

## Research Article

# Positive Solutions to a Generalized Second-Order Difference Equation with Summation Boundary Value Problem

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By using Krasnoselskii's fixed point theorem, we study the existence of positive solutions to the three-point summation boundary value problem  $\Delta^2 u(t-1) + a(t)f(u(t)) = 0$ ,  $t \in \{1, 2, \dots, T\}$ ,  $u(0) = \beta \sum_{s=1}^{\eta} u(s)$ ,  $u(T+1) = \alpha \sum_{s=1}^{\eta} u(s)$ , where  $f$  is continuous,  $T \geq 3$  is a fixed positive integer,  $\eta \in \{1, 2, \dots, T-1\}$ ,  $0 < \alpha < (2T+2)/\eta(\eta+1)$ ,  $0 < \beta < (2T+2 - \alpha\eta(\eta+1))/\eta(2T-\eta+1)$ , and  $\Delta u(t-1) = u(t) - u(t-1)$ . We show the existence of at least one positive solution if  $f$  is either superlinear or sublinear.

## 1. Introduction

The study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential and difference equations was initiated by Ilin and Moiseev [1]. Then Gupta [2] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by many authors; one may see the text books [3, 4] and the papers [5–10]. However, all these papers are concerned with problems with three-point boundary condition restrictions on the difference of the solutions and the solutions themselves, for example,

$$\begin{aligned}u(0) &= 0, & u(T+1) &= 0, \\u(0) &= 0, & au(s) &= u(T+1), \\u(0) &= 0, & u(T+1) - au(s) &= b,\end{aligned}$$

$$\begin{aligned} u(0) - \alpha \Delta u(0) &= 0, & u(T+1) &= \beta u(s), \\ u(0) - \alpha \Delta u(0) &= 0, & \Delta u(T+1) &= 0, \end{aligned} \tag{1.1}$$

and so forth.

In [5], Leggett-Williams developed a fixed point theorem to prove the existence of three positive solutions for Hammerstein integral equations. Since then, this theorem has been reported to be a successful technique for dealing with the existence of three solutions for the two-point boundary value problems of differential and difference equations; see [6, 7]. In [8], X. Lin and W. Liu, using the properties of the associate Green's function and Leggett-Williams fixed point theorem, studied the existence of positive solutions of the problem.

In [9], Zhang and Medina studied the existence of positive solutions for second-order boundary value problems of difference equations by applying Krasnoselskii's fixed point theorem. In [10], Henderson and Thompson used lower and upper solution methods to study the existence of multiple solutions for second-order discrete boundary value problems.

We are interested in the existence of positive solutions of the following second-order difference equation with three-point summation boundary value problem (BVP):

$$\begin{aligned} \Delta^2 u(t-1) + a(t)f(u(t)) &= 0, \quad t \in \{1, 2, \dots, T\}, \\ u(0) = \beta \sum_{s=1}^{\eta} u(s), \quad u(T+1) &= \alpha \sum_{s=1}^{\eta} u(s), \end{aligned} \tag{1.2}$$

where  $f$  is continuous,  $T \geq 3$  is a fixed positive integer,  $\eta \in \{1, 2, \dots, T-1\}$ .

The aim of this paper is to give some results for existence of positive solutions to (1.2), assuming that  $0 < \alpha < (2T+2)/\eta(\eta+1)$ ,  $0 < \beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$ , and  $f$  is either superlinear or sublinear. Set

$$f_0 = \lim_{u \rightarrow 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \rightarrow \infty} \frac{f(u)}{u}. \tag{1.3}$$

Then  $f_0 = 0$  and  $f_\infty = \infty$  correspond to the superlinear case, and  $f_0 = \infty$  and  $f_\infty = 0$  correspond to the sublinear case. Let  $\mathbb{N}$  be the nonnegative integer; we let  $\mathbb{N}_{i,j} = \{k \in \mathbb{N} \mid i \leq k \leq j\}$  and  $\mathbb{N}_p = \mathbb{N}_{0,p}$ . By the positive solution of (1.2), we mean that a function  $u(t) : \mathbb{N}_{T+1} \rightarrow [0, \infty)$  and satisfies the problem (1.2).

Recently, Sitthiwiratham [11] proved the existence of positive solutions for the boundary value problem with summation condition

$$\begin{aligned} \Delta^2 u(t-1) + a(t)f(u(t)) &= 0, \quad t \in \{1, 2, \dots, T\}, \\ u(0) &= 0, \quad u(T+1) = \alpha \sum_{s=1}^{\eta} u(s), \end{aligned} \tag{1.4}$$

where  $f$  is continuous,  $T \geq 3$  is a fixed positive integer,  $\eta \in \{1, 2, \dots, T-1\}$ , and  $0 < \alpha < 2T+2/\eta(\eta+1)$ .

Throughout this paper, we suppose the following conditions hold:

(A1)  $f \in C([0, \infty), [0, \infty))$ ;

(A2)  $a \in C(\mathbb{N}_{T+1}, [0, \infty))$  and there exists  $t_0 \in \mathbb{N}_{\eta, T+1}$  such that  $a(t_0) > 0$ .

The proof of the main theorem is based upon an application of the following Krasnoselskii's fixed point theorem in a cone.

**Theorem 1.1** (see [12]). *Let  $E$  be a Banach space, and let  $K \subset E$  be a cone. Assume  $\Omega_1, \Omega_2$  are open subsets of  $E$  with  $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$ , and let*

$$A : K \cap (\overline{\Omega_2} \setminus \Omega_1) \longrightarrow K \quad (1.5)$$

be a completely continuous operator such that

- (i)  $\|Au\| \leq \|u\|, u \in K \cap \partial\Omega_1$ , and  $\|Au\| \geq \|u\|, u \in K \cap \partial\Omega_2$ , or
- (ii)  $\|Au\| \geq \|u\|, u \in K \cap \partial\Omega_1$ , and  $\|Au\| \leq \|u\|, u \in K \cap \partial\Omega_2$ .

Then  $A$  has a fixed point in  $K \cap (\overline{\Omega_2} \setminus \Omega_1)$ .

## 2. Preliminaries

We now state and prove several lemmas before stating our main results.

**Lemma 2.1.** *Let  $\beta \neq (2T+2 - \alpha\eta(\eta+1))/\eta(2T-\eta+1)$ . Then, for  $y \in C(\mathbb{N}_{T+1}, [0, \infty))$ , the problem*

$$\Delta^2 u(t-1) + y(t) = 0, \quad t \in \mathbb{N}_{1,T}, \quad (2.1)$$

$$u(0) = \beta \sum_{s=1}^{\eta} u(s), \quad u(T+1) = \alpha \sum_{s=1}^{\eta} u(s), \quad (2.2)$$

has a unique solution

$$\begin{aligned} u(t) = & \frac{\beta\eta(\eta+1) + 2t(1-\beta\eta)}{(2T+2 - \alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)y(s) \\ & - \frac{\beta(T+1) - (\beta-\alpha)t}{(2T+2 - \alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s-1)y(s) \\ & - \sum_{s=1}^{t-1} (t-s)y(s), \quad t \in \mathbb{N}_{T+1}. \end{aligned} \quad (2.3)$$

*Proof.* From (2.1), we get

$$\begin{aligned} \Delta u(t) - \Delta u(t-1) &= -y(t), \\ \Delta u(t-1) - \Delta u(t-2) &= -y(t-1), \\ &\vdots \\ \Delta u(1) - \Delta u(0) &= -y(1). \end{aligned} \quad (2.4)$$

We sum the above equations to obtain

$$\Delta u(t) = \Delta u(0) - \sum_{s=1}^t y(s), \quad t \in \mathbb{N}_T, \quad (2.5)$$

we denote  $\sum_{s=p}^q y(s) = 0$ , if  $p > q$ . Similarly, summing the above equation from  $t = 0$  to  $t = h$ , we get

$$u(h+1) = u(0) + (h+1)\Delta u(0) - \sum_{s=1}^h (h+1-s)y(s), \quad h \in \mathbb{N}_T, \quad (2.6)$$

changing the variable from  $h+1$  to  $t$ , we have

$$u(t) = u(0) + t\Delta u(0) - \sum_{s=1}^{t-1} (t-s)y(s) : A + Bt - \sum_{s=1}^{t-1} (t-s)y(s), \quad t \in \mathbb{N}_{T+1}. \quad (2.7)$$

We sum (2.7) from  $s = 1, 2, \dots, \eta$ ,

$$\begin{aligned} \sum_{s=1}^{\eta} u(s) &= \eta A + \frac{\eta(\eta+1)}{2} B - \sum_{s=1}^{\eta-1} \sum_{l=1}^{\eta-s} (l-s)y(s) \\ &= \eta A + \frac{\eta(\eta+1)}{2} B - \frac{1}{2} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)y(s). \end{aligned} \quad (2.8)$$

By (2.2) from  $u(0) = \beta \sum_{s=1}^{\eta} u(s)$ , we get

$$(1 - \beta\eta)A - \frac{\beta\eta(\eta+1)}{2} B = -\frac{\beta}{2} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)y(s), \quad (2.9)$$

and from  $u(T+1) = \alpha \sum_{s=1}^{\eta} u(s)$ , we obtain

$$(1 - \alpha\eta)A + \left( T + 1 - \frac{\alpha\eta(\eta+1)}{2} \right) B = \sum_{s=1}^T (T-s+1)y(s) - \frac{\alpha}{2} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)y(s). \quad (2.10)$$

Therefore,

$$\begin{aligned}
 A &= \frac{\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)y(s) \\
 &\quad - \frac{\beta(T+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)y(s), \\
 B &= \frac{2(1-\beta\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)y(s) \\
 &\quad + \frac{\beta-\alpha}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)y(s).
 \end{aligned} \tag{2.11}$$

Hence, (2.1)-(2.2) has a unique solution

$$\begin{aligned}
 u(t) &= \frac{\beta\eta(\eta+1)+2t(1-\beta\eta)}{(2t+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)y(s) \\
 &\quad - \frac{\beta(T+1)-(\beta-\alpha)t}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)y(s) \\
 &\quad - \sum_{s=1}^{t-1} (t-s)y(s), \quad t \in \mathbb{N}_{T+1}.
 \end{aligned} \tag{2.12}$$

□

**Lemma 2.2.** Let  $0 < \alpha < (2T+2)/\eta(\eta+1)$ ,  $0 < \beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$ . If  $y \in C(\mathbb{N}_{T+1}, [0, \infty))$  and  $y(t) \geq 0$  for  $t \in \mathbb{N}_{1,T}$ , then the unique solution  $u$  of (2.1)-(2.2) satisfies  $u(t) \geq 0$  for  $t \in \mathbb{N}_{T+1}$ .

*Proof.* From the fact that  $\Delta^2 u(t-1) = u(t+1) - 2u(t) + u(t-1) = -y(t) \leq 0$ , we know that  $u(t) \geq (u(t+1) + u(t-1))/2$ , so  $u(t+1)/(t+1) < u(t)/t$ .

Hence

$$\frac{u(T+1)-u(0)}{T+1} < \frac{u(\eta)-u(0)}{\eta+1}, \quad \eta \in \mathbb{N}_{1,T}, \tag{2.13}$$

since  $u(T) \geq 0$  and  $u(0) \geq 0$  imply that  $u(t) \geq 0$  for  $t \in \mathbb{N}_{T+1}$ .

Moreover, from  $u(i) > (i/\eta)u(\eta) + ((\eta - i)/\eta)u(0)$ , we get

$$\begin{aligned} \sum_{s=1}^{\eta} u(s) &> \left[ \frac{1}{\eta}u(\eta) + \frac{\eta-1}{\eta}u(0) \right] + \left[ \frac{2}{\eta}u(\eta) + \frac{\eta-2}{\eta}u(0) \right] + \cdots + \left[ \frac{\eta}{\eta}u(\eta) + \frac{\eta-\eta}{\eta}u(0) \right] \\ &= \frac{1}{\eta}u(\eta)[1+2+\cdots+\eta] + \frac{1}{\eta}u(0)[(\eta-1)+(\eta-2)+\cdots+0] \\ &= \frac{1}{\eta}u(\eta) \left[ \frac{1}{2}\eta(\eta+1) \right] + \frac{1}{\eta}u(0) \left[ \eta^2 - \frac{1}{2}\eta(\eta+1) \right] \\ &= \frac{1}{2}(\eta+1)u(\eta) + \frac{1}{2}(\eta-1)u(0). \end{aligned} \quad (2.14)$$

Combining (2.14) with (2.2), we can get

$$u(0) > \frac{\beta(\eta+1)}{2-\beta(\eta-1)}u(\eta), \quad (2.15)$$

again combining (2.2), (2.14), and (2.15), we obtain

$$u(T+1) > \frac{\alpha(\eta+1)}{2-\beta(\eta-1)}u(\eta), \quad (2.16)$$

such that

$$2-\beta(\eta-1) > 2-\beta\eta > 2-\frac{2T+2-\alpha\eta(\eta+1)}{2T-\eta+1} = \frac{2(T-\eta)+\alpha\eta(\eta+1)}{2T-\eta+1} > 0. \quad (2.17)$$

By using (2.13), (2.15), and (2.16), we obtain

$$\frac{2-2\beta\eta}{\eta}u(\eta) \geq \frac{(\alpha-\beta)(\eta+1)}{T+1}u(\eta). \quad (2.18)$$

If  $u(0) < 0$ , then  $u(\eta) < 0$ . It implies that  $(2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1) \leq \beta$ , a contradiction to  $\beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$ . If  $u(T) < 0$ , then  $u(\eta) < 0$ , and the same contradiction emerges. Thus, it is true that  $u(0) \geq 0$ ,  $u(T) \geq 0$ , together with (2.13), we have

$$u(t) \geq 0, \quad t \in \mathbb{N}_{T+1}. \quad (2.19)$$

This proof is complete.  $\square$

**Lemma 2.3.** Let  $\alpha\eta(\eta+1) > 2T+2$ ,  $\beta > \max\{(2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1), 0\}$ . If  $y \in C(\mathbb{N}_{T+1}, [0, \infty))$  and  $y(t) \geq 0$  for  $t \in \mathbb{N}_{1,T}$ , then problem (2.1)-(2.2) has no positive solutions.

*Proof.* Suppose that problem (2.1)-(2.2) has a positive solution  $u$  satisfying  $u(t) \geq 0$ ,  $t \in \mathbb{N}_{T+1}$ , and there is a  $\tau_0 \in \mathbb{N}_{1,T}$  such that  $u(\tau_0) > 0$ .

If  $u(T + 1) > 0$ , then  $\sum_{s=1}^{\eta} u(s) > 0$ . It implies

$$\begin{aligned}
 u(0) = \beta \sum_{s=1}^{\eta} u(s) &> \frac{2T + 2 - \alpha\eta(\eta + 1)}{\eta(2T - \eta + 1)} \sum_{s=1}^{\eta} u(s) \\
 &\geq \frac{\eta(T + 1)(u(0) + u(\eta)) - \eta(\eta + 1)u(T + 1)}{\eta(2T - \eta + 1)},
 \end{aligned}
 \tag{2.20}$$

that is,

$$\frac{u(T + 1) - u(0)}{T + 1} > \frac{u(\eta) - u(0)}{\eta + 1},
 \tag{2.21}$$

which is a contradiction to (2.13).

If  $u(T + 1) = 0$ , then  $\sum_{s=1}^{\eta} u(s)ds = 0$ . When  $\tau_0 \in \mathbb{N}_{1,\eta-1}$ , we get  $u(\tau_0) > u(T) = 0 > u(\eta)$ , which contradicts to (2.13). When  $\tau_0 \in \mathbb{N}_{\eta+1,T}$ , we get  $u(\eta) \leq 0 = u(0) < u(\tau_0)$ , which contradicts to (2.13) again. Therefore, no positive solutions exist.  $\square$

Let  $E = C(\mathbb{N}_{T+1}, [0, \infty))$ , then  $E$  is a Banach space with respect to the norm

$$\|u\| = \sup_{t \in \mathbb{N}_{T+1}} |u(t)|.
 \tag{2.22}$$

**Lemma 2.4.** *Let  $0 < \alpha < (2T + 2)/\eta(\eta + 1)$ ,  $0 < \beta < (2T + 2 - \alpha\eta(\eta + 1))/\eta(2T - \eta + 1)$ . If  $y \in C(\mathbb{N}_{T+1}, [0, \infty))$  and  $y(t) \geq 0$ , then the unique solution to problem (2.1)-(2.2) satisfies*

$$\inf_{t \in \mathbb{N}_{T+1}} u(t) \geq \gamma \|u\|,
 \tag{2.23}$$

where

$$\gamma := \min \left\{ \frac{\alpha(\eta + 1)(T + 1 - \eta)}{(T + 1)(2 - \beta(\eta - 1)) - \alpha\eta(\eta + 1)}, \frac{\alpha\eta(\eta + 1)}{(2 - \beta(\eta - 1))(T + 1)}, \frac{\beta(\eta + 1)(T + 1 - \eta)}{(2 - \beta(\eta - 1))(T + 1)}, \frac{\beta\eta(\eta + 1)}{(2 - \beta(\eta - 1))(T + 1)} \right\}.
 \tag{2.24}$$

*Proof.* Let  $u(t)$  be maximal at  $t = \tau_1$ , when  $\tau_1 \in \mathbb{N}_{1,T}$  and  $\|u\| = u(\tau_1)$ . We divide the proof into two cases.

Case i. If  $u(0) \geq u(T+1)$  and  $\inf_{t \in \mathbb{N}_{T+1}} u(t) = u(T+1)$ , then either  $0 \leq \tau_1 \leq \eta < T+1$  or  $0 < \eta < \tau_1 < T+1$ , if  $0 \leq \tau_1 \leq \eta < T+1$ , then

$$\begin{aligned} u(\tau_1) &\leq u(T+1) + \frac{u(T+1) - u(\eta)}{T+1-\eta} (\tau_1 - (T+1)) \\ &\leq u(T+1) + \frac{u(T+1) - u(\eta)}{T+1-\eta} (0 - (T+1)) \\ &\leq u(T+1) \left[ 1 - \left( \frac{(T+1) - (T+1)(2-\beta(\eta-1))/\alpha(\eta+1)}{T+1-\eta} \right) \right] \\ &\leq u(T+1) \left[ \frac{(T+1)(2-\beta(\eta-1)) - \alpha\eta(\eta+1)}{\alpha(T+1)(T+1-\eta)} \right]. \end{aligned} \quad (2.25)$$

This implies

$$\inf_{t \in \mathbb{N}_{T+1}} u(t) \geq \frac{\alpha(T+1)(T+1-\eta)}{(T+1)(2-\beta(\eta-1)) - \alpha\eta(\eta+1)} \|u\|. \quad (2.26)$$

Similarly, if  $0 < \eta < \tau_1 < T+1$ , from

$$\frac{u(\eta)}{\eta} \geq \frac{u(\tau_1)}{\tau_1} \geq \frac{u(\tau_1)}{T+1}, \quad (2.27)$$

together with (2.16), we have

$$u(T+1) \geq \frac{\alpha\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)} u(\tau_1). \quad (2.28)$$

This implies

$$\inf_{t \in \mathbb{N}_{T+1}} u(t) \geq \frac{\alpha\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)} \|u\|. \quad (2.29)$$

Case ii. If  $u(0) \leq u(T+1)$  and  $\inf_{t \in \mathbb{N}_{T+1}} u(t) = u(0)$ , then either  $0 < \tau_1 < \eta < T+1$  or  $0 < \eta \leq \tau_1 \leq T+1$ , by (2.13). If  $0 < \tau_1 < \eta < T+1$ , from

$$\frac{u(\eta)}{T+1-\eta} \geq \frac{u(\tau_1)}{T+1-\tau_1} \geq \frac{u(\tau_1)}{T+1}, \quad (2.30)$$

together with (2.15), we have

$$u(0) > \frac{\beta(\eta+1)(T+1-\eta)}{(2-\beta(\eta-1))(T+1)} u(\tau_1). \quad (2.31)$$



Hence,

$$\inf_{t \in \mathbb{N}_{T+1}} u(t) > \frac{\beta(\eta+1)(T+1-\eta)}{(2-\beta(\eta-1))(T+1)} \|u\|. \quad (2.32)$$

If  $0 < \eta \leq \tau_1 \leq T+1$ , from

$$\frac{u(\tau_1)}{T+1} \leq \frac{u(\tau_1)}{\tau_1} \leq \frac{u(\eta)}{\eta}, \quad (2.33)$$

together with (2.15), we have

$$u(0) < \frac{\beta\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)} u(\tau_1). \quad (2.34)$$

This implies

$$\inf_{t \in \mathbb{N}_{T+1}} u(t) < \frac{\beta\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)} \|u\|. \quad (2.35)$$

This completes the proof. □

In the rest of the paper, we assume that  $0 < \alpha < (2T+2)/\eta(\eta+1)$ ,  $T \in \mathbb{N}_{1,T}$ ;  $0 < \beta < (2T+2-\alpha\eta(\eta+1))/\eta(2T-\eta+1)$ . It is easy to see that the BVP (1.2) has a solution  $u = u(t)$  if and only if  $u$  is a solution of the operator equation

$$\begin{aligned} Au(t) \triangleq & \frac{\beta\eta(\eta+1) + 2t(1-\beta\eta)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)u(s)f(u(s)) \\ & - \frac{\beta(T+1) - (\beta-\alpha)t}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)u(s)f(u(s)) \\ & - \sum_{s=1}^{t-1} (t-s)u(s)f(u(s)). \end{aligned} \quad (2.36)$$

Denote

$$K = \left\{ u \in E : u \geq 0, \min_{t \in \mathbb{N}_{T+1}} u(t) \geq \gamma \|u\| \right\}, \quad (2.37)$$

where  $\gamma$  is defined in (2.24).

It is obvious that  $K$  is a cone in  $E$ . Since  $Au = u$  and from Lemmas 2.2 and 2.4, then  $A(K) \subset K$ . It is also easy to check that  $A : K \rightarrow K$  is completely continuous. In the following, for the sake of convenience, set

$$\begin{aligned}\Lambda_1 &= \frac{(2T+2)(1-\beta\eta) + \beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)a(s), \\ \Lambda_2 &= \frac{\gamma(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^T sa(s).\end{aligned}\tag{2.38}$$

### 3. Main Results

Now we are in the position to establish the main result.

**Theorem 3.1.** *The BVP (1.2) has at least one positive solution in the case*

(H<sub>1</sub>)  $f_0 = 0$  and  $f_\infty = \infty$  (superlinear) or

(H<sub>2</sub>)  $f_0 = \infty$  and  $f_\infty = 0$  (sublinear).

*Proof. Superlinear Case*

Let (H<sub>1</sub>) hold. Since  $f_0 = \lim_{u \rightarrow 0^+} (f(u)/u) = 0$  for any  $\varepsilon \in (0, \Lambda_1^{-1}]$ , there exists  $\rho_*$  such that

$$f(u) \leq \varepsilon u \quad \text{for } u \in [0, \rho_*].\tag{3.1}$$

Let  $\Omega_{\rho_*} = \{u \in E : \|u\| < \rho_*\}$  for any  $u \in K \cap \partial\Omega_{\rho_*}$ . From (3.1), we get

$$\begin{aligned}Au(t) &= \frac{2t(1-\beta\eta) + \beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)a(s)f(u(s)) \\ &\quad - \frac{\beta(T+1) - (\beta-\alpha)t}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)a(s)f(u(s)) \\ &\quad - \sum_{s=1}^{t-1} (t-s)a(s)f(u(s)) \\ &\leq \frac{2t(1-\beta\eta) + \beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)a(s)f(u(s)) \\ &\leq \frac{2(T+1)(1-\beta\eta) + \beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)a(s)f(u(s)) \\ &\leq \varepsilon\rho_* \frac{(2T+2)(1-\beta\eta) + \beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)a(s) \\ &= \varepsilon\Lambda_1\rho_* \leq \rho_* = \|u\|,\end{aligned}\tag{3.2}$$

which yields

$$\|Au\| \leq \|u\| \quad \text{for } u \in K \cap \partial\Omega_{\rho_*}. \quad (3.3)$$

Further, since  $f_\infty = \lim_{u \rightarrow \infty} (f(u)/u) = \infty$ , then, for any  $M^* \in [\Lambda_2^{-1}, \infty)$ , there exists  $\rho^* > \rho_*$  such that

$$f(u) \geq M^*u \quad \text{for } u \geq \gamma\rho^*. \quad (3.4)$$

Set  $\Omega_{\rho^*} = \{u \in E : \|u\| < \rho^*\}$  for  $u \in K \cap \partial\Omega_{\rho^*}$ .

Since  $u \in K$ ,  $\min_{t \in N_T} u(t) \geq \gamma\|u\| = \gamma\rho^*$ . Hence, for any  $u \in K \cap \Omega_{\rho^*}$ , from (3.4) and (2.23), we get

$$\begin{aligned} Au(\eta) &= \frac{(2 - \beta\eta + \beta)\eta}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^T (T - s + 1)a(s)f(u(s)) \\ &\quad - \frac{\beta(T + 1) - (\beta - \alpha)\eta}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^{\eta} (\eta - s)(\eta - s + 1)a(s)f(u(s)) \\ &\quad - \sum_{s=1}^{\eta-1} (\eta - s)a(s)f(u(s)) \\ &= \frac{(2 - \beta\eta + \beta)\eta}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^T (T - s + 1)a(s)f(u(s)) \\ &\quad + \frac{1}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \\ &\quad \times \sum_{s=1}^{\eta-1} (\eta - s) [-(2 - \beta\eta + \beta)T + (\beta(T - \eta) + \alpha\eta + 1)s + (\eta - 1)\beta] a(s)f(u(s)) \\ &= \frac{(2 - \beta\eta + \beta)\eta}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^T (T - s + 1)a(s)f(u(s)) \\ &\quad - \frac{T(2 - \beta\eta + \beta)}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^{\eta-1} (\eta - s)a(s)f(u(s)) \\ &\quad + \frac{\beta(t - \eta) + \alpha\eta + 1}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^{\eta-1} (\eta s - s^2)a(s)f(u(s)) \\ &\quad + \frac{(\eta - 1)\beta}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^{\eta-1} (\eta - s)a(s)f(u(s)) \\ &\geq \frac{(2 - \beta\eta + \beta)\eta}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^T (T - s + 1)a(s)f(u(s)) \end{aligned}$$

$$\begin{aligned}
& - \frac{T(2 - \beta\eta + \beta)}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^T (\eta - s)a(s)f(u(s)) \\
& = \frac{(2 - \beta\eta + \beta)(T - \eta)}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^T sa(s)f(u(s)) \\
& \geq \gamma\rho^* M^* \frac{(2 - \beta\eta + \beta)(T - \eta)}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^T sa(s) = M^* \Lambda_2 \rho^* \\
& \geq \rho^* = \|u\|,
\end{aligned} \tag{3.5}$$

which implies

$$\|Au\| \geq \|u\| \quad \text{for } u \in K \cap \partial\Omega_{\rho^*}. \tag{3.6}$$

Therefore, from (3.3), (3.6), and Theorem 1.1, it follows that  $A$  has a fixed point in  $K \cap (\overline{\Omega}_{\rho^*} \setminus \Omega_{\rho_*})$  such that  $\rho_* \leq \|u\| \leq \rho^*$ .

#### Sublinear Case

Let  $(H_2)$  hold. In view of  $f_0 = \lim_{u \rightarrow 0^+} (f(u)/u) = \infty$  for any  $M_* \in [\Lambda_2^{-1}, \infty)$ , there exists  $r_* > 0$  such that

$$f(u) \geq M_* u \quad \text{for } 0 \leq u \leq r_*. \tag{3.7}$$

Set  $\Omega_{r_*} = \{u \in E : \|u\| < r_*\}$  for  $u \in K \cap \partial\Omega_{r_*}$ . Since  $u \in K$ , then  $\min_{t \in \mathbb{N}_{T+1}} u(t) \geq \gamma\|u\| = \gamma r_*$ . Thus, from (3.7) for any  $u \in K \cap \partial\Omega_{r_*}$ , we can get

$$\begin{aligned}
Au(\eta) & = \frac{(2 - \beta\eta + \beta)\eta}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^T (T - s + 1)a(s)f(u(s)) \\
& \quad - \frac{\beta(T + 1) - (\beta - \alpha)\eta}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^{\eta-1} (\eta - s)(\eta - s + 1)a(s)f(u(s)) \\
& \quad - \sum_{s=1}^{\eta-1} (\eta - s)a(s)f(u(s)) \\
& \geq \gamma r_* M_* \frac{(2 - \beta\eta + \beta)(T - \eta)}{(2T + 2 - \alpha\eta(\eta + 1)) - \beta\eta(2T - \eta + 1)} \sum_{s=1}^T sa(s) = M_* \Lambda_2 r_* \geq r_* = \|u\|,
\end{aligned} \tag{3.8}$$

which yields

$$\|Au\| \geq \|u\| \quad \text{for } u \in K \cap \partial\Omega_{r_*}. \tag{3.9}$$

Since  $f_\infty = \lim_{u \rightarrow \infty} (f(u)/u) = 0$ , then, for any  $\varepsilon_1 \in (0, \Lambda_1^{-1}]$ , there exists  $r_0 > r_*$  such that

$$f(u) \leq \varepsilon_1 u \quad \text{for } u \in [r_0, \infty). \quad (3.10)$$

We have the following two cases.

*Case i.* Suppose that  $f(u)$  is unbounded, then, from  $f \in C([0, \infty), [0, \infty))$ , we know that there is  $r^* > r_0$  such that

$$f(u) \leq f(r^*) \quad \text{for } u \in [0, r^*]. \quad (3.11)$$

Since  $r^* > r_0$ , then, from (3.10) and (3.11), one has

$$f(u) \leq f(r^*) \leq \varepsilon_1 r^* \quad \text{for } u \in [0, r^*]. \quad (3.12)$$

For  $u \in K$ ,  $\|u\| = r^*$ , from (3.12), we obtain

$$\begin{aligned} Au(t) &\leq \frac{(2T+2)(1-\beta\eta) + \beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)a(s)f(u(s)) \\ &\leq \varepsilon_1 r^* \frac{(2T+2)(1-\beta\eta) + \beta\eta(\eta+1)}{(2T+2-\alpha(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)a(s) \\ &= \varepsilon_1 \Lambda_1 r^* \leq r^* = \|u\|. \end{aligned} \quad (3.13)$$

*Case ii.* Suppose that  $f(u)$  is bounded, say  $f(u) \leq N$  for all  $u \in [0, \infty)$ . Taking  $r^* \geq \max\{N/\varepsilon_1, r_*\}$ , for  $u \in K$ ,  $\|u\| = r^*$ , we have

$$\begin{aligned} Au(t) &\leq \frac{(2T+2)(1-\beta\eta) + \beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)a(s)f(u(s)) \\ &\leq N \frac{(2T+2)(1-\beta\eta) + \beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)a(s) \\ &\leq \varepsilon_1 r^* \frac{(2T+2)(1-\beta\eta) + \beta\eta(\eta+1)}{(2T-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^T (T-s+1)a(s) \\ &= \varepsilon_1 \Lambda_1 r^* \leq r^* = \|u\|. \end{aligned} \quad (3.14)$$

Hence, in either case, we may always set  $\Omega_{r^*} = \{u \in E : \|u\| < r^*\}$  such that

$$\|Au\| \leq \|u\| \quad \text{for } u \in K \cap \partial\Omega_{r^*}. \quad (3.15)$$

Hence, from (3.9), (3.15), and Theorem 1.1, it follows that  $A$  has a fixed point in  $K \cap (\overline{\Omega}_{\rho^*} \setminus \Omega_{\rho_*})$  such that  $r_* \leq \|u\| \leq r^*$ . The proof is complete.  $\square$

#### 4. Some Examples

In this section, in order to illustrate our result, we consider some examples.

*Example 4.1.* Consider the BVP

$$\begin{aligned} \Delta^2 u(t-1) + t^2 u^k &= 0, \quad t \in N_{1,4}, \\ u(0) &= \frac{1}{3} \sum_{s=1}^2 u(s), \quad u(5) = \frac{2}{3} \sum_{s=1}^2 u(s). \end{aligned} \quad (4.1)$$

Set  $\alpha = 2/3$ ,  $\beta = 1/3$ ,  $\eta = 2$ ,  $T = 4$ ,  $a(t) = t^2$ , and  $f(u) = u^k$ .

We can show that

$$0 < \alpha = \frac{2}{3} < \frac{5}{3} = \frac{2T+2}{\eta(\eta+1)}, \quad 0 < \beta = \frac{1}{3} < \frac{3}{7} = \frac{2T+2-\alpha\eta(\eta+1)}{\eta(2T-\eta+1)}. \quad (4.2)$$

*Case I.*  $k \in (1, \infty)$ . In this case,  $f_0 = 0$ ,  $f_\infty = \infty$ , and  $H_1$  of Theorem 3.1 holds. Then BVP (4.1) has at least one positive solution.

*Case II.*  $k \in (0, 1)$ . In this case,  $f_0 = \infty$ ,  $f_\infty = 0$ , and  $H_2$  of Theorem 3.1 holds. Then BVP (4.1) has at least one positive solution.

*Example 4.2.* Consider the BVP

$$\begin{aligned} \Delta^2 u(t-1) + e^{te} \left( \frac{\pi \sin u + 2 \cos u}{u^2} \right) &= 0, \quad t \in N_{1,4}, \\ u(0) &= \frac{1}{4} \sum_{s=1}^3 u(s), \quad u(5) = \frac{1}{3} \sum_{s=1}^3 u(s). \end{aligned} \quad (4.3)$$

Set  $\alpha = 1/3$ ,  $\beta = 1/4$ ,  $\eta = 3$ ,  $T = 4$ ,  $a(t) = e^{te}$ ,  $f(u) = (\pi \sin u + 2 \cos u)/u^2$ .

We can show that

$$0 < \alpha = \frac{1}{3} < \frac{5}{6} = \frac{2T+2}{\eta(\eta+1)}, \quad 0 < \beta = \frac{1}{4} < \frac{1}{3} = \frac{2T+2-\alpha\eta(\eta+1)}{\eta(2T-\eta+1)}. \quad (4.4)$$

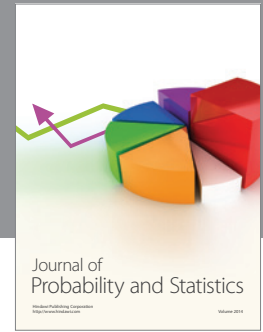
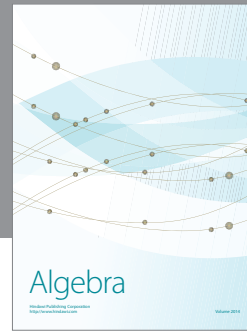
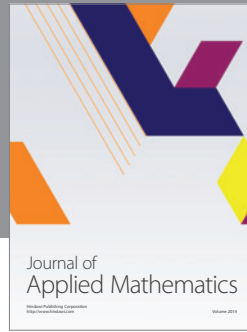
Through a simple calculation we can get  $f_0 = \infty$ ,  $f_\infty = 0$ . Thus, by  $H_2$  of Theorem 3.1, we can get BVP (4.3) has at least one positive solution.

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