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Research Article

Existence of at Least Two Periodic Solutions for a Competition System of Plankton Allelopathy on Time Scales

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We study a competition system of the growth of two species of plankton with competitive and allelopathic effects on each other on time scales. With the help of Mawhin's continuation theorem of coincidence degree theory, a set of easily verifiable criteria is obtained for the existence of at least two periodic solutions for this model. Some new existence results are obtained. An example and numerical simulation are given to illustrate the validity of our results.

1. Introduction

The allelopathic interactions in the phytoplanktonic world have been studied by many researchers. For instance, see [1–4] and references cited therein. Maynard-Smith [2] and Chattopadhyay [3] proposed the following two-species Lotka-Volterra competition system, which describes the changes of size and density of phytoplankton:

$$\begin{aligned} \frac{dN_1(t)}{dt} &= N_1(t)\left[r_1 - a_{11}N_1(t) - a_{12}N_2(t) - b_1N_1(t)N_2(t)\right], \\ \frac{dN_2(t)}{dt} &= N_2(t)\left[r_2 - a_{21}N_1(t) - a_{22}N_2(t) - b_2N_1(t)N_2(t)\right], \end{aligned} \tag{1.1}$$

where b_1 and b_2 are the rates of toxic inhibition of the first species by the second and vice versa, respectively.

Naturally, more realistic models require the inclusion of the periodic changing of environment caused by seasonal effects of weather, food supplies, and so forth. For such systems, as pointed out by Freedman and Wu [5] and Kuang [6], it would be of interest to study the existence of periodic solutions. This motivates us to modify system (1.1) to the form

$$\begin{aligned}\frac{dN_1(t)}{dt} &= N_1(t)[r_1(t) - a_{11}(t)N_1(t) - a_{12}(t)N_2(t) - b_1(t)N_1(t)N_2(t)],\\ \frac{dN_2(t)}{dt} &= N_2(t)[r_2(t) - a_{21}(t)N_1(t) - a_{22}(t)N_2(t) - b_2(t)N_1(t)N_2(t)],\end{aligned} \tag{1.2}$$

where $r_i(t), a_{ij}(t) > 0, b_i(t) > 0$ $(i, j = 1, 2)$ are continuous ω-periodic functions.

If the estimates of the population size and all coefficients in (1.2) are made at equally spaced time intervals, then we can incorporate this aspect in (1.2) and obtain the following discrete analogue of system (1.2):

$$\begin{aligned}N_1(k+1) &= N_1(k)\exp\{r_1(k) - a_{11}(k)N_1(k) - a_{12}(k)N_2(k) - b_1(k)N_1(k)N_2(k)\},\\ N_2(k+1) &= N_2(k)\exp\{r_2(k) - a_{21}(k)N_1(k) - a_{22}(k)N_2(k) - b_2(k)N_1(k)N_2(k)\},\end{aligned} \tag{1.3}$$

where $r_i, a_{ij} > 0, b_i > 0$ $(i, j = 1, 2)$ are ω-periodic, that is,

$$r_i(k+\omega) = r_i(k), \qquad b_i(k+\omega) = b_i(k), \qquad a_{ij}(k+\omega) = a_{ij}(k), \tag{1.4}$$

for any $\mathbb{Z}$ (the set of all integers), ω is a fixed positive integer. System (1.3) was considered by Zhang and Fang [7]. However, dynamics in each equally spaced time interval may vary continuously. So, it may be more realistic to assume that the population dynamics involves the hybrid discrete-continuous processes. For example, Gamarra and Solé pointed out that such hybrid processes appear in the population dynamics of certain species that feature nonoverlapping generations: the change in population from one generation to the next is discrete and so is modelled by a difference equation, while within-generation dynamics vary continuously (due to mortality rates, resource consumption, predation, interaction, etc.) and thus are described by a differential equation [8, page 619]. The theory of calculus on time scales (see [9, 10] and references cited therein) was initiated by Hilger in his Ph.D. thesis in 1988 [11] in order to unify continuous and discrete analysis, and it has become an effective approach to the study of mathematical models involving the hybrid discrete-continuous processes. This motivates us to unify systems (1.2) and (1.3) to a competition system on time scales $\mathbb{T}$ of the form

$$\begin{aligned}x_1^{\Delta}(t) &= r_1(t) - a_{11}(t)\exp\{x_1(t)\} - a_{12}(t)\exp\{x_2(t)\} - b_1(t)\exp\{x_1(t)\}\exp\{x_2(t)\},\\ x_2^{\Delta}(t) &= r_2(t) - a_{21}(t)\exp\{x_1(t)\} - a_{22}(t)\exp\{x_2(t)\} - b_2(t)\exp\{x_1(t)\}\exp\{x_2(t)\},\end{aligned} \tag{1.5}$$

where $r_i(t), a_{ij}(t) > 0, b_i(t) > 0$ $(i, j = 1, 2)$ are rd-continuous ω-periodic functions.

In (1.5), let $N_i(t) = \exp\{x_i(t)\}$, $i = 1, 2$. If $\mathbb{T} = \mathbb{R}$ (the set of all real numbers), then (1.5) reduces to (1.2). If $\mathbb{T} = \mathbb{Z}$ (the set of all integers), then (1.5) reduces to (1.3).

To our knowledge, few papers have been published on the existence of multiple periodic solutions for this model. Motivated by the work of Chen [12], we study the existence of multiple periodic solutions of (1.5) by applying Mawhin's continuation theorem of coincidence degree theory [13]. Some new results are obtained. Even in the special case when $\mathbb{T} = \mathbb{Z}$, our conditions are also easier to verify than that of [7].

2. Preliminaries on Time Scales

In this section, we briefly present some foundational definitions and results from the calculus on time scales so that the paper is self-contained. For more details, one can see [9–11].

Definition 2.1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

Let $\omega > 0$. Throughout this paper, the time scale $\mathbb{T}$ is assumed to be ω-periodic, that is, $t \in \mathbb{T}$ implies $t + \omega \in \mathbb{T}$. In particular, the time scale $\mathbb{T}$ under consideration is unbounded above and below.

Definition 2.2. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, \infty)$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \qquad \mu(t) = \sigma(t) - t \quad \text{for } t \in \mathbb{T}, \tag{2.1}$$

respectively. If $\sigma(t) = t$, then t is called right-dense (otherwise: right-scattered), and if $\rho(t) = t$, then t is called left-dense (otherwise: left-scattered).

Definition 2.3. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left| [f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s] \right| \leq |\sigma(t) - s| \quad \forall s \in U. \tag{2.2}$$

In this case, $f^{\Delta}(t)$ is called the delta (or Hilger) derivative of f at t. Moreover, f is said to be delta or Hilger differentiable on $\mathbb{T}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$. Then we define

$$\int_r^s f(t)\Delta t = F(s) - f(r) \quad \text{for } r, s \in \mathbb{T}. \tag{2.3}$$

Definition 2.4. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{\mathrm{rd}}(\mathbb{T})$.

Lemma 2.5. *Every rd-continuous function has an antiderivative.*

Lemma 2.6. *If $a, b \in \mathbb{T}, \alpha, \beta \in \mathbb{R}$, and $f, g \in C_{\mathrm{rd}}(\mathbb{T})$, then*

(a) $\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$

(b) *if $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t)\Delta t \geq 0$*

(c) *if $|f(t)| \leq g(t)$ on $[a, b) := \{t \in \mathbb{T} : a \leq t < b\}$, then $|\int_a^b f(t)\Delta t| \leq \int_a^b g(t)\Delta t$.*

For convenience, we now introduce some notation to be used throughout this paper. Let

$$\kappa = \min\{[0,\infty)\cap\mathbb{T}\}, \qquad I_\omega = [\kappa,\kappa+\omega]\cap\mathbb{T}, \qquad g^u = \sup_{t\in I_\omega} g(t), \qquad g^l = \inf_{t\in I_\omega} g(t),$$
$$\overline{g} = \frac{1}{\omega}\int_{I_\omega} g(s)\Delta s = \frac{1}{\omega}\int_\kappa^{\kappa+\omega} g(s)\Delta s, \tag{2.4}$$

where $g \in C_{\mathrm{rd}}(\mathbb{T})$ is an ω-periodic real function, that is, $g(t+\omega) = g(t)$ for all $t \in \mathbb{T}$.

Lemma 2.7 (see [14]). *Let $t_1, t_2 \in I_\omega$ and $t \in \mathbb{T}$. If $g : \mathbb{T} \to \mathbb{R}$ is ω- periodic, then*

$$g(t) \le g(t_1) + \int_\kappa^{\kappa+\omega} |g(s)|\Delta s, \qquad g(t) \ge g(t_2) - \int_\kappa^{\kappa+\omega} |g(s)|\Delta s. \tag{2.5}$$

Lemma 2.8 (see [15]). *Assume that $\{f_n\}_{n\in N}$ is a function on J such that*

(i) *$\{f_n\}_{n\in N}$ is uniformly bounded on J,*

(ii) *$\{f_n^\Delta\}_{n\in N}$ is uniformly bounded on J.*

Then there is a subsequence of $\{f_n\}_{n\in N}$ which converges uniformly on J.

3. Existence of Multiple Periodic Solutions

In this section, in order to obtain the existence of multiple periodic solutions of (1.5), we first make the following preparations [13].

Let X, Z be normed vector spaces, let $L : \operatorname{dom} L \subset X \to Z$ be a linear mapping, and let $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim\operatorname{Ker} L = \operatorname{codim}\operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, there then exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I-Q)$. If we define $L_P : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ as the restriction $L|_{\operatorname{dom} L\cap\operatorname{Ker} P}$ of L to $\operatorname{dom} L \cap \operatorname{Ker} P$, then L_P is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact, that is, continuous and such that $K_P(I-Q)N(\overline{\Omega})$ is relatively compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Mawhin's continuation theorem of coincidence degree theory is a very powerful tool to deal with the existence of periodic solutions of differential equations, difference equations and dynamic equations on time scales. For convenience, we introduce Mawhin's continuation theorem [13, page 40] as follows.

Lemma 3.1 (continuation theorem). *Let L be a Fredholm mapping of index zero and let $N : \overline{\Omega} \to Z$ be L-compact on $\overline{\Omega}$. Suppose*

(a) *$Lx \ne \lambda Nx$ for every $x \in \operatorname{dom} L \cap \partial\Omega$ and every $\lambda \in (0,1)$,*

(b) $QNx \neq 0$ *for every* $x \in \partial\Omega \cap \operatorname{Ker} L$, *and Brouwer degree*

$$\deg_B(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0. \tag{3.1}$$

Then $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

In the following, we shall use the notation

$$\begin{aligned}
&\alpha_{ij} = \overline{a}_{ji}\overline{b}_i - \overline{a}_{ii}\overline{b}_j, \qquad \alpha'_{ij} = \overline{a}_{ji}\overline{b}_i - \overline{a}_{ii}\overline{b}_j e^{(\overline{R}_j+\overline{r}_j)\omega},\\
&\alpha''_{ij} = \left(\overline{a}_{ji}\overline{b}_i e^{(\overline{R}_j+\overline{r}_j)\omega} - \overline{a}_{ii}\overline{b}_j\right)e^{(\overline{R}_i+\overline{r}_i)\omega},\\
&\beta_{ij} = \overline{a}_{ii}\overline{a}_{jj} + \overline{b}_i\overline{r}_j - \overline{a}_{ij}\overline{a}_{ji} - \overline{b}_j\overline{r}_i,\\
&\beta'_{ij} = \overline{a}_{ii}\overline{a}_{jj}e^{(\overline{R}_j+\overline{r}_j)\omega} + \overline{b}_i\overline{r}_j - \overline{a}_{ij}\overline{a}_{ji}e^{(\overline{R}_i+\overline{r}_i)\omega} - \overline{b}_j\overline{r}_i e^{(\overline{R}_i+\overline{r}_i+\overline{R}_j+\overline{r}_j)\omega},\\
&\beta''_{ij} = \overline{a}_{ii}\overline{a}_{jj}e^{(\overline{R}_i+\overline{r}_i)\omega} + \overline{b}_i\overline{r}_j e^{(\overline{R}_i+\overline{r}_i+\overline{R}_j+\overline{r}_j)\omega} - \overline{a}_{ij}\overline{a}_{ji}e^{(\overline{R}_j+\overline{r}_j)\omega} - \overline{b}_j\overline{r}_i,\\
&\beta^*_{ij} = \overline{a}_{ii}\overline{a}_{jj} + \overline{b}_i\overline{r}_j - \overline{a}_{ij}\overline{a}_{ji}e^{(\overline{R}_i+\overline{r}_i)\omega} - \overline{b}_j\overline{r}_i e^{(\overline{R}_i+\overline{r}_i+\overline{R}_j+\overline{r}_j)\omega},\\
&\gamma_{ij} = \overline{r}_i\overline{a}_{jj} - \overline{r}_j\overline{a}_{ij}, \qquad \gamma'_{ij} = \left(\overline{r}_i\overline{a}_{jj}e^{(\overline{R}_j+\overline{r}_j)\omega} - \overline{r}_j\overline{a}_{ij}\right)e^{(\overline{R}_i+\overline{r}_i)\omega},\\
&\gamma''_{ij} = \overline{r}_i\overline{a}_{jj} - \overline{r}_j\overline{a}_{ij}e^{(\overline{R}_j+\overline{r}_j)\omega}, \qquad \gamma^*_{ij} = \overline{r}_i\overline{a}_{jj} - \overline{r}_j\overline{a}_{ij}e^{(\overline{R}_i+\overline{r}_i+\overline{R}_j+\overline{r}_j)\omega}, \quad i \neq j,\ i, j = 1, 2,\\
&N_1(\alpha,\beta,\gamma) = \frac{\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}, \qquad N_2(\alpha,\beta,\gamma) = \frac{\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \quad \left(\alpha \neq 0,\ \beta^2 - 4\alpha\gamma > 0\right).
\end{aligned} \tag{3.2}$$

We make the following assumptions.

(H_1) $\overline{R}_i = (1/\omega)\int_\kappa^{\kappa+\omega} |r_i(t)|\Delta t \geq (1/\omega)\int_\kappa^{\kappa+\omega} r_i(t)\Delta t > 0$.

(H_2) $\gamma^*_{ij} = \overline{r}_i\overline{a}_{jj} - \overline{r}_j\overline{a}_{ij}e^{(\overline{R}_i+\overline{r}_i+\overline{R}_j+\overline{r}_j)\omega} > 0,\ i \neq j,\ i, j = 1, 2$.

(H_3) $\alpha_{12} > 0$.

Next, we introduce some lemmas.

Lemma 3.2 (see [16, Lemma 3.2]). *Consider the following algebraic equations:*

$$\begin{aligned}
\overline{a}_{11}N_1 + \overline{a}_{12}N_2 + \overline{b}_1 N_1 N_2 = \overline{r}_1,\\
\overline{a}_{21}N_1 + \overline{a}_{22}N_2 + \overline{b}_2 N_1 N_2 = \overline{r}_2.
\end{aligned} \tag{3.3}$$

Assuming that (H_1), (H_2) *hold, then the following conclusions hold.*

(i) *If* $\alpha_{12} > 0$, *then* (3.3) *have two positive solutions:*

$$\left(N_i(\alpha_{12},\beta_{12},\gamma_{12}), N_1(\alpha_{21},\beta_{21},\gamma_{21})\right), \quad i = 1, 2. \tag{3.4}$$

(ii) *If* $\alpha_{21} > 0$, *then* (3.3) *have two positive solutions:*

$$\left(N_1(\alpha_{12}, \beta_{12}, \gamma_{12}), N_i(\alpha_{21}, \beta_{21}, \gamma_{21})\right), \quad i = 1, 2. \tag{3.5}$$

Lemma 3.3. *Assume that* (H_1)–(H_3) *hold, then the following conclusions hold.*

(i) $\beta_{12} > 0$, $\beta_{12}^2 - 4\alpha_{12}\gamma_{12} > 0$,

(ii) $\beta_{12}^* > 0$, $\beta_{12}^{*2} - 4\alpha_{12}\gamma_{12}' > 0$.

Proof. The proof of (i) is the same as (i) of Lemma 3.5 in [7]. We omit it.

(ii) We have

$$\begin{aligned}
\beta_{12}^* &= \left(\frac{\overline{b}_1}{\overline{a}_{11}} + \frac{\overline{a}_{12}}{\overline{r}_1 e^{(\overline{R}_2+\overline{r}_2)\omega}}\right)\gamma_{21}^* + \left(\frac{\overline{r}_1 \alpha_{12} e^{(\overline{R}_1+\overline{r}_1+\overline{R}_2+\overline{r}_2)\omega}}{\overline{a}_{11}} + \frac{\overline{a}_{11}\gamma_{12}'}{\overline{r}_1 e^{(\overline{R}_1+\overline{r}_1+\overline{R}_2+\overline{r}_2)\omega}}\right) > 0, \\
\beta_{12}^{*2} - 4\alpha_{12}\gamma_{12}' &= \left(\frac{\overline{b}_1}{\overline{a}_{11}} + \frac{\overline{a}_{12}}{\overline{r}_1 e^{(\overline{R}_2+\overline{r}_2)\omega}}\right)^2 \gamma_{21}^{*2} + \left(\frac{\overline{r}_1 \alpha_{12} e^{(\overline{R}_1+\overline{r}_1)\omega}}{\overline{a}_{11}} - \frac{\overline{a}_{11}\gamma_{12}'}{\overline{r}_1 e^{(\overline{R}_1+\overline{r}_1)\omega}}\right)^2 \\
&\quad + 2\left(\frac{\overline{b}_1}{\overline{a}_{11}} + \frac{\overline{a}_{12}}{\overline{r}_1 e^{(\overline{R}_2+\overline{r}_2)\omega}}\right)\left(\frac{\overline{r}_1 \alpha_{12} e^{(\overline{R}_1+\overline{r}_1+\overline{R}_2+\overline{r}_2)\omega}}{\overline{a}_{11}} + \frac{\overline{a}_{11}\gamma_{12}'}{\overline{r}_1 e^{(\overline{R}_1+\overline{r}_1+\overline{R}_2+\overline{r}_2)\omega}}\right)\gamma_{21}^* > 0.
\end{aligned} \tag{3.6}$$

□

Lemma 3.4. *Assume that* (H_1)–(H_3) *hold, then the following conclusions hold.*

$$\begin{aligned}
&N_1(\alpha_{12}, \beta_{12}+m, \gamma_{12}-n) < N_1(\alpha_{12}, \beta_{12}, \gamma_{12}) < N_1(\alpha_{12}, \beta_{12}^*, \gamma_{12}') \\
&< N_2(\alpha_{12}, \beta_{12}^*, \gamma_{12}') < N_2(\alpha_{12}, \beta_{12}, \gamma_{12}) < N_2(\alpha_{12}, \beta_{12}+m, \gamma_{12}-n),
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
m &= \overline{a}_{11}\overline{a}_{22}\left(e^{(\overline{R}_1+\overline{r}_1)\omega} - 1\right) + \overline{b}_1\overline{r}_2\left(e^{(\overline{R}_1+\overline{r}_1+\overline{R}_2+\overline{r}_2)\omega} - 1\right) > 0, \\
n &= \overline{a}_{12}\overline{r}_2\left(e^{(\overline{R}_2+\overline{r}_2)\omega} - 1\right) > 0.
\end{aligned} \tag{3.8}$$

Proof. Under the conditions that $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\beta^2 - 4\alpha\gamma > 0$, we have

$$N_1(\alpha, \beta, \gamma) = \frac{2\gamma}{\beta + \sqrt{\beta^2 - 4\alpha\gamma}}, \qquad N_2(\alpha, \beta, \gamma) = \frac{\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}. \tag{3.9}$$

Thus $N_1(\alpha, \beta, \gamma)$ $(N_2(\alpha, \beta, \gamma))$ is increasing (decreasing) in the first variable, decreasing (increasing) in the second variable, increasing (decreasing) in the third variable. Notice that

$$\alpha_{12}'' > \alpha_{12} > \alpha_{12}' > 0, \qquad \gamma_{12}' > \gamma_{12} > \gamma_{12}'' > \gamma_{12}^* > 0, \qquad \beta_{12} > \beta_{12}^*. \tag{3.10}$$

So from (3.9), (3.10), and (H_1)–(H_3), we obtain that

$$\begin{aligned} &N_1(\alpha_{12}, \beta_{12} + m, \gamma_{12} - n) < N_1(\alpha_{12}, \beta_{12}, \gamma_{12}) < N_1(\alpha_{12}, \beta_{12}^*, \gamma_{12}') \\ &< N_2(\alpha_{12}, \beta_{12}^*, \gamma_{12}') < N_2(\alpha_{12}, \beta_{12}, \gamma_{12}) < N_2(\alpha_{12}, \beta_{12} + m, \gamma_{12} - n). \end{aligned} \tag{3.11}$$

□

Theorem 3.5. *Assume that (H_1)–(H_3) hold. Then system* (1.5) *has at least two ω-periodic solutions.*

Proof. Take

$$\begin{gathered} X = Z = \left\{x = (x_1, x_2)^T : x_i \in C\left(\mathbb{T}, \mathbb{R}^2\right), x_i(t+\omega) = x_i(t)\ \forall t \in \mathbb{T}, i = 1, 2\right\}, \\ \|x\| = \left[\sum_{i=1}^{2}\left(\max_{t\in I_\omega}|x_i(t)|\right)^2\right]^{1/2}, \quad x \in X \text{ (or } Z). \end{gathered} \tag{3.12}$$

It is easy to verify that X and Z are both Banach spaces.

Define the following mappings $L : X \to Z$, $N : X \to Z$, $P : X \to X$ and $Q : Z \to Z$ as follows:

$$\begin{aligned} Nx &= \begin{pmatrix} r_1(t) - \sum_{j=1}^{2} a_{1j}(t)\exp\{x_j(t)\} - b_1(t)\exp\{x_1(t)\}\exp\{x_2(t)\} \\ r_2(t) - \sum_{j=1}^{2} a_{2j}(t)\exp\{x_j(t)\} - b_2(t)\exp\{x_2(t)\}\exp\{x_1(t)\} \end{pmatrix}, \\ Lx &= \begin{pmatrix} x_1^\Delta \\ x_2^\Delta \end{pmatrix}, \\ Px &= \frac{1}{\omega}\int_\kappa^{\kappa+\omega} x(t)\Delta t = Qx, \quad x \in X(\text{or} Z). \end{aligned} \tag{3.13}$$

We first show that L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$. The argument is quite standard. For example, one can see [14, 17, 18]. But for the sake of completeness, we give the details here.

It is easy to see that $\operatorname{Ker}L=\{x \in X : (x_1(t), x_2(t))^T = (h_1, h_2)^T \in \mathbb{R}^2 \text{ for } t \in \mathbb{T}\}$, $\operatorname{Im}L=\{x \in X : \int_\kappa^{\kappa+\omega} x(t)\Delta t = 0\}$ is closed in Z, and $\dim\operatorname{Ker}L = \operatorname{codim}\operatorname{Im}L = 2$. Therefore, L is a Fredholm mapping of index zero. Clearly, P and Q are continuous projectors such that

$$\operatorname{Im}P = \operatorname{Ker}L, \qquad \operatorname{Ker}Q = \operatorname{Im}L = \operatorname{Im}(I - Q). \tag{3.14}$$

On the other hand, K_p: $\operatorname{Im}L \to \operatorname{dom}L \cap \operatorname{Ker}P$, the inverse to L, exists and is given by

$$K_p(x) = \int_\kappa^t x(s)\Delta s - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}\int_\kappa^\eta x(s)\Delta s\Delta\eta. \tag{3.15}$$

Obviously, QN and $K_p(I-Q)N$ are continuous. By Lemma 2.8, it is not difficult to show that $\overline{K_p(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Hence, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$.

Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, one has

$$x_i^{\Delta}(t) = \lambda\left[r_i(t) - \sum_{j=1}^{2} a_{ij}(t)\exp\{x_j(t)\} - b_i(t)\exp\{x_i(t)\}\exp\{x_k(t)\}\right], \tag{3.16}$$

where $i,k = 1,2$, $k \neq i$. Suppose $x(t) = (x_1(t), x_2(t))^T \in X$ is a solution of system (3.16) for some $\lambda \in (0,1)$. Integrating (3.16) over the interval $[\kappa, \kappa+\omega]$, we have

$$\overline{r}_i\omega = \sum_{j=1}^{2}\int_{\kappa}^{\kappa+\omega} a_{ij}(t)\exp\{x_j(t)\}\Delta t + \int_{\kappa}^{\kappa+\omega} b_i(t)\exp\{x_i(t)\}\exp\{x_k(t)\}\Delta t, \tag{3.17}$$

where $i,k = 1,2$, $k \neq i$.

It follows from (3.16) and (3.17) that

$$\begin{aligned}\int_{\kappa}^{\kappa+\omega}\left|x_i^{\Delta}(t)\right|\Delta t &= \lambda\int_{\kappa}^{\kappa+\omega}\left|r_i(t) - \sum_{j=1}^{2} a_{ij}(t)\exp\{x_j(t)\} - b_i(t)\exp\{x_i(t)\}\exp\{x_k(t)\}\right|\Delta t \\ &< \int_{\kappa}^{\kappa+\omega}|r_i(t)|\Delta t + \sum_{j=1}^{2}\int_{\kappa}^{\kappa+\omega} a_{ij}(t)\exp\{x_j(t)\}\Delta t \\ &\quad + \int_{\kappa}^{\kappa+\omega} b_i(t)\exp\{x_i(t)\}\exp\{x_k(t)\}\Delta t = \left(\overline{R}_i + \overline{r}_i\right)\omega, \quad i = 1,2.\end{aligned} \tag{3.18}$$

That is

$$\int_{\kappa}^{\kappa+\omega}\left|x_i^{\Delta}(t)\right|\Delta t < \left(\overline{R}_i + \overline{r}_i\right)\omega, \quad i = 1,2. \tag{3.19}$$

Since $x(t) \in X$, there exist ξ_i, η_i, such that

$$x_i(\xi_i) = \min_{t\in[\kappa,\kappa+\omega]} x_i(t), \qquad x_i(\eta_i) = \max_{t\in[\kappa,\kappa+\omega]} x_i(t), \quad i = 1,2. \tag{3.20}$$

From (3.17), (3.20), one obtains

$$\overline{a}_{11}\exp\{x_1(\eta_1)\} + \overline{a}_{12}\exp\{x_2(\eta_2)\} + \overline{b}_1\exp\{x_1(\eta_1)\}\exp\{x_2(\eta_2)\} \geq \overline{r}_1, \tag{3.21}$$

$$\overline{a}_{21}\exp\{x_1(\xi_1)\} + \overline{a}_{22}\exp\{x_2(\xi_2)\} + \overline{b}_2\exp\{x_1(\xi_1)\}\exp\{x_2(\xi_2)\} \leq \overline{r}_2. \tag{3.22}$$

We can derive from (3.22) that

$$x_2(\eta_2) \le x_2(\xi_2) + \int_{\kappa}^{\kappa+\omega} \left| x_2^{\Delta}(t) \right| \Delta t < \ln \frac{\overline{r}_2 - \overline{a}_{21} \exp\{x_1(\xi_1)\}}{\overline{a}_{22} + \overline{b}_2 \exp\{x_1(\xi_1)\}} + \left(\overline{R}_2 + \overline{r}_2\right)\omega, \tag{3.23}$$

which, together with (3.21), leads to

$$\begin{aligned} \exp\{x_1(\eta_1)\} &\ge \frac{\overline{r}_1 - \overline{a}_{12} \exp\{x_2(\eta_2)\}}{\overline{a}_{11} + \overline{b}_1 \exp\{x_2(\eta_2)\}} \\ &\ge \frac{\overline{r}_1 \left(\overline{a}_{22} + \overline{b}_2 \exp\{x_1(\xi_1)\}\right) - \overline{a}_{12} \exp\left\{\left(\overline{R}_2 + \overline{r}_2\right)\omega\right\} (\overline{r}_2 - \overline{a}_{21} \exp\{x_1(\xi_1)\})}{\overline{a}_{11} \left(\overline{a}_{22} + \overline{b}_2 \exp\{x_1(\xi_1)\}\right) + \overline{b}_1 \exp\left\{\left(\overline{R}_2 + \overline{r}_2\right)\omega\right\} (\overline{r}_2 - \overline{a}_{21} \exp\{x_1(\xi_1)\})}. \end{aligned} \tag{3.24}$$

From (3.19), we have

$$x_1(\xi_1) \ge x_1(\eta_1) - \int_{\kappa}^{\kappa+\omega} \left| x_1^{\Delta}(t) \right| \Delta t > x_1(\eta_1) - \left(\overline{R}_1 + \overline{r}_1\right)\omega. \tag{3.25}$$

That is

$$\exp\{x_1(\xi_1)\} > \exp\{x_1(\eta_1)\} \cdot \exp\left(-\left(\overline{R}_1 + \overline{r}_1\right)\omega\right), \tag{3.26}$$

which, together with (3.24), leads to

$$\begin{aligned} &\exp\left\{\left(\overline{R}_1 + \overline{r}_1\right)\omega\right\} \exp\{x_1(\xi_1)\} \\ &\quad > \frac{\overline{r}_1 \left(\overline{a}_{22} + \overline{b}_2 \exp\{x_1(\xi_1)\}\right) - \overline{a}_{12} \exp\left\{\left(\overline{R}_2 + \overline{r}_2\right)\omega\right\} (\overline{r}_2 - \overline{a}_{21} \exp\{x_1(\xi_1)\})}{\overline{a}_{11} \left(\overline{a}_{22} + \overline{b}_2 \exp\{x_1(\xi_1)\}\right) + \overline{b}_1 \exp\left\{\left(\overline{R}_2 + \overline{r}_2\right)\omega\right\} (\overline{r}_2 - \overline{a}_{21} \exp\{x_1(\xi_1)\})}. \end{aligned} \tag{3.27}$$

Therefore, we have

$$\alpha''_{12} \exp(2x_1(\xi_1)) - \beta''_{12} \exp\{x_1(\xi_1)\} + \gamma''_{12} < 0. \tag{3.28}$$

So from (3.10), one obtains

$$\alpha_{12} \exp(2x_1(\xi_1)) - (\beta_{12} + m) \exp\{x_1(\xi_1)\} + \gamma_{12} - n < 0, \tag{3.29}$$

where

$$\begin{aligned} m &= \overline{a}_{11}\overline{a}_{22}\left(e^{(\overline{R}_1+\overline{r}_1)\omega} - 1\right) + \overline{b}_1\overline{r}_2\left(e^{(\overline{R}_1+\overline{r}_1+\overline{R}_2+\overline{r}_2)\omega} - 1\right) > 0, \\ n &= \overline{a}_{12}\overline{r}_2\left(e^{(\overline{R}_2+\overline{r}_2)\omega} - 1\right) > 0. \end{aligned} \tag{3.30}$$

According to (i) of Lemma 3.3, we obtain

$$N_1(\alpha_{12}, \beta_{12} + m, \gamma_{12} - n) < \exp\{x_1(\xi_1)\} < N_2(\alpha_{12}, \beta_{12} + m, \gamma_{12} - n). \tag{3.31}$$

In a similar way as the above proof, one can conclude from

$$\begin{aligned} \overline{a}_{21} \exp\{x_1(\eta_1)\} + \overline{a}_{22} \exp\{x_2(\eta_2)\} + \overline{b}_2 \exp\{x_1(\eta_1)\} \exp\{x_2(\eta_2)\} \geq \overline{r}_2, \\ \overline{a}_{11} \exp\{x_1(\xi_1)\} + \overline{a}_{12} \exp\{x_2(\xi_2)\} + \overline{b}_1 \exp\{x_1(\xi_1)\} \exp\{x_2(\xi_2)\} \leq \overline{r}_1, \end{aligned} \tag{3.32}$$

that

$$\alpha'_{12} \exp(2x_1(\eta_1)) - \beta'_{12} \exp\{x_1(\eta_1)\} + \gamma'_{12} > 0. \tag{3.33}$$

Noticing that $\alpha_{12} > \alpha'_{12}$, $\beta'_{12} > \beta^*_{12}$, one has

$$\alpha_{12} \exp(2x_1(\eta_1)) - \beta^*_{12} \exp\{x_1(\eta_1)\} + \gamma'_{12} > 0. \tag{3.34}$$

According to (ii) of Lemma 3.3, one has

$$\exp\{x_1(\eta_1)\} > N_2(\alpha_{12}, \beta^*_{12}, \gamma'_{12}), \quad \text{or} \quad \exp\{x_1(\eta_1)\} < N_1(\alpha_{12}, \beta^*_{12}, \gamma'_{12}). \tag{3.35}$$

It follows from (3.19) and (3.31) that

$$\begin{aligned} x_1(\eta_1) &\leq x_1(\xi_1) + \int_{\kappa}^{\kappa+\omega} \left| x_1^{\Delta}(t) \right| \Delta t \\ &< \ln N_2(\alpha_{12}, \beta_{12} + m, \gamma_{12} - n) + \left(\overline{R}_1 + \overline{r}_1 \right) \omega := H. \end{aligned} \tag{3.36}$$

On the other hand, it follows from (3.17) and (3.20) that

$$\overline{a}_{ii} \omega \exp(x_i(\xi_i)) \leq \int_{\kappa}^{\kappa+\omega} a_{ii}(t) \exp\{x_i(t)\} \Delta t < \overline{r}_i \omega, \quad i = 1, 2; \tag{3.37}$$

that is

$$x_i(\xi_i) < \ln \frac{\overline{r}_i}{\overline{a}_{ii}}, \quad i = 1, 2. \tag{3.38}$$

From (3.19) and (3.38), one obtains

$$x_i(t) \leq x_i(\xi_i) + \int_{\kappa}^{\kappa+\omega} \left| x_i^{\Delta}(t) \right| \Delta t < \ln \frac{\overline{r}_i}{\overline{a}_{ii}} + \left(\overline{R}_i + \overline{r}_i \right) \omega, \quad i = 1, 2. \tag{3.39}$$

It follows from (3.17) and (3.20) that

$$\begin{aligned}\overline{r}_2\omega &= \sum_{j=1}^{2}\int_{\kappa}^{\kappa+\omega} a_{2j}(t)\exp\{x_j(t)\}\Delta t \\ &\quad + \int_{\kappa}^{\kappa+\omega} b_2(t)\exp\{x_2(t)\}\exp\{x_1(t)\}\Delta t \\ &\le \sum_{j=1}^{2}\overline{a}_{2j}\omega\exp\{x_j(\eta_j)\} + \overline{b}_2\omega\exp\{x_1(\eta_1)\}\exp\{x_2(\eta_2)\},\end{aligned} \tag{3.40}$$

which implies that

$$\exp\{x_2(\eta_2)\} \ge \frac{\overline{r}_2 - \overline{a}_{21}\exp\{x_1(\eta_1)\}}{\overline{a}_{22} + \overline{b}_2\exp\{x_1(\eta_1)\}}. \tag{3.41}$$

From (3.39) and (3.41), we have

$$x_2(\eta_2) \ge \ln\frac{\overline{a}_{11}\overline{r}_2 - \overline{a}_{21}\overline{r}_1\exp\left\{\left(\overline{R}_1 + \overline{r}_1\right)\omega\right\}}{\overline{a}_{11}\overline{a}_{22} + \overline{b}_2\overline{r}_1\exp\left\{\left(\overline{R}_1 + \overline{r}_1\right)\omega\right\}} := M, \tag{3.42}$$

which leads to

$$x_2(t) \ge x_2(\eta_2) - \int_{\kappa}^{\kappa+\omega}\left|x_2^{\Delta}(t)\right|\Delta t > M - \left(\overline{R}_2 + \overline{r}_2\right)\omega. \tag{3.43}$$

By (3.39) and (3.43), we obtain that

$$|x_2(t)| < \max\left\{\left|\ln\frac{\overline{r}_2}{\overline{a}_{22}} + \left(\overline{R}_2 + \overline{r}_2\right)\omega\right|, \left|M - \left(\overline{R}_2 + \overline{r}_2\right)\omega\right|\right\} := A. \tag{3.44}$$

Now, let us consider QNx with $x = (x_1, x_2)^T \in \mathbb{R}^2$. Note that

$$QN(x_1, x_2) = \begin{pmatrix} \overline{r}_1 - \overline{a}_{11}\exp(x_1) - \overline{a}_{12}\exp(x_2) - \overline{b}_1\exp(x_1)\exp(x_2) \\ \overline{r}_2 - \overline{a}_{21}\exp(x_1) - \overline{a}_{22}\exp(x_2) - \overline{b}_2\exp(x_1)\exp(x_2) \end{pmatrix}. \tag{3.45}$$

According to Lemma 3.2, we can show that $QNx = 0$ has two distinct solutions

$$\widehat{x}^i = (\ln N_i(\alpha_{12}, \beta_{12}, \gamma_{12}), \ln N_1(\alpha_{21}, \beta_{21}, \gamma_{21})), \quad i = 1, 2. \tag{3.46}$$

Choose $C > 0$ such that

$$C > \left|\ln N_1(\alpha_{21}, \beta_{21}, \gamma_{21})\right|. \tag{3.47}$$

Let

$$\Omega_1 = \left\{ x \in X \,\middle|\, \begin{array}{c} x_1(t) \in \left(\ln\ N_1(\alpha_{12}, \beta_{12} + m, \gamma_{12} - n), \ln N_1(\alpha_{12}, \beta_{12}^*, \gamma_{12}')\right), \\ |x_2(t)| < A + C. \end{array} \right\},$$

$$\Omega_2 = \left\{ x \in X \,\middle|\, \begin{array}{c} \min_{t\in I_\omega} x_1(t) \in \left(\ln N_1(\alpha_{12}, \beta_{12} + m, \gamma_{12} - n), \ln N_2(\alpha_{12}, \beta_{12} + m, \gamma_{12} - n)\right), \\ \max_{t\in I_\omega} x_1(t) \in \left(\ln N_2(\alpha_{12}, \beta_{12}^*, \gamma_{12}'), H\right),\ |x_2(t)| < A + C. \end{array} \right\}, \tag{3.48}$$

Then both Ω_1 and Ω_2 are bounded open subsets of X. It follows from Lemma 3.2, Lemma 3.4, and (3.47) that $\hat{x}^i \in \Omega_i$, $i = 1, 2$. With the help of (3.31), (3.35), (3.36), (3.44), and Lemma 3.4, it is easy to see that $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \phi$ and Ω_i satisfies the requirement (a) in Lemma 3.1 for $i = 1, 2$. Moreover, $QNx \neq 0$ for $x \in \partial\Omega_i \cap \operatorname{Ker} L$ $(i = 1, 2)$. A direct computation gives

$$\deg_B(JQN, \Omega_i \cap \operatorname{Ker} L, 0) \neq 0. \tag{3.49}$$

Here J is taken as the identity mapping since $\operatorname{Im} Q = \operatorname{Ker} L$. So far we have proved that Ω_i satisfies all the assumptions in Lemma 3.1. Hence (1.5) has at least two ω-periodic solutions $\check{x}^i$ with $\check{x}^i \in \operatorname{Dom} L \cap \overline{\Omega}_i$ $(i = 1, 2)$. Obviously $\check{x}^i$ $(i = 1, 2)$ are different. The proof is complete. □

Example 3.6. As an application of Theorem 3.5, we consider the following system:

$$\begin{aligned} x_1^{\Delta}(t) &= 0.02 + 0.002\cos(0.4\pi t) - (1 + 0.001\cos(0.4\pi t))\exp\{x_1(t)\} \\ &\quad - (0.1 + 0.0137\cos(0.4\pi t))\exp\{x_2(t)\} \\ &\quad - (2 + 0.95\cos(0.4\pi t))\exp\{x_1(t)\}\exp\{x_2(t)\}, \\ x_2^{\Delta}(t) &= 0.041 + 0.04\cos(0.4\pi t) - (1 + 0.038\cos(0.4\pi t))\exp\{x_1(t)\} \\ &\quad - (1 + 0.0379\cos(0.4\pi t))\exp\{x_2(t)\} \\ &\quad - (1 + 0.039\cos(0.4\pi t))\exp\{x_2(t)\}\exp\{x_1(t)\}. \end{aligned} \tag{3.50}$$

If $\mathbb{T} = \mathbb{R}$, then (3.50) reduces to the following system:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= 0.02 + 0.002\cos(0.4\pi t) - (1 + 0.001\cos(0.4\pi t))\exp\{x_1(t)\} \\ &\quad - (0.1 + 0.0137\cos(0.4\pi t))\exp\{x_2(t)\} \\ &\quad - (2 + 0.95\cos(0.4\pi t))\exp\{x_1(t)\}\exp\{x_2(t)\}, \\ \frac{dx_2(t)}{dt} &= 0.041 + 0.04\cos(0.4\pi t) - (1 + 0.038\cos(0.4\pi t))\exp\{x_1(t)\} \\ &\quad - (1 + 0.0379\cos(0.4\pi t))\exp\{x_2(t)\} \\ &\quad - (1 + 0.039\cos(0.4\pi t))\exp\{x_2(t)\}\exp\{x_1(t)\}. \end{aligned} \tag{3.51}$$

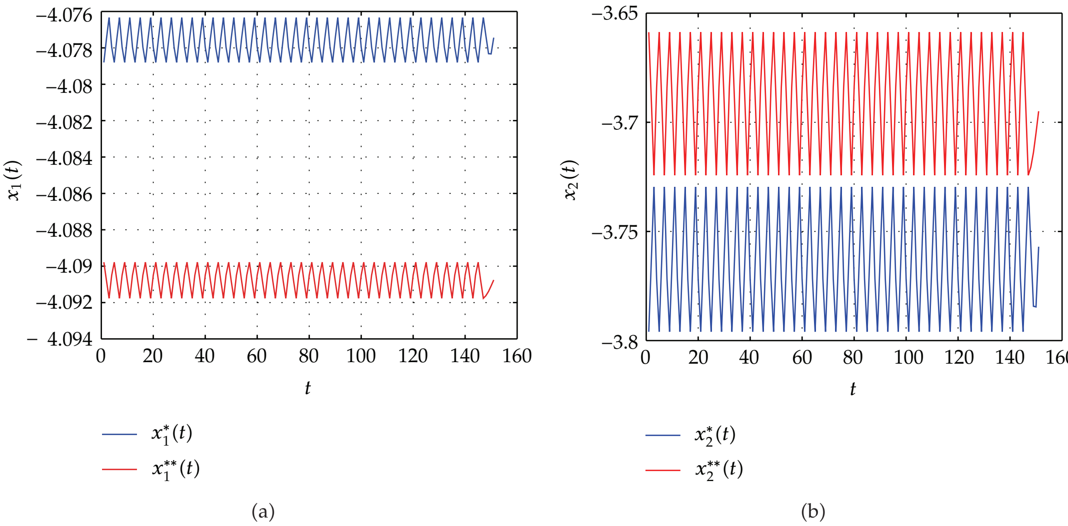

Figure 1: Blue lines stand for $x_1^*(t)$, $x_2^*(t)$, red lines stand for $x_1^{**}(t)$, $x_2^{**}(t)$.

A direct computation gives that

$$\begin{gathered}
\overline{R}_1 = \overline{r}_1 = 0.02, \qquad \overline{R}_2 = \overline{r}_2 = 0.041, \\
\overline{a}_{11} = \overline{a}_{22} = 1, \qquad \overline{a}_{12} = 0.1, \qquad \overline{a}_{21} = 1, \\
\overline{b}_1 = 2, \qquad \overline{b}_2 = 1, \qquad \omega = 5, \\
\alpha_{12} = \overline{a}_{21}\overline{b}_1 - \overline{a}_{11}\overline{b}_2 = 1, \\
\gamma_{12}^* = \overline{r}_1\overline{a}_{22} - \overline{r}_2\overline{a}_{12}e^{(\overline{R}_1+\overline{r}_1+\overline{R}_2+\overline{r}_2)\omega} = 0.02 - 0.0041e^{0.61} > 0, \\
\gamma_{21}^* = \overline{r}_2\overline{a}_{11} - \overline{r}_1\overline{a}_{21}e^{(\overline{R}_2+\overline{r}_2+\overline{R}_1+\overline{r}_1)\omega} = 0.041 - 0.02e^{0.61} > 0.
\end{gathered} \tag{3.52}$$

So according to Theorem 3.5, System (3.51) has at least two 5-periodic solutions $(x_1^*(t), x_2^*(t))$ and $(x_1^{**}(t), x_2^{**}(t))$. The simulation results given in Figure 1 verify the above conclusion. Set $N_i(t) = \exp\{x_i(t)\}$ $(i = 1, 2)$, then (3.51) can be changed into the following system:

$$\begin{aligned}
\frac{dN_1(t)}{dt} &= N_1(t)[0.02 + 0.002\cos(0.4\pi t) - (1 + 0.001\cos(0.4\pi t))N_1(t) \\
&\qquad - (0.1 + 0.0137\cos(0.4\pi t))N_2(t) - (2 + 0.95\cos(0.4\pi t))N_1(t)N_2(t)], \\
\frac{dN_2(t)}{dt} &= N_2(t)[0.041 + 0.04\cos(0.4\pi t) - (1 + 0.038\cos(0.4\pi t))N_1(t) \\
&\qquad - (1 + 0.0379\cos(0.4\pi t))N_2(t) - (1 + 0.039\cos(0.4\pi t))N_2(t)N_1(t)].
\end{aligned} \tag{3.53}$$

Therefore, System (3.53) has at least two positive 5-periodic solutions. Similar to the proof of Theorem 3.5, we can prove the following result.

Theorem 3.7. *In addition to* (H_1) *and* (H_2)*, assume further that system* (1.5) *satisfies*

$(\mathrm{H}_3)'$ $\alpha_{21} > 0$.

Then system (1.5) *has at least two* ω*-periodic solutions.*

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