end{aligned} \tag{2.17}$$

so that (2.16) can be rewritten as

$$\delta_{k+1} - \delta_k - \frac{1}{2}\|x_k - x_{k+1}\|^2 \le -\alpha_k\big\langle (I-f)x_k, x_k - x^*\big\rangle - \frac{\omega_k}{2}(1-\alpha_k)\|x_k - Tx_k\|^2. \tag{2.18}$$

Now using (2.12) again, we have

$$\begin{aligned}\|x_{k+1}-x_k\|^2 &= \left\|\alpha_k(f(x_k)-x_k)+(1-\alpha_k)(T_{\omega_k}x_k-x_k\right\|^2\\ &\le \left(\alpha_k\left\|f(x_k)-x_k\right\|+(1-\alpha_k)\|T_{\omega_k}x_k-x_k\|\right)^2\\ &\le 2\alpha_k^2\left\|f(x_k)-x_k\right\|^2+2(1-\alpha_k)^2\|T_{\omega_k}x_k-x_k\|^2\\ &\le 2\alpha_k^2\left\|f(x_k)-x_k\right\|^2+2(1-\alpha_k)\omega_k^2\|Tx_k-x_k\|^2,\end{aligned} \tag{2.19}$$

which yields

$$\frac{1}{2}\|x_{k+1}-x_k\|^2 \le \alpha_k^2\left\|f(x_k)-x_k\right\|^2+(1-\alpha_k)\omega_k^2\|Tx_k-x_k\|^2. \tag{2.20}$$

From (2.18) and (2.20), we obtain

$$\delta_{k+1}-\delta_k+\omega_k(1-\alpha_k)\left(\frac{1}{2}-\omega_k\right)\|Tx_k-x_k\|^2 \le \alpha_k\left[\alpha_k\left\|f(x_k)-x_k\right\|^2-\left\langle(I-f)x_k,x_k-x^*\right\rangle\right]. \tag{2.21}$$

It follows from Remark 1.2 that

$$\left\langle(I-f)x_k-(I-f)x^*,x_k-x^*\right\rangle \ge (1-\rho)\|x_k-x^*\|^2 = 2(1-\rho)\delta_k, \tag{2.22}$$

and hence

$$2(1-\rho)\delta_k+\left\langle(I-f)x^*,x_k-x^*\right\rangle \le \left\langle(I-f)x_k,x_k-x^*\right\rangle. \tag{2.23}$$

The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists k_0 such that $\{\delta_k\}_{k\ge k_0}$ is nonincreasing. In this situation, $\{\delta_k\}$ is convergent because it is nonnegative, so that $\lim_{k\to\infty}(\delta_{k+1}-\delta_k)=0$; hence, in light of (2.21) together with $\alpha_k \to 0$, the boundedness of $\{x_k\}$, and $0<\liminf_{k\to\infty}\omega_k \le \limsup_{k\to\infty}\omega_k < 1/2$, we obtain

$$\lim_{k\to\infty}\|x_k-Tx_k\|=0. \tag{2.24}$$

From (2.21) again, we have

$$\alpha_k\left[-\alpha_k\left\|f(x_k)-x_k\right\|^2+\left\langle(I-f)x_k,x_k-x^*\right\rangle\right] \le \delta_k-\delta_{k+1}. \tag{2.25}$$

By $\sum_k \alpha_k=\infty$, we deduce that

$$\liminf_{k\to\infty}\left(-\alpha_k\left\|f(x_k)-x_k\right\|^2+\left\langle(I-f)x_k,x_k-x^*\right\rangle\right) \le 0 \tag{2.26}$$

and hence (as $\alpha_k\|f(x_k)-x_k\|^2 \to 0$)

$$\liminf_{k\to\infty}\langle (I-f)x_k, x_k-x^*\rangle \le 0. \tag{2.27}$$

By (2.23) and (2.27), we have

$$\liminf_{k\to\infty}\left(2(1-\rho)\delta_k + \langle (I-f)x^*, x_k-x^*\rangle\right) \le 0; \tag{2.28}$$

recalling that $\lim_{k\to\infty}\delta_k$ exists, we obtain

$$2(1-\rho)\lim_{k\to\infty}\delta_k + \liminf_{k\to\infty}\langle (I-f)x^*, x_k-x^*\rangle \le 0. \tag{2.29}$$

Now we prove that

$$\liminf_{k\to\infty}\langle (I-f)x^*, x_k-x^*\rangle \ge 0. \tag{2.30}$$

It follows from (2.7) and (2.24) that

$$\begin{aligned}
\gamma(1-\lambda\gamma)\|(S-I)Ax_k\|^2 &\le \|x_k-y\|^2-\|Tx_k-y\|^2\\
&= \left(\|x_k-y\|-\|Tx_k-y\|\right)\left(\|x_k-y\|+\|Tx_k-y\|\right)\\
&\le \|x_k-Tx_k\|\left(\|x_k-y\|+\|Tx_k-y\|\right)\\
&\longrightarrow 0 \quad (k\longrightarrow\infty),
\end{aligned} \tag{2.31}$$

and hence

$$\lim_{k\to\infty}\|(S-I)Ax_k\| = 0. \tag{2.32}$$

Taking $y\in\omega_w(x_k)$, from the demiclosedness of $S-I$ at 0, we obtain

$$S(Ay) = Ay. \tag{2.33}$$

Now, by setting $u_k = x_k+\gamma A^*(S-I)Ax_k$, it follows that $y\in\omega_w(u_k)$. On the other hand,

$$\|U(u_k)-u_k\| = \left\|Tx_k-x_k-\gamma A^*(S-I)Ax_k\right\| \le \|Tx_k-x_k\|+\gamma\|A^*\|\cdot\|(S-I)Ax_k\| \longrightarrow 0, \tag{2.34}$$

which, combined with the demiclosedness of $U-I$ at 0, yields

$$Uy = y. \tag{2.35}$$

Hence, $y \in C$ and $y \in \Gamma$. We can take subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j} \rightharpoonup y$ as j $\to \infty$ and

$$\liminf_{k\to\infty}\langle (I-f)x^*, x_k - x^*\rangle = \lim_{j\to\infty}\left\langle (I-f)x^*, x_{k_j} - x^*\right\rangle, \tag{2.36}$$

which leads to

$$\liminf_{k\to\infty}\langle (I-f)x^*, x_k - x^*\rangle = \langle (I-f)x^*, y - x^*\rangle \geq 0. \tag{2.37}$$

By (2.29), we have $\lim_{k\to\infty}\delta_k = 0$, and hence $\{x_k\}$ converges strongly to x^*.

Case 2. Suppose there exists a subsequence $\{\delta_{k_j}\}_{j\geq 0}$ of $\{\delta_k\}$ such that $\delta_{k_j} < \delta_{k_j+1}$ for all $j \geq 0$. In this situation, we consider the sequence of indices $\{\tau(k)\}$ as defined in Lemma 1.3. It follows that $\delta_{\tau(k)+1} - \delta_{\tau(k)} > 0$, which by (2.21) amounts to

$$\begin{aligned}\omega_k(1-\alpha_{\tau(k)})\left(\frac{1}{2}-\omega_k\right)\left\|Tx_{\tau(k)} - x_{\tau(k)}\right\|^2 \leq \alpha_{\tau(k)}\Big[&\alpha_{\tau(k)}\left\|f(x_{\tau(k)}) - x_{\tau(k)}\right\|^2 \\ &-\langle (I-f)x_{\tau(k)}, x_{\tau(k)} - x^*\rangle\Big].\end{aligned} \tag{2.38}$$

By the boundedness of $\{x_k\}$ and $\alpha_k \to 0$, we immediately obtain

$$\lim_{k\to\infty}\left\|Tx_{\tau(k)} - x_{\tau(k)}\right\| = 0. \tag{2.39}$$

Similar to Case 1, we have

$$\liminf_{k\to\infty}\langle (I-f)x^*, x_{\tau(k)} - x^*\rangle \geq 0. \tag{2.40}$$

It follows from (2.38) that

$$\langle (I-f)x_{\tau(k)}, x_{\tau(k)} - x^*\rangle \leq \alpha_{\tau(k)}\left\|f(x_{\tau(k)}) - x_{\tau(k)}\right\|^2, \tag{2.41}$$

which in the light of (2.23) yields

$$2(1-\rho)\delta_{\tau(k)} + \langle (I-f)x^*, x_{\tau(k)} - x^*\rangle \leq \alpha_{\tau(k)}\left\|f(x_{\tau(k)}) - x_{\tau(k)}\right\|^2; \tag{2.42}$$

hence (as $\alpha_{\tau(k)}\|f(x_{\tau(k)}) - x_{\tau(k)}\|^2 \to 0$) it follows that

$$2(1-\rho)\limsup_{k\to\infty}\delta_{\tau(k)} \leq -\liminf_{k\to\infty}\langle (I-f)x^*, x_{\tau(k)} - x^*\rangle. \tag{2.43}$$

From (2.40) we have $\limsup_{k\to\infty}\delta_{\tau(k)} = 0$, so that $\lim_{k\to\infty}\delta_{\tau(k)} = 0$, and hence $\lim_{k\to\infty}\|x_{\tau(k)} - x^*\| = 0$. On the other hand, it follows that

$$\begin{aligned}\left\|x_{\tau(k)+1} - x_{\tau(k)}\right\| &= \left\|\alpha_{\tau(k)}\left(f\left(x_{\tau(k)}\right) - x_{\tau(k)}\right) + \left(1-\alpha_{\tau(k)}\right)\left(T_{\omega_k}x_{\tau(k)} - x_{\tau(k)}\right)\right\| \\ &\leq \alpha_{\tau(k)}\left\|f\left(x_{\tau(k)}\right) - x_{\tau(k)}\right\| + \left(1-\alpha_{\tau(k)}\right)\omega_k\left\|Tx_{\tau(k)} - x_{\tau(k)}\right\|,\end{aligned} \tag{2.44}$$

which, by (2.39), implies that

$$\lim_{k\to\infty}\left\|x_{\tau(k)+1} - x_{\tau(k)}\right\| = 0. \tag{2.45}$$

So we have

$$\lim_{k\to\infty}\delta_{\tau(k)+1} = \frac{1}{2}\left\|x_{\tau(k)+1} - x^*\right\| = 0. \tag{2.46}$$

Then, recalling that $\delta_k \leq \delta_{\tau(k)+1}$ (by Lemma 1.3), we get $\lim_{k\to\infty}\delta_k = 0$, so that the sequence $\{x_k\}$ converges strongly to x^*.

□

Theorem 2.2. *Given a bounded linear operator $A : H_1 \to H_2$, let $U : H_1 \to H_1$ and $S : H_2 \to H_2$ be quasi-nonexpansive mappings with nonempty fixed-point set $F(U) = C$ and $F(S) = Q$. Assume that $U - I$ and $S - I$ are demiclosed at origin. Let $x_0 \in H$ be arbitrary and $\{x_k\}$ the sequence given by*

$$x_{k+1} = \alpha_k f(x_k) + (1-\alpha_k)((1-\omega)x_k + \omega T x_k), \tag{2.47}$$

where $T = U(I + \gamma A^*(S-I)A)$, $f : H \to H$ a contraction of modulus ρ, $\gamma \in (0, 1/\lambda)$, $\omega \in (0, 1/2)$, and $\{\alpha_k\} \subset (0,1)$ such that $\lim_{k\to\infty}\alpha_k = 0$ and $\sum_k \alpha_k = \infty$. If $\Gamma \neq \emptyset$, then the sequence $\{x_k\}$ strongly converges to a split common fixed-point $x^* \in \Gamma$, verifying $x^* = P_\Gamma f(x^*)$ which equivalently solves the following variational inequality problem:

$$x^* \in \Gamma, \qquad \left\langle (I-f)x^*, v - x^*\right\rangle \geq 0, \quad \forall v \in \Gamma. \tag{2.48}$$

Acknowledgments

The research was supported by Fundamental Research Funds for the Central Universities (Program No. ZXH2011D005); it was also supported by science research foundation program in Civil Aviation University of China (2011kys02).

References

[1] Y. Censor and T. Elfving, "A multiprojection algorithm using Bregman projections in a product space," *Numerical Algorithms*, vol. 8, no. 2-4, pp. 221–239, 1994.

[2] C. Byrne, "Iterative oblique projection onto convex sets and the split feasibility problem," *Inverse Problems*, vol. 18, no. 2, pp. 441–453, 2002.

[3] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.

[4] P.-E. Maingé, "The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces," *Computers & Mathematics with Applications*, vol. 59, no. 1, pp. 74–79, 2010.

[5] B. Qu and N. Xiu, "A note on the *CQ* algorithm for the split feasibility problem," *Inverse Problems*, vol. 21, no. 5, pp. 1655–1665, 2005.

[6] H.-K. Xu, "A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem," *Inverse Problems*, vol. 22, no. 6, pp. 2021–2034, 2006.

[7] Q. Yang, "The relaxed *CQ* algorithm solving the split feasibility problem," *Inverse Problems*, vol. 20, no. 4, pp. 1261–1266, 2004.

[8] Q. Yang and J. Zhao, "Generalized KM theorems and their applications," *Inverse Problems*, vol. 22, no. 3, pp. 833–844, 2006.

[9] Y. Yao, W. Jigang, and Y.-C. Liou, "Regularized methods for the split feasibility problem," *Abstract and Applied Analysis*, vol. 2012, Article ID 140679, 13 pages, 2012.

[10] H.-K. Xu, "Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces," *Inverse Problems*, vol. 26, no. 10, Article ID 105018, 2010.

[11] Y. Yao, Y.-C. Liou, and S. M. Kang, "Two-step projection methods for a system of variational inequality problems in Banach spaces," *Journal of Global Optimization*. In press.

[12] Y. Yao, R. Chen, and Y.-C. Liou, "A unified implicit algorithm for solving the triple-hierarchical constrained optimization problem," *Mathematical and Computer Modelling*, vol. 55, no. 3-4, pp. 1506–1515, 2012.

[13] Y. Yao, Y.-J. Cho, and Y.-C. Liou, "Hierarchical convergence of an implicit doublenet algorithm for nonexpansive semigroups and variational inequalities," *Fixed Point Theory and Applications*, vol. 2011, article 101, 2011.

[14] J. Zhao and Q. Yang, "Several solution methods for the split feasibility problem," *Inverse Problems*, vol. 21, no. 5, pp. 1791–1799, 2005.

[15] Y. Censor and A. Segal, "The split common fixed point problem for directed operators," *Journal of Convex Analysis*, vol. 16, no. 2, pp. 587–600, 2009.

[16] S. Măruşter and C. Popirlan, "On the Mann-type iteration and the convex feasibility problem," *Journal of Computational and Applied Mathematics*, vol. 212, no. 2, pp. 390–396, 2008.

[17] A. Moudafi, "A note on the split common fixed-point problem for quasi-nonexpansive operators," *Nonlinear Analysis*, vol. 74, no. 12, pp. 4083–4087, 2011.

[18] P.-E. Maingé, "Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization," *Set-Valued Analysis*, vol. 16, no. 7-8, pp. 899–912, 2008.