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Research Article

Constrained Solutions of a System of Matrix Equations

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We derive the necessary and sufficient conditions of and the expressions for the orthogonal solutions, the symmetric orthogonal solutions, and the skew-symmetric orthogonal solutions of the system of matrix equations $AX = B$ and $XC = D$, respectively. When the matrix equations are not consistent, the least squares symmetric orthogonal solutions and the least squares skew-symmetric orthogonal solutions are respectively given. As an auxiliary, an algorithm is provided to compute the least squares symmetric orthogonal solutions, and meanwhile an example is presented to show that it is reasonable.

1. Introduction

Throughout this paper, the following notations will be used. $\mathbb{R}^{m\times n}$, $O\mathbb{R}^{n\times n}$, $S\mathbb{R}^{n\times n}$, and $AS\mathbb{R}^{n\times n}$ denote the set of all $m \times n$ real matrices, the set of all $n \times n$ orthogonal matrices, the set of all $n \times n$ symmetric matrices, and the set of all $n \times n$ skew-symmetric matrices, respectively. I_n is the identity matrix of order n. $(\cdot)^T$ and $\mathrm{tr}(\cdot)$ represent the transpose and the trace of the real matrix, respectively. $\|\cdot\|$ stands for the Frobenius norm induced by the inner product. The following two definitions will also be used.

Definition 1.1 (see [1]). A real matrix $X \in \mathbb{R}^{n\times n}$ is said to be a symmetric orthogonal matrix if $X^T = X$ and $X^TX = I_n$.

Definition 1.2 (see [2]). A real matrix $X \in \mathbb{R}^{2m\times 2m}$ is called a skew-symmetric orthogonal matrix if $X^T = -X$ and $X^TX = I_n$.

The set of all $n \times n$ symmetric orthogonal matrices and the set of all $2m \times 2m$ skew-symmetric orthogonal matrices are, respectively, denoted by $SO\mathbb{R}^{n\times n}$ and $SSO\mathbb{R}^{2m\times 2m}$. Since

the linear matrix equation(s) and its optimal approximation problem have great applications in structural design, biology, control theory, and linear optimal control, and so forth, see, for example, [3–5], there has been much attention paid to the linear matrix equation(s). The well-known system of matrix equations

$$AX = B, \qquad XC = D, \tag{1.1}$$

as one kind of linear matrix equations, has been investigated by many authors, and a series of important and useful results have been obtained. For instance, the system (1.1) with unknown matrix X being bisymmetric, centrosymmetric, bisymmetric nonnegative definite, Hermitian and nonnegative definite, and (P,Q)-(skew) symmetric has been, respectively, investigated by Wang et al. [6, 7], Khatri and Mitra [8], and Zhang and Wang [9]. Of course, if the solvability conditions of system (1.1) are not satisfied, we may consider its least squares solution. For example, Li et al. [10] presented the least squares mirrorsymmetric solution. Yuan [11] got the least-squares solution. Some results concerning the system (1.1) can also be found in [12–18].

Symmetric orthogonal matrices and skew-symmetric orthogonal matrices play important roles in numerical analysis and numerical solutions of partial differential equations. Papers [1, 2], respectively, derived the symmetric orthogonal solution X of the matrix equation $XC = D$ and the skew-symmetric orthogonal solution X of the matrix equation $AX = B$. Motivated by the work mentioned above, we in this paper will, respectively, study the orthogonal solutions, symmetric orthogonal solutions, and skew-symmetric orthogonal solutions of the system (1.1). Furthermore, if the solvability conditions are not satisfied, the least squares skew-symmetric orthogonal solutions and the least squares symmetric orthogonal solutions of the system (1.1) will be also given.

The remainder of this paper is arranged as follows. In Section 2, some lemmas are provided to give the main results of this paper. In Sections 3, 4, and 5, the necessary and sufficient conditions of and the expression for the orthogonal, the symmetric orthogonal, and the skew-symmetric orthogonal solutions of the system (1.1) are, respectively, obtained. In Section 6, the least squares skew-symmetric orthogonal solutions and the least squares symmetric orthogonal solutions of the system (1.1) are presented, respectively. In addition, an algorithm is provided to compute the least squares symmetric orthogonal solutions, and meanwhile an example is presented to show that it is reasonable. Finally, in Section 7, some concluding remarks are given.

2. Preliminaries

In this section, we will recall some lemmas and the special C-S decomposition which will be used to get the main results of this paper.

Lemma 2.1 (see [1, Lemmas 1 and 2]). *Given $C \in \mathbb{R}^{2m\times n}$, $D \in \mathbb{R}^{2m\times n}$. The matrix equation $YC = D$ has a solution $Y \in O\mathbb{R}^{2m\times 2m}$ if and only if $D^TD = C^TC$. Let the singular value decompositions of C and D be, respectively,*

$$C = U\begin{pmatrix}\Pi & 0\\ 0 & 0\end{pmatrix}V^T, \qquad D = W\begin{pmatrix}\Pi & 0\\ 0 & 0\end{pmatrix}V^T, \tag{2.1}$$

where

$$\Pi = \operatorname{diag}(\sigma_1, \ldots, \sigma_k) > 0, \quad k = \operatorname{rank}(C) = \operatorname{rank}(D),$$
$$U = (U_1 \;\; U_2) \in O\mathbb{R}^{2m\times 2m}, \quad U_1 \in \mathbb{R}^{2m\times k}, \tag{2.2}$$
$$W = (W_1 \;\; W_2) \in O\mathbb{R}^{2m\times 2m}, \quad W_1 \in \mathbb{R}^{2m\times k}, \qquad V = (V_1 \;\; V_2) \in O\mathbb{R}^{n\times n}, \quad V_1 \in \mathbb{R}^{n\times k}.$$

Then the orthogonal solutions of $YC = D$ can be described as

$$Y = W\begin{pmatrix} I_k & 0 \\ 0 & P \end{pmatrix} U^T, \tag{2.3}$$

where $P \in O\mathbb{R}^{(2m-k)\times(2m-k)}$ is arbitrary.

Lemma 2.2 (see [2, Lemmas 1 and 2]). *Given $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{n\times m}$. The matrix equation $AX = B$ has a solution $X \in O\mathbb{R}^{m\times m}$ if and only if $AA^T = BB^T$. Let the singular value decompositions of A and B be, respectively,*

$$A = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^T, \qquad B = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} Q^T, \tag{2.4}$$

where

$$\Sigma = \operatorname{diag}(\delta_1, \ldots, \delta_l) > 0, \quad l = \operatorname{rank}(A) = \operatorname{rank}(B), \qquad U = (U_1 \;\; U_2) \in O\mathbb{R}^{n\times n}, \quad U_1 \in \mathbb{R}^{n\times l},$$
$$V = (V_1 \;\; V_2) \in O\mathbb{R}^{m\times m}, \quad V_1 \in \mathbb{R}^{m\times l}, \qquad Q = (Q_1 \;\; Q_2) \in O\mathbb{R}^{m\times m}, \quad Q_1 \in \mathbb{R}^{m\times l}. \tag{2.5}$$

Then the orthogonal solutions of $AX = B$ can be described as

$$X = V\begin{pmatrix} I_l & 0 \\ 0 & W \end{pmatrix} Q^T, \tag{2.6}$$

where $W \in O\mathbb{R}^{(m-l)\times(m-l)}$ is arbitrary.

Lemma 2.3 (see [2, Theorem 1]). *If*

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in O\mathbb{R}^{2m\times 2m}, \quad X_{11} \in AS\mathbb{R}^{k\times k}, \tag{2.7}$$

then the C-S decomposition of X can be expressed as

$$\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}^T \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & R_2 \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \tag{2.8}$$

where $D_1 \in O\mathbb{R}^{k\times k}$, $D_2, R_2 \in O\mathbb{R}^{(2m-k)\times(2m-k)}$,

$$\Sigma_{11} = \begin{pmatrix} \tilde{I} & 0 & 0 \\ 0 & \tilde{C} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Sigma_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & I \end{pmatrix},$$

$$\Sigma_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad \Sigma_{22} = \begin{pmatrix} I & 0 & 0 \\ 0 & -\tilde{C}^T & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{I} = \operatorname{diag}\left(\tilde{I}_1, \ldots, \tilde{I}_{r_1}\right), \quad \tilde{I}_1 = \cdots = \tilde{I}_{r_1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tilde{C} = \operatorname{diag}(C_1, \ldots, C_{l_1}), \tag{2.9}$$

$$C_i = \begin{pmatrix} 0 & c_i \\ -c_i & 0 \end{pmatrix}, \quad i = 1, \ldots, l_1; \qquad S = \operatorname{diag}(S_1, \ldots, S_{l_1}),$$

$$S_i = \begin{pmatrix} s_i & 0 \\ 0 & s_i \end{pmatrix}, \quad S_i^T S_i + C_i^T C_i = I_2, \quad i = 1, \ldots, l_1;$$

$$2r_1 + 2l_1 = \operatorname{rank}(X_{11}), \qquad \Sigma_{21} = \Sigma_{12}^T, \qquad k - 2r_1 = \operatorname{rank}(X_{21}).$$

Lemma 2.4 (see [1, Theorem 1]). *If*

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \in O\mathbb{R}^{m\times m}, \quad K_{11} \in S\mathbb{R}^{l\times l}, \tag{2.10}$$

then the C-S decomposition of K can be described as

$$\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}^T \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & R_2 \end{pmatrix} = \begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{pmatrix}, \tag{2.11}$$

where $D_1 \in O\mathbb{R}^{l\times l}$, D_2, $R_2 \in O\mathbb{R}^{(m-l)\times(m-l)}$,

$$\Pi_{11} = \begin{pmatrix} \tilde{I} & 0 & 0 \\ 0 & \tilde{C} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Pi_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & I \end{pmatrix},$$

$$\Pi_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad \Pi_{22} = \begin{pmatrix} I & 0 & 0 \\ 0 & -\tilde{C} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.12}$$

$$\tilde{I} = \operatorname{diag}(i_1, \ldots, i_r), \qquad i_j = \pm 1, \quad j = 1, \ldots, r;$$

$$S = \operatorname{diag}(s_{1'}, \ldots, s_{l'}), \qquad \tilde{C} = \operatorname{diag}(c_{1'}, \ldots, c_{l'}),$$

$$s_{i'}^2 = \sqrt{1 - c_{i'}^2}, \quad i = 1', \ldots, l'; \qquad r + l' = \operatorname{rank}(K_{11}), \qquad \Pi_{21} = \Pi_{12}^T.$$

Remarks 2.5. In order to know the C-S decomposition of an orthogonal matrix with a $k \times k$ leading (skew-) symmetric submatrix for details, one can deeply study the proof of Theorem 1 in [1] and [2].

Lemma 2.6. *Given $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{n\times m}$. Then the matrix equation $AX = B$ has a solution $X \in SO\mathbb{R}^{m\times m}$ if and only if $AA^T = BB^T$ and $AB^T = BA^T$. When these conditions are satisfied, the general symmetric orthogonal solutions can be expressed as*

$$X = \tilde{V}\begin{pmatrix} I_{2l-r} & 0 \\ 0 & G \end{pmatrix}\tilde{Q}^T, \tag{2.13}$$

where

$$\tilde{Q} = JQ\,\mathrm{diag}(I, D_2) \in O\mathbb{R}^{m\times m}, \qquad \tilde{V} = JV\,\mathrm{diag}(I, R_2) \in O\mathbb{R}^{m\times m},$$
$$J = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}, \tag{2.14}$$

and $G \in SO\mathbb{R}^{(m-2l+r)\times(m-2l+r)}$ is arbitrary.

Proof. The Necessity. Assume $X \in SO\mathbb{R}^{m\times m}$ is a solution of the matrix equation $AX = B$, then we have

$$BB^T = AXX^TA^T = AA^T,$$
$$BA^T = AXA^T = AX^TA^T = AB^T. \tag{2.15}$$

The Sufficiency. Since the equality $AA^T = BB^T$ holds, then by Lemma 2.2, the singular value decompositions of A and B can be, respectively, expressed as (2.4). Moreover, the condition $AB^T = BA^T$ means

$$U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}V^TQ\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}U^T = U\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}Q^TV\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}U^T, \tag{2.16}$$

which can be written as

$$V_1^TQ_1 = Q_1^TV_1. \tag{2.17}$$

From Lemma 2.2, the orthogonal solutions of the matrix equation $AX = B$ can be described as (2.6). Now we aim to find that X in (2.6) is also symmetric. Suppose that X is symmetric, then we have

$$\begin{pmatrix} I_l & 0 \\ 0 & W^T \end{pmatrix}V^TQ = Q^TV\begin{pmatrix} I_l & 0 \\ 0 & W \end{pmatrix}, \tag{2.18}$$

together with the partitions of the matrices Q and V in Lemma 2.2, we get

$$V_1^T Q_1 = Q_1^T V_1, \tag{2.19}$$

$$Q_1^T V_2 W = V_1^T Q_2, \tag{2.20}$$

$$Q_2^T V_2 W = W^T V_2^T Q_2. \tag{2.21}$$

By (2.17), we can get (2.19). Now we aim to find the orthogonal solutions of the system of matrix equations (2.20) and (2.21). Firstly, we obtain from (2.20) that $Q_1^T V_2 (Q_1^T V_2)^T = V_1^T Q_2 (V_1^T Q_2)^T$, then by Lemma 2.2, (2.20) has an orthogonal solution W. By (2.17), the $l \times l$ leading principal submatrix of the orthogonal matrix $V^T Q$ is symmetric. Then we have, from Lemma 2.4,

$$V_1^T Q_2 = D_1 \Pi_{12} R_2^T, \tag{2.22}$$

$$Q_1^T V_2 = D_1 \Pi_{12} D_2^T, \tag{2.23}$$

$$V_2^T Q_2 = D_2 \Pi_{22} R_2^T. \tag{2.24}$$

From (2.20), (2.22), and (2.23), the orthogonal solution W of (2.20) is

$$W = D_2 \begin{pmatrix} G & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} R_2^T, \tag{2.25}$$

where $G \in O\mathbb{R}^{(m-2l+r)\times(m-2l+r)}$ is arbitrary. Combining (2.21), (2.24), and (2.25) yields $G^T = G$, that is, G is a symmetric orthogonal matrix. Denote

$$\widehat{V} = V \operatorname{diag}(I, D_2), \qquad \widehat{Q} = Q \operatorname{diag}(I, R_2), \tag{2.26}$$

then the symmetric orthogonal solutions of the matrix equation $AX = B$ can be expressed as

$$X = \widehat{V} \begin{pmatrix} I & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & I \end{pmatrix} \widehat{Q}^T. \tag{2.27}$$

Let the partition matrix J be

$$J = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}, \tag{2.28}$$

compatible with the block matrix

$$\begin{pmatrix} I & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & I \end{pmatrix}. \tag{2.29}$$

Put

$$\widetilde{V} = J\widehat{V}, \qquad \widetilde{Q} = J\widehat{Q}, \tag{2.30}$$

then the symmetric orthogonal solutions of the matrix equation $AX = B$ can be described as (2.13). □

Setting $A = C^T$, $B = D^T$, and $X = Y^T$ in [2, Theorem 2], and then by Lemmas 2.1 and 2.3, we can have the following result.

Lemma 2.7. *Given $C \in \mathbb{R}^{2m\times n}$, $D \in \mathbb{R}^{2m\times n}$. Then the equation has a solution $Y \in SSO\mathbb{R}^{2m\times 2m}$ if and only if $D^TD = C^TC$ and $D^TC = -C^TD$. When these conditions are satisfied, the skew-symmetric orthogonal solutions of the matrix equation $YC = D$ can be described as*

$$Y = \widetilde{W}\begin{pmatrix} I & 0 \\ 0 & H \end{pmatrix}\widetilde{U}^T, \tag{2.31}$$

where

$$\widetilde{W} = J'W\,\mathrm{diag}(I, -D_2) \in O\mathbb{R}^{2m\times 2m}, \qquad \widetilde{U} = J'U\,\mathrm{diag}(I, R_2) \in O\mathbb{R}^{2m\times 2m},$$
$$J' = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}, \tag{2.32}$$

and $H \in SSO\mathbb{R}^{2r\times 2r}$ is arbitrary.

3. The Orthogonal Solutions of the System (1.1)

The following theorems give the orthogonal solutions of the system (1.1).

Theorem 3.1. *Given A, $B \in \mathbb{R}^{n\times m}$ and C, $D \in \mathbb{R}^{m\times n}$, suppose the singular value decompositions of A and B are, respectively, as* (2.4). *Denote*

$$Q^TC = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \qquad V^TD = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \tag{3.1}$$

where $C_1, D_1 \in \mathbb{R}^{l\times n}$, and $C_2, D_2 \in \mathbb{R}^{(m-l)\times n}$. Let the singular value decompositions of C_2 and D_2 be, respectively,

$$C_2 = \widetilde{U}\begin{pmatrix}\Pi & 0\\ 0 & 0\end{pmatrix}\widetilde{V}^T, \qquad D_2 = \widetilde{W}\begin{pmatrix}\Pi & 0\\ 0 & 0\end{pmatrix}\widetilde{V}^T, \tag{3.2}$$

where $\widetilde{U}, \widetilde{W} \in O\mathbb{R}^{(m-l)\times(m-l)}$, $\widetilde{V} \in O\mathbb{R}^{n\times n}$, $\Pi \in \mathbb{R}^{k\times k}$ is diagonal, whose diagonal elements are nonzero singular values of C_2 or D_2. Then the system (1.1) *has orthogonal solutions if and only if*

$$AA^T = BB^T, \qquad C_1 = D_1, \qquad D_2^T D_2 = C_2^T C_2. \tag{3.3}$$

In which case, the orthogonal solutions can be expressed as

$$X = \widehat{V}\begin{pmatrix}I_{k+l} & 0\\ 0 & G'\end{pmatrix}\widehat{Q}^T, \tag{3.4}$$

where

$$\widehat{V} = V\begin{pmatrix}I_l & 0\\ 0 & \widetilde{W}\end{pmatrix} \in O\mathbb{R}^{m\times m}, \qquad \widehat{Q} = Q\begin{pmatrix}I_l & 0\\ 0 & \widetilde{U}\end{pmatrix} \in O\mathbb{R}^{m\times m}, \tag{3.5}$$

and $G' \in O\mathbb{R}^{(m-k-l)\times(m-k-l)}$ is arbitrary.

Proof. Let the singular value decompositions of A and B be, respectively, as (2.4). Since the matrix equation $AX = B$ has orthogonal solutions if and only if

$$AA^T = BB^T, \tag{3.6}$$

then by Lemma 2.2, its orthogonal solutions can be expressed as (2.6). Substituting (2.6) and (3.1) into the matrix equation $XC = D$, we have $C_1 = D_1$ and $WC_2 = D_2$. By Lemma 2.1, the matrix equation $WC_2 = D_2$ has orthogonal solution W if and only if

$$D_2^T D_2 = C_2^T C_2. \tag{3.7}$$

Let the singular value decompositions of C_2 and D_2 be, respectively,

$$C_2 = \widetilde{U}\begin{pmatrix}\Pi & 0\\ 0 & 0\end{pmatrix}\widetilde{V}^T, \qquad D_2 = \widetilde{W}\begin{pmatrix}\Pi & 0\\ 0 & 0\end{pmatrix}\widetilde{V}^T, \tag{3.8}$$

where $\widetilde{U}, \widetilde{W} \in O\mathbb{R}^{(m-l)\times(m-l)}$, $\widetilde{V} \in O\mathbb{R}^{n\times n}$, $\Pi \in \mathbb{R}^{k\times k}$ is diagonal, whose diagonal elements are nonzero singular values of C_2 or D_2. Then the orthogonal solutions can be described as

$$W = \widetilde{W}\begin{pmatrix}I_k & 0\\ 0 & G'\end{pmatrix}\widetilde{U}^T, \tag{3.9}$$

where $G' \in O\mathbb{R}^{(m-k-l)\times(m-k-l)}$ is arbitrary. Therefore, the common orthogonal solutions of the system (1.1) can be expressed as

$$X = V\begin{pmatrix} I_l & 0 \\ 0 & W \end{pmatrix}Q^T = V\begin{pmatrix} I_l & 0 \\ 0 & \widetilde{W} \end{pmatrix}\begin{pmatrix} I_l & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & G' \end{pmatrix}\begin{pmatrix} I_l & 0 \\ 0 & \widetilde{U}^T \end{pmatrix}Q^T = \widehat{V}\begin{pmatrix} I_{k+l} & 0 \\ 0 & G' \end{pmatrix}\widehat{Q}^T, \tag{3.10}$$

where

$$\widehat{V} = V\begin{pmatrix} I_l & 0 \\ 0 & \widetilde{W} \end{pmatrix} \in O\mathbb{R}^{m\times m}, \qquad \widehat{Q} = Q\begin{pmatrix} I_l & 0 \\ 0 & \widetilde{U} \end{pmatrix} \in O\mathbb{R}^{m\times m}, \tag{3.11}$$

and $G' \in O\mathbb{R}^{(m-k-l)\times(m-k-l)}$ is arbitrary. □

The following theorem can be shown similarly.

Theorem 3.2. *Given A, $B \in \mathbb{R}^{n\times m}$ and C, $D \in \mathbb{R}^{m\times n}$, let the singular value decompositions of C and D be, respectively, as* (2.1). *Partition*

$$AW = \begin{pmatrix} A_1 & A_2 \end{pmatrix}, \qquad BU = \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \tag{3.12}$$

where $A_1, B_1 \in \mathbb{R}^{n\times k}$, $A_2, B_2 \in \mathbb{R}^{n\times(m-k)}$. Assume the singular value decompositions of A_2 and B_2 are, respectively,

$$A_2 = \widetilde{U}\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}\widetilde{V}^T, \qquad B_2 = \widetilde{U}\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}\widetilde{Q}^T, \tag{3.13}$$

where $\widetilde{V}$, $\widetilde{Q} \in O\mathbb{R}^{(m-k)\times(m-k)}$, $\widetilde{U} \in O\mathbb{R}^{n\times n}$, $\Sigma \in \mathbb{R}^{l'\times l'}$ is diagonal, whose diagonal elements are nonzero singular values of A_2 or B_2. Then the system (1.1) *has orthogonal solutions if and only if*

$$D^TD = C^TC, \qquad A_1 = B_1, \qquad A_2A_2^T = B_2B_2^T. \tag{3.14}$$

In which case, the orthogonal solutions can be expressed as

$$X = \widehat{W}\begin{pmatrix} I_{k+l'} & 0 \\ 0 & H' \end{pmatrix}\widehat{U}^T, \tag{3.15}$$

where

$$\widehat{W} = W\begin{pmatrix} I_k & 0 \\ 0 & \widetilde{W} \end{pmatrix} \in O\mathbb{R}^{m\times m}, \qquad \widehat{U} = U\begin{pmatrix} I_k & 0 \\ 0 & \widetilde{U} \end{pmatrix} \in O\mathbb{R}^{m\times m}, \tag{3.16}$$

and $H' \in O\mathbb{R}^{(m-k-l')\times(m-k-l')}$ is arbitrary.

4. The Symmetric Orthogonal Solutions of the System (1.1)

We now present the symmetric orthogonal solutions of the system (1.1).

Theorem 4.1. *Given $A, B \in \mathbb{R}^{n\times m}$, $C, D \in \mathbb{R}^{m\times n}$. Let the symmetric orthogonal solutions of the matrix equation $AX = B$ be described as in Lemma 2.6. Partition*

$$\tilde{Q}^T C = \begin{pmatrix} C_1' \\ C_2' \end{pmatrix}, \qquad \tilde{V}^T D = \begin{pmatrix} D_1' \\ D_2' \end{pmatrix}, \tag{4.1}$$

where $C_1', D_1' \in \mathbb{R}^{(2l-r)\times n}$, $C_2', D_2' \in \mathbb{R}^{(m-2l+r)\times n}$. Then the system (1.1) has symmetric orthogonal solutions if and only if

$$AA^T = BB^T, \qquad AB^T = BA^T, \qquad C_1' = D_1', \qquad D_2'^T D_2' = C_2'^T C_2', \qquad D_2'^T C_2' = C_2'^T D_2'. \tag{4.2}$$

In which case, the solutions can be expressed as

$$X = \hat{V} \begin{pmatrix} I & 0 \\ 0 & G'' \end{pmatrix} \hat{Q}^T, \tag{4.3}$$

where

$$\hat{V} = V \begin{pmatrix} I_{2l-r} & 0 \\ 0 & \widetilde{W} \end{pmatrix} \in O\mathbb{R}^{m\times m}, \qquad \hat{Q} = Q \begin{pmatrix} I_{2l-r} & 0 \\ 0 & \tilde{U} \end{pmatrix} \in O\mathbb{R}^{m\times m}, \tag{4.4}$$

and $G'' \in SO\mathbb{R}^{(m-2l+r-2l'+r')\times(m-2l+r-2l'+r')}$ is arbitrary.

Proof. From Lemma 2.6, we obtain that the matrix equation $AX = B$ has symmetric orthogonal solutions if and only if $AA^T = BB^T$ and $AB^T = BA^T$. When these conditions are satisfied, the general symmetric orthogonal solutions can be expressed as

$$X = \tilde{V} \begin{pmatrix} I_{2l-r} & 0 \\ 0 & G \end{pmatrix} \tilde{Q}^T, \tag{4.5}$$

where $G \in SO\mathbb{R}^{(m-2l+r)\times(m-2l+r)}$ is arbitrary, $\tilde{Q} \in O\mathbb{R}^{m\times m}$, $\tilde{V} \in O\mathbb{R}^{m\times m}$. Inserting (4.1) and (4.5) into the matrix equation $XC = D$, we get $C_1' = D_1'$ and $GC_2' = D_2'$. By [1, Theorem 2], the matrix equation $GC_2' = D_2'$ has a symmetric orthogonal solution if and only if

$$D_2'^T D_2' = C_2'^T C_2', \qquad D_2'^T C_2' = C_2'^T D_2'. \tag{4.6}$$

In which case, the solutions can be described as

$$G = \widetilde{W} \begin{pmatrix} I_{2l'-r'} & 0 \\ 0 & G'' \end{pmatrix} \tilde{U}^T, \tag{4.7}$$

where $G'' \in SO\mathbb{R}^{(m-2l+r-2l'+r')\times(m-2l+r-2l'+r')}$ is arbitrary, $\widetilde{W} \in O\mathbb{R}^{(m-2l+r)\times(m-2l+r)}$, and $\widetilde{U} \in O\mathbb{R}^{(m-2l+r)\times(m-2l+r)}$. Hence the system (1.1) has symmetric orthogonal solutions if and only if all equalities in (4.2) hold. In which case, the solutions can be expressed as

$$X = \widetilde{V}\begin{pmatrix} I_{2l-r} & 0 \\ 0 & G \end{pmatrix}\widetilde{Q}^T = \widetilde{V}\begin{pmatrix} I_{2l-r} & 0 \\ 0 & \widetilde{W} \end{pmatrix}\begin{pmatrix} I_{2l-r} & 0 \\ 0 & \begin{pmatrix} I & 0 \\ 0 & G'' \end{pmatrix} \end{pmatrix}\begin{pmatrix} I_{2l-r} & 0 \\ 0 & \widetilde{U}^T \end{pmatrix}\widetilde{Q}^T, \tag{4.8}$$

that is, the expression in (4.3). □

The following theorem can also be obtained by the method used in the proof of Theorem 4.1.

Theorem 4.2. *Given $A, B \in \mathbb{R}^{n\times m}$, $C, D \in \mathbb{R}^{m\times n}$. Let the symmetric orthogonal solutions of the matrix equation $XC = D$ be described as*

$$X = \widetilde{M}\begin{pmatrix} I_{2k-r} & 0 \\ 0 & G \end{pmatrix}\widetilde{N}^T, \tag{4.9}$$

where $\widetilde{M}$, $\widetilde{N} \in O\mathbb{R}^{m\times m}$, $G \in SO\mathbb{R}^{(m-2k+r)\times(m-2k+r)}$. Partition

$$A\widetilde{M} = \begin{pmatrix} M_1 & M_2 \end{pmatrix}, \quad B\widetilde{N} = \begin{pmatrix} N_1 & N_2 \end{pmatrix}, \quad M_1, N_1 \in \mathbb{R}^{n\times(2k-r)},\ M_2, N_2 \in \mathbb{R}^{n\times(m-2k+r)}. \tag{4.10}$$

Then the system (1.1) *has symmetric orthogonal solutions if and only if*

$$D^TD = C^TC, \qquad D^TC = C^TD, \qquad M_1 = N_1, \qquad M_2M_2^T = N_2N_2^T, \qquad M_2N_2^T = N_2M_2^T. \tag{4.11}$$

In which case, the solutions can be expressed as

$$X = \widehat{M}\begin{pmatrix} I & 0 \\ 0 & H'' \end{pmatrix}\widehat{N}^T, \tag{4.12}$$

where

$$\widehat{M} = \widetilde{M}\begin{pmatrix} I_{2k-r} & 0 \\ 0 & \widetilde{W}_1 \end{pmatrix} \in O\mathbb{R}^{m\times m}, \qquad \widehat{N} = \widetilde{N}\begin{pmatrix} I_{2k-r} & 0 \\ 0 & \widetilde{U}_1 \end{pmatrix} \in O\mathbb{R}^{m\times m}, \tag{4.13}$$

and $H'' \in SO\mathbb{R}^{(m-2k+r-2k'+r')\times(m-2k+r-2k'+r')}$ is arbitrary.

5. The Skew-Symmetric Orthogonal Solutions of the System (1.1)

In this section, we show the skew-symmetric orthogonal solutions of the system (1.1).

Theorem 5.1. *Given $A, B \in \mathbb{R}^{n\times 2m}$, $C, D \in \mathbb{R}^{2m\times n}$. Suppose the matrix equation $AX = B$ has skew-symmetric orthogonal solutions with the form*

$$X = \widetilde{V}_1 \begin{pmatrix} I & 0 \\ 0 & H \end{pmatrix} \widetilde{Q}_1^T, \tag{5.1}$$

where $H \in SSO\mathbb{R}^{2r\times 2r}$ is arbitrary, $\widetilde{V}_1$, $\widetilde{Q}_1 \in O\mathbb{R}^{2m\times 2m}$. Partition

$$\widetilde{Q}_1^T C = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \qquad \widetilde{V}_1^T D = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \tag{5.2}$$

where $Q_1, V_1 \in \mathbb{R}^{(2m-2r)\times n}$, $Q_2, V_2 \in \mathbb{R}^{2r\times n}$. Then the system (1.1) has skew-symmetric orthogonal solutions if and only if

$$AA^T = BB^T, \qquad AB^T = -BA^T, \qquad Q_1 = V_1, \qquad Q_2^T Q_2 = V_2^T V_2, \qquad Q_2^T V_2 = -V_2^T Q_2. \tag{5.3}$$

In which case, the solutions can be expressed as

$$X = \widehat{V}_1 \begin{pmatrix} I & 0 \\ 0 & J' \end{pmatrix} \widehat{Q}_1^T, \tag{5.4}$$

where

$$\widehat{V} = \widetilde{V}_1 \begin{pmatrix} I_{2m-2r} & 0 \\ 0 & \widetilde{W} \end{pmatrix} \in O\mathbb{R}^{2m\times 2m}, \qquad \widehat{Q} = \widetilde{Q}_1 \begin{pmatrix} I_{2m-2r} & 0 \\ 0 & \widetilde{U} \end{pmatrix} \in O\mathbb{R}^{2m\times 2m}, \tag{5.5}$$

and $J' \in SSO\mathbb{R}^{2k'\times 2k'}$ is arbitrary.

Proof. By [2, Theorem 2], the matrix equation $AX = B$ has the skew-symmetric orthogonal solutions if and only if $AA^T = BB^T$ and $AB^T = -BA^T$. When these conditions are satisfied, the general skew-symmetric orthogonal solutions can be expressed as (5.1). Substituting (5.1) and (5.2) into the matrix equation $XC = D$, we get $Q_1 = V_1$ and $HQ_2 = V_2$. From Lemma 2.7, equation $HQ_2 = V_2$ has a skew-symmetric orthogonal solution H if and only if

$$Q_2^T Q_2 = V_2^T V_2, \qquad Q_2^T V_2 = -V_2^T Q_2. \tag{5.6}$$

When these conditions are satisfied, the solution can be described as

$$H = \widetilde{W} \begin{pmatrix} I & 0 \\ 0 & J' \end{pmatrix} \widetilde{U}^T, \tag{5.7}$$

where $J' \in SSO\mathbb{R}^{2k'\times 2k'}$ is arbitrary, $\widetilde{W}, \widetilde{U} \in O\mathbb{R}^{2r\times 2r}$. Inserting (5.7) into (5.1) yields that the system (1.1) has skew-symmetric orthogonal solutions if and only if all equalities in (5.3) hold. In which case, the solutions can be expressed as (5.4). □

Similarly, the following theorem holds.

Theorem 5.2. *Given $A, B \in \mathbb{R}^{n\times 2m}$, $C, D \in \mathbb{R}^{2m\times n}$. Suppose the matrix equation $XC = D$ has skew-symmetric orthogonal solutions with the form*

$$X = \widetilde{W}\begin{pmatrix} I & 0 \\ 0 & K \end{pmatrix}\widetilde{U}^T, \tag{5.8}$$

where $K \in SSO\mathbb{R}^{2p\times 2p}$ is arbitrary, $\widetilde{W}, \widetilde{U} \in O\mathbb{R}^{2m\times 2m}$. Partition

$$A\widetilde{W} = (W_1 \;\; W_2), \qquad B\widetilde{U} = (U_1 \;\; U_2), \tag{5.9}$$

where $W_1, U_1 \in \mathbb{R}^{n\times(2m-2p)}$, $W_2, U_2 \in \mathbb{R}^{n\times 2p}$. Then the system (1.1) *has skew-symmetric orthogonal solutions if and only if*

$$D^T D = C^T C, \qquad D^T C = -C^T D, \qquad W_1 = U_1, \qquad W_2 W_2^T = U_2 U_2^T, \qquad U_2 W_2^T = -W_2 U_2^T. \tag{5.10}$$

In which case, the solutions can be expressed as

$$X = \widehat{W}_1\begin{pmatrix} I & 0 \\ 0 & J'' \end{pmatrix}\widehat{U}_1^T, \tag{5.11}$$

where

$$\widehat{W}_1 = \widetilde{W}\begin{pmatrix} I_{2m-2p} & 0 \\ 0 & \widetilde{W}_1 \end{pmatrix} \in O\mathbb{R}^{2m\times 2m}, \qquad \widehat{U}_1 = \begin{pmatrix} I_{2m-2p} & 0 \\ 0 & \widetilde{U}_1^T \end{pmatrix}\widetilde{U} \in O\mathbb{R}^{2m\times 2m}, \tag{5.12}$$

and $J'' \in SSO\mathbb{R}^{2q\times 2q}$ is arbitrary.

6. The Least Squares (Skew-) Symmetric Orthogonal Solutions of the System (1.1)

If the solvability conditions of a system of matrix equations are not satisfied, it is natural to consider its least squares solution. In this section, we get the least squares (skew-) symmetric orthogonal solutions of the system (1.1), that is, seek $X \in SSO\mathbb{R}^{2m\times 2m}(SO\mathbb{R}^{n\times n})$ such that

$$\min_{X\in SSO\mathbb{R}^{2m\times 2m}(SO\mathbb{R}^{n\times n})} \|AX - B\|^2 + \|XC - D\|^2. \tag{6.1}$$

With the help of the definition of the Frobenius norm and the properties of the skew-symmetric orthogonal matrix, we get that

$$\|AX - B\|^2 + \|XC - D\|^2 = \|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2 - 2\,\mathrm{tr}\left(X^T\left(A^TB + DC^T\right)\right). \tag{6.2}$$

Let

$$\begin{gathered} A^TB + DC^T = K, \\ \frac{K^T - K}{2} = T. \end{gathered} \tag{6.3}$$

Then, it follows from the skew-symmetric matrix X that

$$\mathrm{tr}\left(X^T\left(A^TB + DC^T\right)\right) = \mathrm{tr}\left(X^TK\right) = \mathrm{tr}(-XK) = \mathrm{tr}\left(X\left(\frac{K^T - K}{2}\right)\right) = \mathrm{tr}(XT). \tag{6.4}$$

Therefore, (6.1) holds if and only if (6.4) reaches its maximum. Now, we pay our attention to find the maximum value of (6.4). Assume the eigenvalue decomposition of T is

$$T = E\begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}E^T \tag{6.5}$$

with

$$\Lambda = \mathrm{diag}(\Lambda_1, \ldots, \Lambda_l), \qquad \Lambda_i = \begin{pmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{pmatrix}, \quad \alpha_i > 0,\ i = 1, \ldots, l; \qquad 2l = \mathrm{rank}(T). \tag{6.6}$$

Denote

$$E^TXE = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \tag{6.7}$$

partitioned according to

$$\begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}, \tag{6.8}$$

then (6.4) has the following form:

$$\mathrm{tr}(XT) = \mathrm{tr}\left(E^TXE\begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}\right) = \mathrm{tr}(X_{11}\Lambda). \tag{6.9}$$

Thus, by

$$X_{11} = \widetilde{I} = \operatorname{diag}\left(\widetilde{I}_1, \ldots, \widetilde{I}_l\right), \tag{6.10}$$

where

$$\widetilde{I}_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad i = 1, \ldots, l. \tag{6.11}$$

Equation (6.9) gets its maximum. Since $E^T X E$ is skew-symmetric, it follows from

$$X = E \begin{pmatrix} \widetilde{I} & 0 \\ 0 & G \end{pmatrix} E^T, \tag{6.12}$$

where $G \in SSO\mathbb{R}^{(2m-2l)\times(2m-2l)}$ is arbitrary, that (6.1) obtains its minimum. Hence we have the following theorem.

Theorem 6.1. *Given $A, B \in \mathbb{R}^{n\times 2m}$ and $C, D \in \mathbb{R}^{2m\times n}$, denote*

$$A^T B + DC^T = K, \qquad \frac{K^T - K}{2} = T, \tag{6.13}$$

and let the spectral decomposition of T be (6.5). *Then the least squares skew-symmetric orthogonal solutions of the system* (1.1) *can be expressed as* (6.12).

If X in (6.1) is a symmetric orthogonal matrix, then by the definition of the Frobenius norm and the properties of the symmetric orthogonal matrix, (6.2) holds. Let

$$A^T B + DC^T = H, \qquad \frac{H^T + H}{2} = N. \tag{6.14}$$

Then we get that

$$\min_{X\in SO\mathbb{R}^{n\times n}} \|AX - B\|^2 + \|XC - D\|^2 = \min_{X\in SO\mathbb{R}^{n\times n}} \left[\|A\|^2 + \|B\|^2 + \|C\|^2 + \|D\|^2 - 2\operatorname{tr}(XN)\right]. \tag{6.15}$$

Thus (6.15) reaches its minimum if and only if $\operatorname{tr}(XN)$ obtains its maximum. Now, we focus on finding the maximum value of $\operatorname{tr}(XN)$. Let the spectral decomposition of the symmetric matrix N be

$$N = M \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} M^T, \tag{6.16}$$

where

$$\begin{aligned}
&\Sigma = \begin{pmatrix} \Sigma^{+} & 0 \\ 0 & \Sigma^{-} \end{pmatrix}, \quad \Sigma^{+} = \operatorname{diag}(\lambda_1, \ldots, \lambda_s), \\
&\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0, \quad \Sigma^{-} = \operatorname{diag}(\lambda_{s+1}, \ldots, \lambda_t), \\
&\lambda_t \leq \lambda_{t-1} \leq \cdots \leq \lambda_{s+1} < 0, \quad t = \operatorname{rank}(N).
\end{aligned} \tag{6.17}$$

Denote

$$M^T X M = \begin{pmatrix} \overline{X_{11}} & \overline{X_{12}} \\ \overline{X_{21}} & \overline{X_{22}} \end{pmatrix} \tag{6.18}$$

being compatible with

$$\begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.19}$$

Then

$$\operatorname{tr}(XN) = \operatorname{tr}\left(M^T X M \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \right) = \operatorname{tr}\left(\begin{pmatrix} \overline{X_{11}} & \overline{X_{12}} \\ \overline{X_{21}} & \overline{X_{22}} \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \right) = \operatorname{tr}\left(\overline{X_{11}} \Sigma \right). \tag{6.20}$$

Therefore, it follows from

$$\overline{X_{11}} = \widehat{I} = \begin{pmatrix} I_s & 0 \\ 0 & -I_{t-s} \end{pmatrix} \tag{6.21}$$

that (6.20) reaches its maximum. Since $M^T X M$ is a symmetric orthogonal matrix, then when X has the form

$$X = M \begin{pmatrix} \widehat{I} & 0 \\ 0 & L \end{pmatrix} M^T, \tag{6.22}$$

where $L \in SO\mathbb{R}^{(n-t)\times(n-t)}$ is arbitrary, (6.15) gets its minimum. Thus we obtain the following theorem.

Theorem 6.2. *Given $A, B \in \mathbb{R}^{n\times m}$, $C, D \in \mathbb{R}^{m\times n}$, denote*

$$A^T B + D C^T = H, \qquad \frac{H^T + H}{2} = N, \tag{6.23}$$

and let the eigenvalue decomposition of N be (6.16). *Then the least squares symmetric orthogonal solutions of the system* (1.1) *can be described as* (6.22).

Algorithm 6.3. Consider the following.

Step 1. Input $A, B \in \mathbb{R}^{n\times m}$ and $C, D \in \mathbb{R}^{m\times n}$.

Step 2. Compute

$$A^T B + DC^T = H,$$
$$N = \frac{H^T + H}{2}. \tag{6.24}$$

Step 3. Compute the spectral decomposition of N with the form (6.16).

Step 4. Compute the least squares symmetric orthogonal solutions of (1.1) according to (6.22).

Example 6.4. Assume

$$A = \begin{pmatrix} 12.2 & 8.4 & -5.6 & 6.3 & 9.4 & 10.7 \\ 11.8 & 2.9 & 8.5 & 6.9 & 9.6 & -7.8 \\ 10.6 & 2.3 & 11.5 & 7.8 & 6.7 & 8.9 \\ 3.6 & 7.8 & 4.9 & 11.9 & 9.4 & 5.9 \\ 4.5 & 6.7 & 7.8 & 3.1 & 5.6 & 11.6 \end{pmatrix}, \quad B = \begin{pmatrix} 8.5 & 9.4 & 3.6 & 7.8 & 6.3 & 4.7 \\ 2.7 & 3.6 & 7.9 & 9.4 & 5.6 & 7.8 \\ 3.7 & 6.7 & 8.6 & 9.8 & 3.4 & 2.9 \\ -4.3 & 6.2 & 5.7 & 7.4 & 5.4 & 9.5 \\ 2.9 & 3.9 & -5.2 & 6.3 & 7.8 & 4.6 \end{pmatrix},$$
$$C = \begin{pmatrix} 5.4 & 6.4 & 3.7 & 5.6 & 9.7 \\ 3.6 & 4.2 & 7.8 & -6.3 & 7.8 \\ 6.7 & 3.5 & -4.6 & 2.9 & 2.8 \\ -2.7 & 7.2 & 10.8 & 3.7 & 3.8 \\ 1.9 & 3.9 & 8.2 & 5.6 & 11.2 \\ 8.9 & 7.8 & 9.4 & 7.9 & 5.6 \end{pmatrix}, \quad D = \begin{pmatrix} 7.9 & 9.5 & 5.4 & 2.8 & 8.6 \\ 8.7 & 2.6 & 6.7 & 8.4 & 8.1 \\ 5.7 & 3.9 & -2.9 & 5.2 & 1.9 \\ 4.8 & 5.8 & 1.8 & -7.2 & 5.8 \\ 2.8 & 7.9 & 4.5 & 6.7 & 9.6 \\ 9.5 & 4.1 & 3.4 & 9.8 & 3.9 \end{pmatrix}. \tag{6.25}$$

It can be verified that the given matrices A, B, C, and D do not satisfy the solvability conditions in Theorem 4.1 or Theorem 4.2. So we intend to derive the least squares symmetric orthogonal solutions of the system (1.1). By Algorithm 6.3, we have the following results:

(1) the least squares symmetric orthogonal solution

$$X = \begin{pmatrix} 0.01200 & 0.06621 & -0.24978 & 0.27047 & 0.91302 & -0.16218 \\ 0.06621 & 0.65601 & 0.22702 & -0.55769 & 0.24587 & 0.37715 \\ -0.24978 & 0.22702 & 0.80661 & 0.40254 & 0.04066 & -0.26784 \\ 0.27047 & -0.55769 & 0.40254 & 0.06853 & 0.23799 & 0.62645 \\ 0.91302 & 0.24587 & 0.04066 & 0.23799 & -0.12142 & -0.18135 \\ -0.16218 & 0.37715 & -0.26784 & 0.62645 & -0.181354 & 0.57825 \end{pmatrix}, \tag{6.26}$$

(2)

$$\min_{X\in SO\mathbb{R}^{m\times m}} \|AX - B\|^2 + \|XC - D\|^2 = 1.98366,$$
$$\left\|X^T X - I_6\right\| = 2.84882 \times 10^{-15}, \qquad \left\|X^T - X\right\| = 0.00000. \tag{6.27}$$

Remark 6.5. (1) There exists a unique symmetric orthogonal solution such that (6.1) holds if and only if the matrix

$$N = \frac{H^T + H}{2}, \tag{6.28}$$

where

$$H = A^T B + DC^T, \tag{6.29}$$

is invertible. Example 6.4 just illustrates it.

(2) The algorithm about computing the least squares skew-symmetric orthogonal solutions of the system (1.1) can be shown similarly; we omit it here.

7. Conclusions

This paper is devoted to giving the solvability conditions of and the expressions of the orthogonal solutions, the symmetric orthogonal solutions, and the skew-symmetric orthogonal solutions to the system (1.1), respectively, and meanwhile obtaining the least squares symmetric orthogonal and skew-symmetric orthogonal solutions of the system (1.1). In addition, an algorithm and an example have been provided to compute its least squares symmetric orthogonal solutions.

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