Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2012, Article ID 627419, 30 pages
doi:10.1155/2012/627419

Research Article

Uniqueness and Multiplicity of a Prey-Predator Model with Predator Saturation and Competition

Xiaozhou Feng[1,2] and Lifeng Li[2]

[1] *College of Science, Xi'an Technological University, Xi'an, Shaanxi 710032, China*
[2] *College of Science, Xi'an University of Posts and Telecommunications, Xi'an, Shaanxi 710062, China*

Correspondence should be addressed to Xiaozhou Feng, flxzfxz8@163.com

Received 15 September 2012; Revised 26 November 2012; Accepted 10 December 2012

Academic Editor: Junjie Wei

We investigate positive solutions of a prey-predator model with predator saturation and competition under homogeneous Dirichlet boundary conditions. First, the existence of positive solutions and some sufficient and necessary conditions is established by using the standard fixed point index theory in cones. Second, the changes of solution branches, multiplicity, uniqueness, and stability of positive solutions are obtained by virtue of bifurcation theory, perturbation theory of eigenvalues, and the fixed point index theory. Finally, the exact number and type of positive solutions are proved when k or m converges to infinity.

1. Introduction

Considering the destabilizing force of predator saturation and the stabilizing force of competition for prey, Bazykin [1] proposed the function response $f(u,v) = 1/(1+mu)(1+kv)$ in the prey-predator model instead of the classical Holling-type II functional response. For this functional response, the prey-predator model is taken as the following form:

$$\begin{gathered}
u_t - \Delta u = u\left(a - u - \frac{bv}{(1+mu)(1+kv)}\right), \quad x \in \Omega,\ t > 0, \\
v_t - \Delta v = v\left(c - v + \frac{du}{(1+mu)(1+kv)}\right), \quad x \in \Omega,\ t > 0, \\
u = v = 0, \quad x \in \partial\Omega,\ t > 0, \\
u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, \quad x \in \Omega.
\end{gathered} \tag{1.1}$$

In this paper, we are concerned with the positive solution of the boundary value problem of the following elliptic system corresponding to the system (1.1):

$$\begin{aligned} -\Delta u &= u\left(a - u - \frac{bv}{(1+mu)(1+kv)}\right), \quad x \in \Omega, \\ -\Delta v &= v\left(c - v + \frac{du}{(1+mu)(1+kv)}\right), \quad x \in \Omega, \\ u &= v = 0, \quad x \in \partial\Omega, \end{aligned} \tag{1.2}$$

where Ω is a bounded domain in R^N $(N \geq 1)$ with smooth boundary $\partial\Omega$; a, b, c are positive constants; d is constant; m and k are nonnegative constants.

If $m = k = 0$, then (1.2) is reduced to the classical Lotka-Volterra prey-predator model which has received extensive study in the last decade, see [2–8]. In particular, the existence of positive solutions for this case was completely understood, see Dancer [8]. It has been conjectured that there is at most one positive solution, but this was shown only for the case the space dimension n is one, see [9]. For space dimension greater than one, this is still an open problem; we also refer to [10, 11] for some partial results on uniqueness. The stability of positive solutions was studied in [10, 11], but the results are still far from being complete.

The case when $m > 0$ and $k = 0$ was first studied by Blat and Brown [12]. In this case, the term $f(u, v)$ is known as the Holling-Tanner interaction term, and we refer to [5, 12–17] for more discussion on this model. In [12], Blat and Brown studied the existence of positive solutions to (1.2) by making use of both local and global bifurcation theories. The case when m goes to infinity was extensively studied by Du and Lou in [13, 14, 18]. They gave a good understanding of the existence, stability, and number of positive solutions for large m.

However, the case when $m > 0$ and $k > 0$ was first studied by Bazykin in the paper [1], more detailed background on this case, we can refer to [1]. And more works can refer to [19], Wang studies the existence, multiplicity, and stability of positive solutions of (1.2). However, Our work is more specific and meticulous than theirs. In particular. Firstly, the changes of solution branches, uniqueness, and stability of positive solutions are obtained by virtue of bifurcation theory, perturbation theory of eigenvalues, and the fixed point index theory. Secondly, the exact number and type of positive solutions are proved when k or m is large.

This paper is organized as follows: in Section 2, we give sufficient and necessary conditions for the existence of coexistence states of (1.2) by using index theory. In Section 3, by using a as a main bifurcation parameter, the multiplicity of coexistence stats to (1.2) is investigated in the gap between the sufficient and necessary conditions for the existence of coexistence states which are found in Section 2. In Section 4, the multiplicity, uniqueness, and stability of coexistence states of (1.2) are investigated when k or m converges to infinity.

2. Existence and Nonexistence of Coexistence States

In this section, we will obtain existence and nonexistence of coexistence states. Firstly, we present some basic results which will be used in this paper.

Let $\lambda_1(q) < \lambda_2(q) \leq \lambda_3(q) \leq \cdots$ be all eigenvalues of the following problem:

$$-\Delta\phi + q(x)\phi = \lambda\phi, \quad \phi\big|_{\partial\Omega} = 0, \tag{2.1}$$

where $q(x) \in C(\overline{\Omega})$. It is easy to see that $\lambda_1(q)$ is simple and $\lambda_1(q)$ is strictly increasing in the sense that $q_1 \leq q_2$ and $q_1 \not\equiv q_2$ imply $\lambda_1(q_1) < \lambda_1(q_2)$. When $q(x) \equiv 0$, we denote by $\lambda_i \lambda_i(0)$. Moreover, we denote by Φ_1 the eigenfunction corresponding to λ_1 with normalization $\|\Phi_1\|_\infty = 1$ and positive in Ω.

Define $C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) \mid u = 0 \text{ on } \partial\Omega\}$. It is well known that for any $a > \lambda_1$, the problem

$$-\Delta u = au - u^2, \quad u|_{\partial\Omega} = 0 \tag{2.2}$$

has a unique positive solution which we denote by θ_a. It is well known that the mapping $a \to \theta_a$ is strictly increasing, continuously differentiable from (λ_1, ∞) to $C^2(\Omega) \cap C_0(\overline{\Omega})$ and that $\theta_a \to 0$ uniformly on $\overline{\Omega}$ as $a \to \lambda_1$. Moreover, we have $0 < \theta_a < a$ in Ω. It follows that (1.2) has two semitrivial solutions $(\theta_a, 0)$ and $(0, \theta_c)$ if $a, c > \lambda_1$.

Next, we give an a priori estimate based on maximum principle. Its proof will be omitted.

Lemma 2.1. *Any coexistence state (u, v) of* (1.2) *has an a priori boundary, that is,*

$$u \leq a, \qquad v \leq B := c + \frac{ad}{1 + am}. \tag{2.3}$$

In the following, we set up the fixed point index theory for later use. Let E be a Banach space. $W \subset E$ is called a wedge if W is a closed convex set and $\beta W \subset W$ for all $\beta \geq 0$. For $y \in W$, we define $W_y = \{x \in E : \exists r = r(x) > 0, \text{s.t. } y + rx \in W\}$, $S_y = \{x \in \overline{W}_y : -x \in \overline{W}_y\}$: we always assume that $E = \overline{W - W}$. Let $T : W_y \to W_y$ be a compact linear operator on E. We say that T has property α on $\overline{W}_y$ if there exists $t \in (0,1)$ and $\omega \in \overline{W}_y \setminus S_y$, such that $\omega - tT\omega \in S_y$.

For any $\delta > 0$ and $y \in W$, we denote $B_\delta^+(y) = B_\delta(y) \cap W$. Assume that $F : B_\delta^+(y) \to W$ is a compact operator and y is an isolated fixed point of F. If F' is Fréchet differentiable at y, then the derivative $F'(y)$ has the property that $F'(y) : \overline{W}_y \to \overline{W}_y$. We denote by $\text{index}_W(F, y)$ the fixed point index of F at y relative to W.

We state a general result of Dancer [20] on the fixed point index with respect to the positive cone W (see also [6]).

Lemma 2.2. *Suppose that $I - L$ is invertible on $\overline{W}_y$.*

(i) *If L has property α on $\overline{W}_y$, then* $\text{index}_W(F, y) = 0$.

(ii) *If L does not have property α on $\overline{W}_y$, then* $\text{index}_W(F, y) = (-1)^\sigma$, *where σ is the sum of algebra multiplicities of the eigenvalues of L which are greater than* 1.

We introduce some notations as follows:

$X = C_0^1(\overline{\Omega}) \bigoplus C_0^1(\overline{\Omega})$, where $C_0^1(\overline{\Omega}) = \{\omega \in C^1(\overline{\Omega}) : \omega|_{\partial\Omega} = 0\}$,

$W = K \bigoplus K$, where $K = \{\varphi \in C(\overline{\Omega}) : \varphi(x) \geq 0\}$,

$D := \{(u,v) \in X : \ u \leq a+1, v \leq B+1 \ \}$, $D' := (\operatorname{int} D) \bigcap W$.

Define $F_t : D' \to W$ by

$$F_t(u,v) = (-\Delta + P)^{-1} \begin{pmatrix} tu\left(a - u - \dfrac{bv}{(1+mu)(1+kv)}\right) + Pu \\ tv\left(c - v + \dfrac{du}{(1+mu)(1+kv)}\right) + Pv \end{pmatrix}, \tag{2.4}$$

where $t \in [0,1]$ and $P > \max\{a + bB, c + da/(1+ma)\}$. It follows from Maximum Principle that $(-\Delta + P)^{-1}$ is a compact positive operator, F_t is complete continuous and Fréchet differentiable. Denote $F_1 = F$, observe that (1.2) has a positive solution in W if and only if $F_t = F_1$ has a positive fixed point in D'.

If $a > \lambda_1$ and $c > \lambda_1$, then $(0,0)$, $(\theta_a, 0)$, and $(0, \theta_c)$ are the only nonnegative fixed points of F. Then $\operatorname{index}_W(F,(0,0))$, $\operatorname{index}_W(F,(\theta_a,0))$, and $\operatorname{index}_W(F,(0,\theta_c))$ are well defined. We calculate the Fréchet operator of F as follows:

$$(-\Delta + P)^{-1} \begin{pmatrix} a - 2u - \dfrac{bv}{(1+mu)^2(1+kv)} + P & -\dfrac{bu}{(1+mu)(1+kv)^2} \\ \dfrac{dv}{(1+mu)^2(1+kv)} & c - 2v + \dfrac{du}{(1+mu)(1+kv)^2} + P \end{pmatrix}. \tag{2.5}$$

We can obtain the following lemmas by similar methods to those in the proofs of Lemmas 1 and 2 in [19].

Lemma 2.3. *Suppose that $a > \lambda_1$, one has*

(i) $\deg_W(I - F, D') = 1$, *where* $\deg_W(I - F, D')$ *is the degree of $T - F$ in D' relative to W*

(ii) *if $c \neq \lambda_1$, then* $\operatorname{index}_W(F,(0,0)) = 0$.

(iii) *if $c > \lambda_1(-d\theta_a/(1+m\theta_a))$, then* $\operatorname{index}_W(F,(\theta_a,0)) = 0$.

(iv) *if $c < \lambda_1(-d\theta_a/(1+m\theta_a))$, then* $\operatorname{index}_W(F,(\theta_a,0)) = 1$.

Lemma 2.4. *Suppose that $c > \lambda_1$, one has*

(i) *if $a > \lambda_1(b\theta_c/(1+k\theta_c))$, then* $\operatorname{index}_W(F,(0,\theta_c)) = 0$.

(ii) *if $a < \lambda_1(b\theta_c/(1+k\theta_c))$, then* $\operatorname{index}_W(F,(0,\theta_c)) = 1$.

Next, we will show some results of existence and nonexistence of positive solutions of (1.2).

Theorem 2.5. (i) *If $a \leq \lambda_1$, then* (1.2) *has no positive solution; if $a \leq \lambda_1$ and $c \leq \lambda_1$, then* (1.2) *has no nonnegative nonzero solution.*

(ii) *If $c \leq \lambda_1$ and* (1.2) *has a positive solution, then $a > \lambda_1$, $c + da/(1+ma) > \lambda_1$.*

(iii) *If $c > \lambda_1$ and* (1.2) *has a positive solution, then $a > \lambda_1(b\theta_c/(1+m\theta_a)(1+k\theta_c))$.*

Proof. (i) First assume that (u,v) is a positive solution of (1.2), then (u,v) satisfies

$$-\Delta u = u\left(a - u - \frac{bv}{(1+mu)(1+kv)}\right), \quad x \in \Omega,\ u = 0,\ x \in \partial\Omega, \tag{2.6}$$

and so $a = \lambda_1(u + bv/(1+mu)(1+kv))$ by the eigenvalue problem. Due to the comparison principle for eigenvalues, we have $a > \lambda_1$, a contradiction. Next, assume that (u,v) is a nonnegative nonzero solution of (1.2). If $u \not\equiv 0$ and $v \equiv 0$, then $a > \lambda_1$ by the previous proof. We can also similarly derive $c > \lambda_1$ when $u \equiv 0$ and $v \not\equiv 0$, which is a contradiction again.

(ii) Assume that (u,v) is a positive solution of (1.2). Then $a > \lambda_1$ by (i), and so the positive semitrivial solution θ_a exists. Since

$$-\Delta u = u\left(a - u - \frac{bv}{(1+mu)(1+kv)}\right) \le u(a-u), \quad x \in \Omega,\ u = 0,\ x \in \partial\Omega, \tag{2.7}$$

u is a lower solution of (1.2). By the uniqueness of θ_a, $u \le \theta_a$. Furthermore, since v satisfies the equation

$$-\Delta v = v\left(c - v + \frac{du}{(1+mu)(1+kv)}\right), \quad x \in \Omega,\ v = 0,\ x \in \partial\Omega, \tag{2.8}$$

one has $0 = \lambda_1(-c+v-du/(1+mu)(1+kv)) > \lambda_1(-c-da/(1+ma))$, which implies the result.

(iii) Let (u,v) be a positive solution of (1.2); then θ_a exists with $u \le \theta_a$ as in (ii). Similarly, the given assumption $c > \lambda_1$ implies the existence of positive solution θ_c of (1.2) with $\theta_c \le v$. Similar to the proof of (i), we have $a = \lambda_1(u+bv/(1+mu)(1+kv)) > \lambda_1(b\theta_c/(1+m\theta_a)(1+k\theta_c))$. This follows since the function $bv/(1+mu)(1+kv)$ has a minimum at $u = \theta_a$ and $v = \theta_c$ (for $u \le \theta_a$ and $v \ge \theta_c$). □

Theorem 2.6. (i) *If $c > \lambda_1$ and $a > \lambda_1(b\theta_c/(1+k\theta_c))$. Then* (1.2) *has at least a positive solution.*

(ii) *Suppose that $c < \lambda_1$. Then* (1.2) *has positive solution if and only if $a > \lambda_1$ and $c > \lambda_1(-d\theta_a/(1+m\theta_a))$.*

Proof. (i) By Lemmas 2.3 and 2.4, we have

$$\deg_W(I-F,D) - \operatorname{index}_W\big(f,(0,0)\big) - \operatorname{index}_W\big(f,(\theta_a,0)\big) - \operatorname{index}_W\big(f,(0,\theta_c)\big) = 1. \tag{2.9}$$

So (1.2) has at least one positive solution.

(ii) We first prove the sufficiency. Since $c < \lambda_1$, (1.2) has no solution taking the form $(0,v)$ with $v > 0$. If $a > \lambda_1$ and $c > \lambda_1(-d\theta_a/(1+m\theta_a))$, note that $c < \lambda_1$; from Lemma 2.4, we have

$$\deg_W(I-F,D) - \operatorname{index}_W\big(f,(0,0)\big) - \operatorname{index}_W\big(f,(\theta_a,0)\big) = 1. \tag{2.10}$$

Hence (1.2) has at least one positive solution.

Conversely, suppose that (u, v) is a positive solution of (1.2). Then $a > \lambda_1$, and $u < \theta_a$. Since (u, v) satisfies

$$-\Delta v = v(c - v + du/(1 + mu)(1 + kv)), \quad x \in \Omega,\ v = 0,\ x \in \partial\Omega. \tag{2.11}$$

It follows that $0 = \lambda_1(-c + v - du/(1 + mu)(1 + kv)) > \lambda_1(-c - da/(1 + ma))$. □

Theorem 2.7. *If one of the following conditions holds, then* (1.2) *has no positive solutions:*

(i) $b \geq (1 + ma)(1 + kB)$ *and* $a \leq c$,

(ii) $b < (1 + ma)(1 + kB)$ *and* $c - a \geq (1 - ba/(1 + ma)(1 + kB))\ B$.

Proof. Since the proof of Theorem 2.7 is similar to the proof of Theorem 3 of [19], we omit it. □

3. Global Bifurcation and Stability of Positive Solution

In this section, we consider a positive solution bifurcates from the semitrivial nonnegative branch $\{(0, \theta_c, a)\}$ by taking a as a bifurcation parameter and fixing $c > \lambda_1$. Furthermore, we show that the existence of global bifurcation of (1.2) with respect to parameter a and its stability. Moreover, the multiplicity, uniqueness, and stability of positive solutions are obtained by means of perturbation theory of eigenvalues and the fixed point theory.

Let $\tilde{a}$ be the principal eigenvalue of the following problem:

$$-\Delta\phi + \frac{b\theta_c}{1 + k\theta_c}\phi = a\phi, \quad \phi|_{\partial\Omega} = 0, \tag{3.1}$$

and Φ is the corresponding eigenfunction with $\|\Phi\|_\infty = 1$.

Let $\omega = u, \chi = v - \theta_c$; then $0 \leq \omega \leq \theta_a,\ \chi \geq 0$, and ω, χ satisfies

$$\begin{gathered} -\Delta\omega = \left(a - \frac{b\theta_c}{1 + k\theta_c}\right)\omega + F_1(\omega, \chi), \quad x \in \Omega, \\ -\Delta\chi = (c - 2\theta_c)\chi + \frac{d\theta_c}{1 + k\theta_c}\omega + F_2(\omega, \chi), \quad x \in \Omega, \\ \omega = \chi = 0, \quad x \in \partial\Omega, \end{gathered} \tag{3.2}$$

where

$$\begin{aligned} F_1(\omega, \chi) &= \frac{b\omega\theta_c}{1 + k\theta_c} - \frac{b\omega(\chi + \theta_c)}{(1 + m\omega)(1 + k(\chi + \theta_c))} - \omega^2, \\ F_2(\omega, \chi) &= \frac{d\omega(\chi + \theta_c)}{(1 + m\omega)(1 + k(\chi + \theta_c))} - \frac{d\omega\theta_c}{1 + k\theta_c} - \chi^2. \end{aligned} \tag{3.3}$$

Clearly, $F = (F_1, F_2)$ is continuous, $F(0,0) = 0$, and the Fréchet derivative $D_{(\omega,\chi)}F|_{(0,0)} = 0$. Let K be the inverse of $-\Delta$ with Dirichlet boundary condition. Then, we have

$$\begin{gathered}\omega = aK\omega - bK\left(\frac{\omega\theta_c}{1+k\theta_c}\right) + KF_1(\omega,\chi), \quad x \in \Omega,\\ \chi = cK\chi - 2K(\chi\theta_c) + dK\left(\frac{\omega\theta_c}{1+k\theta_c}\right) + KF_2(\omega,\chi), \quad x \in \Omega,\\ \omega = \chi = 0, \quad x \in \partial\Omega.\end{gathered} \tag{3.4}$$

Define the operator $T : R^+ \times X \to X$ as follows:

$$T(a;\omega,\chi) = \begin{pmatrix} aK\omega - bK\left(\dfrac{\omega\theta_c}{1+k\theta_c}\right) + KF_1(\omega,\chi) \\ cK\chi - 2K(\chi\theta_c) + dK\left(\dfrac{\omega\theta_c}{1+k\theta_c}\right) + KF_2(\omega,\chi) \end{pmatrix}, \tag{3.5}$$

then $T(a;\omega,\chi)$ is a compact operator on X. Let $G(a;\omega,\chi) = (\omega,\chi)^T - T(a;\omega,\chi)$; then G is continuous, and $G(a;0,0) = 0$. $G(a;\omega,\chi) = 0$ with $0 \le \omega \le \theta_a, \chi \ge 0$ if and only if $(\omega, \chi + \theta_c, a)$ is a nonnegative solution of (1.2).

Lemma 3.1. *Assume that $c > \lambda_1$. Then $(a;u,v) = (\tilde{a};0,\theta_c)$ is a bifurcation point of* (3.2)*, and there exist positive solutions of* (3.2) *in the neighborhood of $(\tilde{a};0,\theta_c)$, where $\tilde{a} = \lambda_1(b\theta_c/(1+k\theta_c))$.*

Remark 3.2. The proof of Lemma 3.1 is similar to the proof of Theorem 9 in [19]. The proof of Lemma 3.1 shows that there exist $\delta > 0$ and C^1 continuous curve $(a(s);\phi(s),\varphi(s)) : (-\delta,\delta) \to R \times Z$ such that $a(0) = \tilde{a}, \phi(0) = 0, \varphi(0) = 0$, and $(a(s);\omega(s),\chi(s)) = (a(s);s(\Phi + \phi(s)), s(\Psi + \varphi(s)))$ satisfies $G(a(s);\omega(s),\chi(s)) = 0$, where $X = Z \oplus \operatorname{span}\{(\Phi,\Psi)\}$. Hence $(a(s);U(s),V(s))$ $(|s| < \delta)$ is a bifurcation solution of (3.2), where $U(s) = s(\Phi + \phi(s)), V(s) = \theta_c + s(\Psi + \varphi(s))$, $\Psi = (-\Delta - c + 2\theta_c)^{-1}(d\theta_c/(1+k\theta_c)\Phi)$.

If we take $0 < s < \delta$, then the nontrivial nonnegative solutions of (1.2) close to $(\tilde{a};0,\theta_c)$ are either on the branch $\{(a;0,\theta_c) : a \in R^+\}$ or the branch $\{(a(s);U(s),V(s)) : 0 < s < \delta\}$.

Let $T : X \times R \to X$ be a compact continuously differentiable operator such that $T(0,a) = 0$. Suppose that we can write T as $T(u,a) = K(a)u + R(u,a)$, where $K(a)$ is a linear compact operator and the Fréchet derivative $R_u(0,0) = 0$. If x_0 is an isolated fixed point of T, then we can define the index if T at x_0 as $\operatorname{index}(T,x_0) = \deg(I - T, U_\delta(x_0), x_0)$, where $U_\delta(x_0)$ is a ball with center at x_0 such that x_0 is the only fixed point of T in $U_\delta(x_0)$. If $I - T'(x_0)$ is invertible, then x_0 is an isolated fixed point of T and $\operatorname{index}(T,x_0) = \deg(I - T, U_\delta(x_0), x_0) = \deg(I - T'(x_0), U_\delta(x_0), 0)$. If $x_0 = 0$, then it is well known that the Leray-Schauder degree $\deg(I - K(a), U_\delta(x_0), 0) = (-1)^\sigma$, where σ is equal to the algebraic multiplicities of the eigenvalue of K which is greater than one.

Next, we will extend the local bifurcation solution $\{(a(s);U(s),V(s)) : 0 < s < \delta\}$ given by Lemma 3.1 to the global bifurcation.

Let $P_1 = \{u \in C_0^1(\overline{\Omega}) : u(x) > 0, x \in \Omega, \partial u/\partial n < 0, x \in \partial\Omega\}$, $P = \{(u,v,a) \in X \times R^+ : u,v \in P_1\}$.

Theorem 3.3. *Suppose that $c > \lambda_1$; then the global bifurcation C of bifurcating branch of positive solutions of* (1.2) *becomes unbounded by a going to infinity in P.*

Proof. Let

$$\begin{aligned} T'(a)\cdot(\omega,\chi) &= D_{(\omega,\chi)}T(a;0,0)\cdot(\omega,\chi) \\ &= \left(aK\omega - bK\left(\frac{\omega\theta_c}{1+k\theta_c}\right), cK\chi - 2K(\chi\theta_c) + dK\frac{\omega\theta_c}{1+k\theta_c}\right). \end{aligned} \tag{3.6}$$

Suppose that $\mu \geq 1$ is an eigenvalue of $T'(a)$. Then we have

$$\begin{gathered} -\mu\Delta\omega = \left(a - \frac{b\theta_c}{1+k\theta_c}\right)\omega, \quad x\in\Omega, \\ -\mu\Delta\chi = (c-2\theta_c)\chi + \frac{d\theta_c}{1+k\theta_c}\omega, \quad x\in\Omega, \\ \omega = \chi = 0, \quad x\in\partial\Omega. \end{gathered} \tag{3.7}$$

Clearly $\omega \not\equiv 0$, otherwise, $\omega \equiv 0$, since all eigenvalue of the operator $(-\mu\Delta - c + 2\theta_c)$ is greater than 0, so $\chi \equiv 0$, a contradiction. Therefore, for some i such that $a = a_i(\mu)$ is the eigenvalue of the following problem:

$$-\mu\Delta\omega + \frac{b\theta_c}{1+k\theta_c}\omega = a\omega, \quad \omega|_{\partial\Omega} = 0. \tag{3.8}$$

It is well known that $a_i(\mu)$ is increasing with respect to μ on $[1,+\infty)$ and can be ordered as

$$0 < a_1(\mu) < a_2(\mu) \leq a_3(\mu) \leq \cdots \longrightarrow \infty, \quad a_1(1) = \tilde{a}. \tag{3.9}$$

On the other hand, if $\mu \geq 1$, then all eigenvalues of $(-\mu\Delta - c + 2\theta_c)$ are greater than 0; furthermore, $\chi = (-\mu\Delta - c + 2\theta_c)^{-1}(d\theta_c/(1+k\theta_c)\omega)$. Thus, $\mu \geq 1$ is the eigenvalue of $T'(a)$ if and only if there exists some i, such that $a = a_i(\mu)$.

Suppose that $a < \tilde{a}$. Then for any $\mu \geq 1, i \geq 1, a < a_1(1) \leq a_i(\mu)$. Hence, $T'(a)$ has no eigenvalue greater than 1, and $\operatorname{index}(T(a;\cdot),0) = 1$ as $a < \tilde{a}$.

Suppose that $\tilde{a} < a < a_2(1)$. Then for any $\mu \geq 1, i \geq 2$, $a < a_i(\mu)$. Since $a_1(1) = \tilde{a}$, $\lim_{\mu\to\infty} a_1(\mu) = +\infty$, and $a_1(\mu)$ is increasing with respect to μ. Hence, there exists a unique $\mu_1 > 1$, such that $a = a_1(\mu_1)$. So $N(\mu_1 I - T'(a)) = \operatorname{span}\{(\overline{\omega},\overline{\chi})\}$, $\dim N(\mu_1 I - T'(a)) = 1$, where $\overline{\omega} > 0$ is the principal eigenvalue of the following problem:

$$\mu_1\Delta\overline{\omega} + \left(a - \frac{b\theta_c}{1+k\theta_c}\right)\overline{\omega} = 0, \quad \overline{\omega}|_{\partial\Omega} = 0, \tag{3.10}$$

where $\overline{\chi} = (-\mu_1\Delta - c + 2\theta_c)^{-1}(d\theta_c/(1+k\theta_c)\overline{\omega})$.

In the following, we will prove that $R(\mu_1 I - T'(a)) \cap N(\mu_1 I - T'(a)) = 0$. In fact, if the assertion is false, we may assume that $(\overline{\omega}, \overline{\chi}) \in R(\mu_1 I - T'(a))$. Then there exists $(\omega, \chi) \in X$, such that $(\mu_1 I - T'(a))(\omega, \chi) = (\overline{\omega}, \overline{\chi})$, that is,

$$\mu_1 \Delta\omega + \left(a - \frac{b\theta_c}{1 + k\theta_c}\right)\omega = \Delta\overline{\omega}, \quad \omega|_{\partial\Omega} = 0. \tag{3.11}$$

Multiplying the equation by $\overline{\omega}$, integrating over Ω, and using Green's formula, we obtain

$$\int_\Omega \overline{\omega}\Delta\overline{\omega} = \int_\Omega \left(\mu_1 \Delta\omega + a\omega - \frac{b\omega\theta_c}{1 + k\theta_c}\right)\overline{\omega} = \int_\Omega \left(\mu_1 \Delta\overline{\omega} + a\overline{\omega} - \frac{b\overline{\omega}\theta_c}{1 + k\theta_c}\right)\omega = 0, \tag{3.12}$$

which leads to $\int_\Omega (1/\mu_1)(a - b\theta_c/(1 + k\theta_c))\overline{\omega}^2 = 0$, a contradiction. This proves the assertion, so it is verified that the multiplicity of μ_1 is one and $\operatorname{index}(T(a;\cdot), 0) = -1$ for $\tilde{a} < a < a_2(1)$. According to global bifurcation theory [12], there exists a continuum C_0 of zeros of $G(a;\omega,\chi) = 0$ in $R^+ \times X$ bifurcating from $(\tilde{a}; 0, 0)$, and all zeros of $G(a;\omega,\chi)$ close to $(\tilde{a}; 0, 0)$ lie on the curve whose existence was proved by Lemma 3.1. Let C_1 be the maximal continuum defined by $C_1 = C_0 - \{(a(s); s(\Phi + \phi(s)), s(\Psi + \varphi(s))) : -\delta < s < 0\}$. Then, C_1 consists of the curve $\{(a(s); s(\Phi + \phi(s)), s(\Psi + \varphi(s))) : -\delta < s < 0\}$ in the neighborhood of the bifurcation point $(\tilde{a}; 0, 0)$. Let $C = \{(a; u, v) : U = \omega, V = \theta_c + \chi, (\omega, \chi) \in C_1\}$. Then C is the solution branch of (1.2) which bifurcates from $(\tilde{a}; 0, \theta_c)$ and remains positive in a small neighborhood of $(\tilde{a}; 0, \theta_c)$ and $C \subset P$. Thus the continuum $C - \{(\tilde{a}; 0, \theta_c)\}$ must satisfy one of the following three alternatives.

(i) C contains in its closure points $(\tilde{a}; 0, \theta_c)$ and $(\overline{a}; 0, \theta_c)$, where $I - T'(\overline{a})$ is not invertible, and $\tilde{a} \neq \overline{a}$.

(ii) C is joining up from $(\tilde{a}; 0, \theta_c)$ to ∞ in $R \times X$.

(iii) C is containing points of the form $(a; u, \theta_c + v)$ and $(a; -u, \theta_c - v)$, where $(u, v) \neq (0, 0)$.

Next, we prove that $C - \{(\tilde{a}; 0, \theta_c)\} \subset P$. Assume that $C - \{(\tilde{a}; 0, \theta_c)\} \not\subseteq P$. Then there exists $(\hat{a}; \hat{u}, \hat{v}) \in (C - \{(\tilde{a}; 0, \theta_c)\}) \cap \partial P$ and sequence $\{(a_n; u_n, v_n)\} \subset C \cap P, u_n > 0, v_n > 0$ such that $(a_n; u_n, v_n) \to (\hat{a}; \hat{u}, \hat{v})$ when $n \to \infty$. It is easy to get that $\hat{u} \in \partial P_1$ or $\hat{v} \in \partial P_1$. Suppose $\hat{u} \in \partial P_1$, then $\hat{u} \geq 0, x \in \overline{\Omega}$. Hence, we find either $x_0 \in \Omega$ such that $\hat{u}(x_0) = 0$ or $x_0 \in \partial\Omega$ such that $\partial\hat{u}/\partial n|_{x_0} = 0$. Since $\hat{u}$ satisfies

$$-\Delta\hat{u} = \left(\hat{a} - \hat{u} - \frac{b\hat{v}}{(1 + m\hat{u})(1 + k\hat{v})}\right)\hat{u}, \quad \hat{u}|_{\partial\Omega} = 0. \tag{3.13}$$

It follows from the maximum principle that $\hat{u} \equiv 0$. Similarly, we can show that $\hat{v} \equiv 0$ for $\hat{v} \in \partial P_1$.

Thus we only need the following three cases:

$$\text{(i)}\,(\hat{u}, \hat{v}) \equiv (\theta_{\hat{a}}, 0), \qquad \text{(ii)}\,(\hat{u}, \hat{v}) \equiv (0, \theta_c), \qquad \text{(iii)}\ (\hat{u}, \hat{v}) \equiv (0, 0). \tag{3.14}$$

Suppose that $(\hat{u},\hat{v}) \equiv (\theta_{\hat{a}},0)$. Then $(a_n;u_n,v_n) \to (\hat{a};\theta_{\hat{a}},0)$ when $n \to \infty$. Let $V_n = v_n/\|v_n\|_\infty$; then V_n satisfies

$$-\Delta V_n = \left(c - v_n + \frac{du_n}{(1+mu_n)(1+kv_n)}\right)V_n, \quad V_n|_{\partial\Omega} = 0. \tag{3.15}$$

Thanks to L^P estimates and Sobolev embedding theorem, there exists a convergent subsequence of V_n, which we still denote by V_n, such that $V_n \to V$ in $C_0^1(\overline{\Omega})$ as $n \to \infty$, and $V \geq 0, \not\equiv 0, x \in \Omega$ because of $\|V\| = 1$. So taking the limit in (3.15) as $n \to \infty$, we get

$$-\Delta V = \left(c + \frac{d\theta_{\hat{a}}}{1+m\theta_{\hat{a}}}\right)V, \quad V|_{\partial\Omega} = 0. \tag{3.16}$$

It follows from the maximum principle that $V > 0, x \in \Omega$, which implies $c = \lambda_1(-d\theta_{\hat{a}}/(1+m\theta_{\hat{a}}))$. This contradicts $c > \lambda_1$.

Suppose that $(\hat{u},\hat{v}) \equiv (0,\theta_c)$. Then $(a_n;u_n,v_n) \to (\hat{a};0,\theta_c)$ as $n \to \infty$. Let $U_n = u_n/\|u_n\|_\infty$; then U_n satisfies

$$-\Delta U_n = \left(a_n - u_n - \frac{bv_n}{(1+mu_n)(1+kv_n)}\right)U_n, \quad U_n|_{\partial\Omega} = 0. \tag{3.17}$$

Similarly, By L^P estimates and and Sobolev embedding theorem, there exists a convergent subsequence of U_n, which we still denote by U_n, such that $U_n \to U$ in $C_0^1(\overline{\Omega})$ as $n \to \infty$, and $U \geq 0, \not\equiv 0, x \in \Omega$ because of $\|V\| = 1$. So taking limit in (3.17) as $n \to \infty$, we obtain

$$-\Delta U = \left(\hat{a} - \frac{b\theta_c}{1+k\theta_c}\right)U, \quad U|_{\partial\Omega} = 0. \tag{3.18}$$

It follows from the maximum principle that $U > 0, x \in \Omega$. Hence $\hat{a} = \lambda_1(b\theta_c/(1+k\theta_c))$. A contradiction with $\hat{a} \neq \tilde{a}$.

Suppose that $(\hat{u},\hat{v}) \equiv (0,0)$. Similar to the previously mentioned, we can get contradiction.

Thus $C - \{(\tilde{a};0,\theta_c)\} \subset P$. By Lemma 2.1, we have $0 \leq U \leq a, \theta_c \leq V \leq c + da/(1+\alpha a)$. Thanks to L^P estimates and and Sobolev embedding theorem, then there exists a constant $M > 0$ such that $\|U\|_{C^1}, \|V\|_{C^1} \leq M$. Hence the global bifurcation C of positive solutions of (1.2) bifurcating at $(\tilde{a};0,\theta_c)$ contains points with a is arbitrarily large in P. □

In the following, we will study the stability of the bifurcation solution. Let $X_1 = [C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega}) \cap X]$, $Y = [C^\alpha(\overline{\Omega}) \times C^\alpha(\overline{\Omega})]$, where $0 < \alpha < 1$. $i : X_1 \to Y$ is the inclusion mapping. Since L_1 is the linearized operator at $(\tilde{a};0,\theta_c)$ for (1.2). By the proof of Lemma 3.1, we have $N(L_1) = \text{span}\{(\Phi,\Psi)\}$, $\text{Codim}R(L_1) = 1$, and $R(L_0) = \{(u,v) \in X : \int_\Omega u\Phi dx = 0\}$. Since $i(\Phi,\Psi) \in R(L_1)$, so it follows from [21] that 0 is an i-simple eigenvalue of L_1.

Lemma 3.4. *0 is the eigenvalue of L_1 with the largest real part, and all the other eigenvalue of L_1 lie in the left half complex plane.*

Proof. We assume that μ_0 is the eigenvalue of L_1 with the largest real part Re $\mu_0 > 0$, and (ξ, η) is the corresponding eigenfunction; then $L_1(\xi, \eta) = \mu_0(\xi, \eta)$, equivalently,

$$\begin{gathered}\Delta\xi + \left(\tilde{a} - \frac{b\theta_c}{1+k\theta_c}\right)\xi = \mu_0\xi, \quad x \in \Omega, \\ \Delta\eta + (c - 2\theta_c)\eta + \frac{d\theta_c}{1+k\theta_c}\xi = \mu_0\eta, \quad x \in \Omega, \\ \eta = \xi = 0, \quad x \in \partial\Omega.\end{gathered} \tag{3.19}$$

Suppose that $\xi \equiv 0$. Then μ_0 is an eigenvalue of the operator $(\Delta + (c - 2\theta_c)I)$, and then $\mu_0 \in R$ and $\mu_0 < 0$, a contradiction. Thus $\Phi \equiv 0$. It follows that μ_0 is an eigenvalue of the operator $(\Delta + (\tilde{a} - b\theta_c/(1+k\theta_c))I)$. Since $\tilde{a} = \lambda_1(b\theta_c/(1+k\theta_c))$, 0 is the principal eigenvalue of the operator $(\Delta + (\tilde{a} - b\theta_c/(1+k\theta_c))I)$; furthermore, $\mu_0 \leq 0$. This contradicts the assumption, so assumption does not hold. Which proves our conclusion. □

We will use the linearized stability theory from [22]. Let $L(u(s), v(s), a(s))$, and $L(a; 0, \theta_c)$ be the linearized operators of (1.2) at $(u(s), v(s), a(s))$, and $(a; 0, \theta_c)$, respectively. It follows that from Lemma 7, Corollary 1.13, and Theorem 1.16 [22, 23] that Lemma 3.5 holds.

Lemma 3.5. *There exists C^1-function: $a \to (M(a), \gamma(a))$, and $s \to (N(s), \pi(s))$, defined from the neighborhood of $\tilde{a}$ and 0 into $X_1 \times R$, respectively, such that $\gamma(\tilde{a}) = \pi(0) = 0$, $M(\tilde{a}) = N(0) = (\Phi, \Psi)$ and*

$$\begin{gathered}L(a; 0, \theta_c)M(a) = \gamma(a)M(a), \quad for\ |a - \tilde{a}| \ll 1, \\ L(u(s), v(s), a(s))N(s) = \pi(s)N(s), \quad for\ |s| \ll 1,\end{gathered} \tag{3.20}$$

where $M(a) = (\phi_1(a), \phi_2(a))$, $N(s) = (\varphi_1(a)\varphi_2(a))$. Moreover $\gamma'(\tilde{a}) \neq 0$, whence $\pi(s) \neq 0$, $\pi(s)$ and $-sa'(s)\gamma'(\tilde{a})$ have the same sign for $|s| \ll 1$. Where $\gamma'(\tilde{a})$ is the derivative of $\gamma(a)$ with respect to a at $a = \tilde{a}$, and $a'(s)$ is the derivative of $a(s)$ with respect to s.

Lemma 3.6. *The derivative of $\gamma(s)$ with respect to a at $\tilde{a}$ is positive.*

Proof. It follows from $L(a; 0, \theta_c)M(a) = \gamma(a)M(a)$, for, $|a - \tilde{a}| \ll 1$, that is,

$$\begin{gathered}\Delta\phi_1 + \left(a - \frac{b\theta_c}{1+k\theta_c}\right)\phi_1 = \gamma(a)\phi_1, \quad x \in \Omega, \\ \Delta\phi_2 + (c - 2\theta_c)\phi_2 + \frac{d\theta_c}{1+k\theta_c}\phi_1 = \gamma(a)\phi_2, \quad x \in \Omega, \\ \phi_1 = \phi_2 = 0, \quad x \in \partial\Omega.\end{gathered} \tag{3.21}$$

Since $|a - \tilde{a}| \ll 1$, so $|\gamma(a)| \ll 1$. Clearly, $\phi_1 \not\equiv 0$, otherwise, $\phi_1 \equiv 0$, then $\phi_2 \equiv 0$, a contradiction. Hence $\gamma(a)$ is an eigenvalue of the operator $(\Delta + (a - b\theta_c/(1+k\theta_c))I)$. We consider $\Phi > 0$, then $\phi_1 = \phi_1(a) > 0$, as $|a - \tilde{a}| \ll 1$. So $\gamma(a)$ is the principal eigenvalue of the operator $(\Delta + (a - b\theta_c/(1+k\theta_c))I)$, and $\gamma(a)$ is increasing with respect to a as $|a - \tilde{a}| \ll 1$. Moreover, $\gamma'(\tilde{a}) \neq 0$. Hence $\gamma'(\tilde{a}) > 0$. □

Lemma 3.7. *The derivative of $a(s)$ with respect to s at $s = 0$ satisfies*

$$a'(0)\int_\Omega \Phi^2 dx = \int_\Omega \left(1 - \frac{bm\theta_c}{1+k\theta_c}\right)\Phi^3 dx + \int_\Omega \frac{b\Psi}{(1+k\theta_c)^2}\Phi^2 dx. \tag{3.22}$$

Proof. By substituting $(u(s), v(s), a(s))$ into (1.2), differentiating with respect to s, and then setting $s = 0$, we find that

$$-\Delta\phi'(0) = \left(\tilde{a} - \frac{b\theta_c}{1+k\theta_c}\right)\phi'(0) + \left[a'(0) - \Phi - b\frac{\Psi - m\theta_c\Phi(1+k\theta_c)}{(1+k\theta_c)^2}\right]\Phi, \tag{3.23}$$

where $\phi'(0)$ is the derivative of ϕ with respect to s at $s = 0$.

Taking the inner product with Φ, using Green's formula, and noting the definition of Φ, we have

$$a'(0)\int_\Omega \Phi^2 dx = \int_\Omega \left(1 - \frac{bm\theta_c}{1+k\theta_c}\right)\Phi^3 dx + \int_\Omega \frac{b\Psi}{(1+k\theta_c)^2}\Phi^2 dx. \tag{3.24}$$

□

It follows from Lemmas 3.4–3.7 that we obtain the following Theorem.

Theorem 3.8. *Let $\sigma = \int_\Omega (1 - bm\theta_c/(1+k\theta_c))\Phi^3 dx + \int_\Omega (b\Psi/(1+k\theta_c)^2)\Phi^2 dx$. If $\sigma > 0$, then bifurcation solution $(u(s), v(s))$ is stable; If $\sigma < 0$, then bifurcation solution $(u(s), v(s))$ is unstable.*

In Section 2, from Theorems 2.5–2.6, we can obtain the sufficient condition and necessary condition on the existence of positive solution and find that there exists a gap between $a > \lambda_1(b\theta_c/(1+k\theta_c))$ and $a > \lambda_1(b\theta_c/(1+m\theta_a)(1+k\theta_c))$ when $c > \lambda_1$. Next, we will consider the multiplicity, stability, and uniqueness of positive solutions in the gap.

Theorem 3.9. *Assume $c > \lambda_1$ and $\int_\Omega (1 - bm\theta_c/(1+k\theta_c))\Phi^3 dx < 0$. Then there exist $\varepsilon > 0$ such that bifurcation positive solution $(u(s), v(s))$ is nondegenerate and unstable for $a \in (\tilde{a} - \varepsilon, \tilde{a})$ and $d \ll 1$. Moreover, Problem (1.2) has at least two positive solutions.*

Proof. We first prove that bifurcation positive solution $(u(s), v(s))$ is nondegenerate and unstable. To this end, it suffices to show that there exists a sufficiently small $\varepsilon > 0$ such that for $a \in (\tilde{a} - \varepsilon, \tilde{a})$, any positive solution $(u(s), v(s))$ of (1.2) is nondegenerate and the linearized eigenvalue problem:

$$\begin{gathered}
-\Delta\xi - \left[a(s) - u(s) - \frac{bv(s)}{(1+mu(s))^2(1+kv(s))}\right]\xi + \frac{bu(s)}{(1+mu(s))(1+kv(s))^2}\eta = \mu\xi, \quad x \in \Omega, \\
-\Delta\eta - \left[c - 2v(s) + \frac{du(s)}{(1+mu(s))(1+kv(s))^2}\right]\eta - \frac{dv(s)}{(1+mu(s))^2(1+kv(s))} = \mu\eta, \quad x \in \Omega, \\
\xi = \eta = 0, \quad x \in \partial\Omega,
\end{gathered} \tag{3.25}$$

has a unique eigenvalue μ_* and $\mathrm{Re}(\mu_*) < 0$ with algebra multiplicity one.

Let $\{\varepsilon_i > 0\}$ and $\{d_i > 0\}$ be sequences which approach 0 as $i \to \infty$. Due to $a = \tilde{a} + a'(0)s + O(s^2)$, we can set sequences $\{\varepsilon_i > 0\}$ and $\{a_i\}$ such that $a_i \in (\tilde{a} - \varepsilon_i, \tilde{a})$ and $s_i \to 0$ as $i \to \infty$. It follows that (u_i, v_i) is a solution of (1.2). Then the corresponding linearized problem (3.25) can become the following form:

$$L_i \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} = \mu_i \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}, \qquad L_i = \begin{pmatrix} M_i^{11} & M_i^{12} \\ M_i^{21} & M_i^{22} \end{pmatrix}, \tag{3.26}$$

where $(\xi_i, \eta_i) \not\equiv (0,0)$ and

$$\begin{aligned} M_i^{11} &= -\Delta - \left[a_i - 2u_i - \frac{bv_i}{(1+mu_i)^2(1+kv_i)} \right], \qquad M_i^{12} = \frac{bu_i}{(1+mu_i)(1+kv_i)^2}, \\ M_i^{21} &= -\frac{dv_i}{(1+mu_i)^2(1+kv_i)}, \qquad M_i^{22} = -\Delta - \left[c - 2v_i + \frac{du_i}{(1+mu_i)(1+kv_i)^2} \right]. \end{aligned} \tag{3.27}$$

Observe that as $i \to \infty$, $L_i \left(\begin{smallmatrix} \xi_i \\ \eta_i \end{smallmatrix} \right)$ converges to

$$L_0 \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} -\Delta\xi - \left(\tilde{a} - \dfrac{b\theta_c}{1+k\theta_c} \right)\xi & 0 \\ 0 & -\Delta\eta - (c - 2\theta_c)\eta \end{pmatrix}. \tag{3.28}$$

It is easy to get that 0 is a simple eigenvalue of the operator L_0 with corresponding eigenfunction $(\xi, \eta)^T = (\Phi, 0)^T$. Moreover, all the other eigenvalues of L_0 are positive and stand apart from 0. Therefore, using perturbation theory [24], we get that for large i, L_i has a unique eigenvalue μ_i which is close to zero. In addition, all the other eigenvalues of L_i have positive real parts and stand apart from 0. Note that μ_i is simple real eigenvalue which converges to zero, and we can take the corresponding eigenfunction $(\xi_i, \eta_i)^T$ such that $(\xi_i, \eta_i) \to (\Phi, 0)$ as $i \to \infty$. If we show that $\operatorname{Re} \mu_i < 0$ for large i, then the result follows. By multiplying Φ to the first equation of $L_i(\xi_i, \eta_i)^T = \mu_i(\xi_i, \eta_i)^T$ and integrating over Ω, we obtain

$$-\int_\Omega \Phi\Delta\xi_i - \int_\Omega \left(a_i - 2u_i - \frac{bv_i}{(1+mu_i)^2(1+kv_i)} \right)\Phi\xi_i + \int_\Omega \frac{bu_i\Phi\eta_i}{(1+mu_i)(1+kv_i)^2} = \int_\Omega \mu_i\Phi\xi_i. \tag{3.29}$$

Multiplying the first equation of (1.2) with $(a, u, v) = (a_i, u_i, v_i)$ by ξ_i and integrating, we have

$$-\int_\Omega \xi_i\Delta u_i - \int_\Omega \left(a_i - u_i - \frac{bv_i}{(1+mu_i)(1+kv_i)} \right)u_i\xi_i. \tag{3.30}$$

Due to $u_i = s_i\Phi + O(s_i^2)$, the previous equation becomes

$$-\int_\Omega \Phi\Delta\xi_i - \int_\Omega \xi_i\Phi\left(a_i - u_i - \frac{bv_i}{(1+mu_i)(1+kv_i)} \right) + O\left(s_i^2\right). \tag{3.31}$$

Using (3.29) and (3.31), we have

$$\int_\Omega \left[1 - \frac{bmv_i}{(1+mu_i)^2(1+kv_i)}\right] u_i\Phi\xi_i + \int_\Omega \frac{bu_i\Phi\eta_i}{(1+mu_i)(1+kv_i)^2} = \int_\Omega u_i\Phi\xi_i. \tag{3.32}$$

Recall that $(u_i, v_i) = (s_i\Phi + O(s_i^2), \theta_c + s_i\Psi_{d_i} + O(s_i^2))$, here Ψ_{d_i} is Ψ defined in Remark 3.2, and so dividing the previous equation by s_i and taking the limit, we have

$$\lim_{i\to\infty} \frac{\mu_i}{s_i} = \frac{\int_\Omega (1 - bm\theta_c/(1+k\theta_c))\Phi^3}{\int_\Omega \Phi^2 < 0}, \tag{3.33}$$

which implies that $\operatorname{Re}\mu_i < 0$ for large i. This proved our claim.

Next, we apply the method in [25] to show the remaining part of Theorem 3.9. A contradiction argument will be used; we assume that (1.2) has a unique positive solution $(\tilde{u}, \tilde{v})$, then this solution must be bifurcated from $(0, \theta_c)$. Since there exists a positive solution near $\tilde{a}$ by the local bifurcation theory. So $(\tilde{u}, \tilde{v})$ is nondegenerate, and the corresponding linearized eigenvalue problem has a unique eigenvalue $\tilde{\mu}$ with algebra multiplicity one such that $\operatorname{Re}\tilde{\mu} < 0$. Due to these facts, it is easy to show that $I - F'(\tilde{u}, \tilde{v})$ is invertible and does not have property α on $\overline{W}_{(\tilde{u},\tilde{v})}$; it follows that $\operatorname{index}(F, (\tilde{u}, \tilde{v})) = (-1)^1 = -1$ by Lemma 2.2 (ii). Finally, Using Lemmas 2.3–2.4 and the additivity property of the index, we obtain

$$\begin{aligned} 1 = \operatorname{index}_W(F, D) &= \operatorname{index}_W(F, (0,0)) + \operatorname{index}_W(F, (\theta_a, 0)) \\ &\quad + \operatorname{index}_W(F, (0, \theta_c)) + \operatorname{index}_W(F, (\tilde{u}, \tilde{v})) = 0 + 0 + 1 - 1 = 0, \end{aligned} \tag{3.34}$$

which derives a contradiction. Hence the Proof is complete. □

Remark 3.10. In Theorem 3.9, the multiplicity can be shown easily when $m \geq 0$. Note that $a'(0) < 0$ for a sufficiently small d, since $\int_\Omega (1 - bm\theta_c/(1+k\theta_c))\Phi^3 < 0$, and so $a = a(s) \in (\lambda_1, \tilde{a})$. Since there is no positive solution of (1.2) if $a \leq \lambda_1(b\theta_c/(1+m\theta_a)(1+k\theta_c))$ by Theorem 2.5 (i) when $m \geq 0$ and $c > \lambda_1$. Therefore we easily see that there must be at least two positive solutions for $a \in (a^*, \tilde{a})$ and some $a^* \in (\lambda_1(b\theta_c/(1+m\theta_a)(1+k\theta_c)), \lambda_1(b\theta_c/(1+k\theta_c)))$.

4. Multiplicity, Uniqueness and Stability as k or m Is Large

In this section, taking k or m as a parameter, we investigate the multiplicity, stability, and uniqueness of positive solutions of (1.2) as k or m is large. In the following, we will always assume that $c > \lambda_1$, and let b, c, d be fixed, unless otherwise specified.

Firstly, we consider the case that k is large and a is bounded away from λ_1. Hence the upper solution θ_a for u and the lower solution θ_c for v do not depend on k when $a > \lambda_1$ and $c > \lambda_1$.

Lemma 4.1. *For any small $\varepsilon > 0$, there exists $K(\varepsilon)$ such that for $k \geq K(\varepsilon)$,* (1.2) *has at least one positive solution (u, v) which satisfies*

$$\theta_{(a-\varepsilon)} \leq u \leq \theta_a, \qquad \theta_c \leq v \leq \theta_{(c+\varepsilon)}. \tag{4.1}$$

Proof. Since the proof of Lemma 4.1 is similar to the proof of Lemma 3 in [19], we omit it. □

Lemma 4.2. (i) *If $k \to \infty$, then any positive solution of* (1.2) *approaches* (θ_a, θ_c).

(ii) *There exists a large $K(\epsilon)$ such that any positive solution of* (1.2) *is nondegenerate and linearly stable when $k > K(\epsilon)$.*

Proof. (i) Let $k \to \infty$, we show that the compact operator $F(u,v)$ converges to $\widehat{F}(u,v)$, where

$$\widehat{F}(u,v) = (-\Delta + P)^{-1}\begin{pmatrix} u(a-u) + Pu \\ v(c-v) + Pv \end{pmatrix}. \tag{4.2}$$

It follows that any positive solution of (1.2) converges to the fixed point of $\widehat{F}(u,v)$ in this case. It is easy to see that (θ_a, θ_c) is a unique fixed point of $\widehat{F}(u,v)$, so the positive solutions of (1.2) could not converge to semitrivial solutions when $k \to \infty$. Thus the conclusion is complete.

(ii) We use a contradiction method. Assume that there exists $k_i \to \infty$, μ_i with $\operatorname{Re}\mu_i \le 0$ and $(\xi_i, \eta_i) \not\equiv (0,0)$ with $\|\xi_i\|_{L^2}^2 + \|\eta_i\|_{L^2}^2 = 1$ such that

$$\begin{gathered} -\Delta\xi_i - \left[a - 2u_i - \frac{bv_i}{(1+mu_i)^2(1+k_iv_i)}\right]\xi_i + \frac{bu_i}{(1+mu_i)(1+k_iv_i)^2}\eta_i = \mu_i\xi_i, \quad x \in \Omega, \\ -\Delta\eta_i - \left[c - 2v_i + \frac{du_i}{(1+mu_i)(1+k_iv_i)^2}\right]\eta_i - \frac{dv_i}{(1+mu_i)^2(1+k_iv_i)}\xi_i = \mu_i\eta_i, \quad x \in \Omega, \\ \xi_i = \eta_i = 0, \quad x \in \partial\Omega, \end{gathered} \tag{4.3}$$

where (u_i, v_i) is a positive solution of (1.2) with $k = k_i$. By computing, we have

$$\begin{aligned} \mu_i = {} & \int_\Omega |\nabla\xi_i|^2 - \int_\Omega \left(a - 2u_i - \frac{bv_i}{(1+mu_i)^2(1+k_iv_i)}\right)|\xi_i|^2 + \int_\Omega \frac{bu_i\eta_i\overline{\xi}_i}{(1+mu_i)(1+k_iv_i)^2} \\ & + \int_\Omega |\nabla\eta_i|^2 - \int_\Omega \frac{dv_i\xi_i\overline{\eta}_i}{(1+mu_i)^2(1+k_iv_i)} - \int_\Omega \left(c - 2v_i + \frac{du_i}{(1+mu_i)(1+k_iv_i)^2}\right)|\eta_i|^2, \end{aligned} \tag{4.4}$$

where $\overline{\xi}_i, \overline{\eta}_i$ are the complex conjugates of ξ_i, η_i. From Lemma 2.1, we know that $0 \le u_n \le \theta_a$ and $\theta_c \le v_i \le c + da/(1+ma)$. It follows that $\operatorname{Im}\mu_i$ is bounded and $\operatorname{Re}\mu_i$ is bounded from the following. Thus μ_i is bounded as we assume $\operatorname{Re}\mu_i \le 0$. Hence we can suppose that $\mu_i \to \mu$ and $\operatorname{Re}\mu \le 0$. Thanks to L^p estimate, we obtain $\|\xi_i\|_{W^{2,2}}$ and $\|\eta_i\|_{W^{2,2}}$ are bounded. Hence we may assume that $\xi_i \to \xi$ and $\eta_i \to \eta$ in H_0^1 strongly, here $(\xi, \eta) \not\equiv (0,0)$. Setting $i \to \infty$ in (3.1) and (3.2), we know that ξ, η satisfy the following two single equations weakly (then strongly):

$$\begin{gathered} -\Delta\xi - (a - 2\theta_a)\xi = \mu\xi, \quad x \in \Omega, \\ -\Delta\zeta - (c - 2\theta_c)\eta = \mu\eta, \quad x \in \Omega, \\ \xi = \eta = 0, \quad x \in \partial\Omega. \end{gathered} \tag{4.5}$$

Clearly $\mu \in R$. If $\xi \not\equiv 0$, then $\mu = \lambda_1(-a + 2\theta_a) > \lambda_1(-a + \theta_a) = 0$. However, $\operatorname{Re}\mu \leq 0$. Hence $\xi \equiv 0$. Similarly, we have $\eta \equiv 0$, a contradiction; hence the proof is complete. □

Theorem 4.3. (i) *Assume that* $\lambda_1 < a < \lambda_1(b\theta_c/(1 + k\theta_c))$. *Then* (1.2) *has no positive solution when* k *is sufficiently large.*

(ii) *Assume that* $a > \lambda_1(b\theta_c/(1 + k\theta_c))$ *and* k *is sufficiently large. Then* (1.2) *has a unique positive solution, and it is asymptotically stable.*

Proof. (i) Assume that there exists a positive solution (u_*, v_*) of (1.2) for sufficiently large k. It is easy to show that $\operatorname{index}_W(F, (u_*, v_*)) = 1$ by Lemma 4.2. By Lemmas 2.3–2.4 and the additivity property of the index, we obtain

$$\begin{aligned} 1 = \operatorname{index}_W(F, D') &= \operatorname{index}_W(F, (0,0)) + \operatorname{index}_W(F, (\theta_a, 0)) \\ &\quad + \operatorname{index}_W(F, (0, \theta_c)) + \operatorname{index}_W(F, (u_*, v_*)) = 0 + 0 + 1 + 1 = 2, \end{aligned} \tag{4.6}$$

which gives a contradiction.

(ii) From Theorem 2.6, the existence is trivial. Since $a > \lambda_1(b\theta_c/(1 + k\theta_c))$ and $c > \lambda_1$, for k sufficiently large, the nondegenerate positive solutions may not converge to semitrivial solutions by the proof of Lemma 4.2 (i). We need only to show the uniqueness. It follows from compactness and nondegeneracy that F has at most finitely many positive fixed points in the region D'. We denote them by (u_i, v_i) for $i = 1, 2, \ldots, l$. From the proof of (i), we obtain $\operatorname{index}_W(F, (u_i, v_i)) = 1$. Applying Lemmas 2.3–2.4 and the additivity property of the index again, we have

$$\begin{aligned} 1 = \operatorname{index}_W(F, D') &= \operatorname{index}_W(F, (0,0)) + \operatorname{index}_W(F, (\theta_a, 0)) \\ &\quad + \operatorname{index}_W(F, (0, \theta_c)) + \sum_{i=1}^{l} \operatorname{index}_W(F, (u_i, v_i)) = 0 + 0 + 0 + l = l. \end{aligned} \tag{4.7}$$

Hence the uniqueness is obtained. The stability has been given in Lemma 4.2. □

In the following, we investigate the case when m is large. This part is motivated by the work of Du and Lou in [13, 14, 18], and many of our methods used nextly come from their work.

Theorem 4.4. *For any* $\epsilon > 0$ *to be small, there that exists* $M = M(\epsilon)$ *is large such that for* $m \geq M$,

(i) *if* $a \in [\lambda_1 + \epsilon,\ \lambda_1(b\theta_c/(1 + k\theta_c)))$, *then* (1.2) *has at least two positive solutions;*

(ii) *if* $a \geq \lambda_1(b\theta_c/(1 + k\theta_c))$, *then* (1.2) *has a unique positive solution and it is asymptotically stable.*

Theorem 4.4 is the main result we will prove that when m is large. The cases $a \in [\lambda_1 + \epsilon,\ \lambda_1(b\theta_c/(1 + k\theta_c)))$ and $a \geq \lambda_1(b\theta_c/(1 + k\theta_c))$ will be treated separately. First, we deal with the multiplicity in Theorem 4.4. Similar to the method of [19], if $a \in [\lambda_1 + \epsilon,\ \infty)$, and m is sufficiently large, then we can get the following result on (1.2).

Lemma 4.5. *For any given small ϵ, there exists that $M = M(\epsilon)$ such that if $a \geq \lambda_1 + \epsilon$ and $m \geq M$,* (1.2) *has a positive solution $(\tilde{u}, \tilde{v})$, which satisfy*

$$\theta_{(a-\epsilon/2)} \leq \tilde{u} \leq \theta_a, \qquad \theta_c \leq \tilde{v} \leq \theta_{(c+\epsilon/2)}. \tag{4.8}$$

Lemma 4.6. *For any $\epsilon > 0$ to be small and any $\widehat{A} > \lambda_1$, there exists that $M = M(\epsilon, \widehat{A}) > 0$ is large such that if $a \in (\lambda_1 + \epsilon, \widehat{A}]$ and $m \geq M$, then any positive solution which satisfies* (4.8) *is nondegenerate and linearly stable.*

Proof. Assume that $a \in (\lambda_1 + \epsilon, \widehat{A}]$ and m is large, then we can easily get that (1.2) is a regular perturbation of

$$\begin{aligned} -\Delta u - (a-u)u &= 0, \quad x \in \Omega, \\ -\Delta v - (c-v)v &= 0, \quad x \in \Omega, \\ u = v &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{4.9}$$

It is well know that (4.9) has a unique positive solution (θ_a, θ_c) which is linearly stable. So the positive solutions cannot bifurcate from semitrivial ones. Hence, Lemma 4.6 is proved by a standard regular perturbation argument. We omit the details. □

Proof of (i) of Theorem 4.4. For any $\epsilon > 0$ to be small, let

$$M = \max\left\{ M(\epsilon), M\left(\epsilon, \lambda_1\left(\frac{b\theta_c}{1+k\theta_c}\right)\right)\right\}, \tag{4.10}$$

where $M(\epsilon)$ and $M(\epsilon, \lambda_1(b\theta_c/(1+k\theta_c)))$ are defined in Lemmas 4.5 and 4.6, respectively. Assume that for some $m \geq M$ and some $a \in [\lambda_1 + \epsilon,\ \lambda_1(b\theta_c/(1+k\theta_c)))$, Thus the unique positive solution $(\tilde{u}, \tilde{v})$ must be the one found in Lemma 4.5. It follows from Lemma 4.5 that $I - F'(\tilde{u}, \tilde{v})$ is invertible in X and $F'(\tilde{u}, \tilde{v})$ has no eigenvalue greater than one. Hence $\operatorname{index}_W(F, (\tilde{u}, \tilde{v})) = (-1)^0 = 1$. Applying Lemmas 2.3–2.4 and the additivity property of the index, we obtain

$$\begin{aligned} 1 = \operatorname{index}_W(F, D') &= \operatorname{index}_W(F, (0,0)) + \operatorname{index}_W(F, (\theta_a, 0)) \\ &\quad + \operatorname{index}_W(F, (0, \theta_c)) + \operatorname{index}_W(F, (\tilde{u}, \tilde{v})) = 0 + 0 + 1 + 1 = 2, \end{aligned} \tag{4.11}$$

a contradiction. Hence the proof if complete. □

Part (i) in Theorem 4.4 implies that when $a \in [\lambda_1 + \epsilon,\ \lambda_1(b\theta_c/(1+k\theta_c)))$ and m is large, (1.2) has at least two positive solutions. In the following, we will show that (1.2) has only two types of positive solutions in this case, one of which is close to (θ_a, θ_c) and asymptotically stable and the other is close to $(0, \theta_c)$ which is not stable.

Theorem 4.7. *For any $\epsilon, \delta > 0$ to be small, there exists that $M = M(\epsilon, \delta) > 0$ is large such that if $m \geq M$ and $a \in [\lambda_1 + \epsilon, \lambda_1(b\theta_c/(1+k\theta_c)))$, one obtains either (i) $\|u - \theta_a\|_{C^1} + \|v - \theta_c\|_{C^1} \leq \delta$ or (ii) $\|u\|_{C^1} + \|v - \theta_c\|_{C^1} \leq \delta$, where (u, v) is any positive solution of* (1.2). *In particular, if (ii) occurs,*

by choosing $M(\epsilon,\delta)$ suitably larger, one gets $\|mu-w\|_{C^1}\le\delta$, where w is a positive solution of the following equation:

$$-\Delta w-\left(a-\frac{b\theta_c}{(1+w)(1+k\theta_c)}\right)w=0,\quad w|_{\partial\Omega}=0. \tag{4.12}$$

Proof. Suppose that the conclusion does not hold. Then there exist $m_i\to\infty$ $a_i\in[\lambda_1+\epsilon,\lambda_1(b\theta_c/(1+k\theta_c)))$ and a positive solution (u_i,v_i) of (1.2) with $(a,m)=(a_i,m_i)$ such that (u_i,v_i) is bounded away from (θ_a,θ_c) and $(0,\theta_c)$. We may assume that $a_i\to a\in[\lambda_1+\epsilon,\lambda_1(b\theta_c/(1+k\theta_c))]$ and $(1/(1+m_iu_i)(1+kv_i))\xrightarrow{L^2}h$ with $0<h<1$. Since $\theta_c\le v_i\le\theta_{(c+d/m_i)}$, we have $v_i\xrightarrow{C^1}\theta_c$ as $i\to\infty$. Thanks to L^p estimate and Sobolev embedding theorems, we may assume that $u_i\xrightarrow{C^1}u$ and u satisfies

$$\Delta u+(a-u-b\theta_ch)u=0,\quad u\ge0,\ u|_{\partial\Omega}=0. \tag{4.13}$$

If $u\equiv0$, then $(u_i,v_i)\xrightarrow{C^1}(u,\theta_c)\equiv(0,\theta_c)$, which contradicts our assumption that (u_i,v_i) is bounded away from $(0,\theta_c)$. If $u\ge0,\not\equiv0$, from maximum principle, we have $u>0$ in Ω. Hence $h\equiv0$ and $u\equiv\theta_a$, which also contradicts our assumption. Thus, the first part of the proof is complete.

To complete the proof, it suffices to prove that if $m_i\to\infty$, $a_i\in[\lambda_1+\epsilon,\lambda_1(b\theta_c/(1+k\theta_c)))$ and $\|u_i\|_{C^1}+\|v_i-\theta_c\|_{C^1}\to0$, then m_iu_i approaches some positive solution of (4.12) with $a=a_i$ in the C^1 norm. It is easy to see that (4.12) has a positive solution when $a\in[\lambda_1+\epsilon,\lambda_1(b\theta_c/(1+k\theta_c)))$. First we claim that $m_i\|u_i\|_\infty$ is uniformly bounded. If this is not true, we may assume that $m_i\|u_i\|_\infty\to\infty$. Let $\tilde{u}_i=u_i/\|u_i\|_\infty$. Then we have

$$\Delta\tilde{u}_i+\left(a_i-u_i-\frac{bv_i}{(1+m_iu_i)(1+kv_i)}\right)\tilde{u}_i=0,\quad \|\tilde{u}_i\|_\infty=1,\ \tilde{u}_i|_{\partial\Omega}=0. \tag{4.14}$$

Thanks to standard elliptic regularity theory, we may assume that $\tilde{u}_i\xrightarrow{C^1}\tilde{u}$, $(1/((1+m_iu_i)(1+kv_i)))\xrightarrow{L^2}h$ and $a_i\to a\in[\lambda_1+\epsilon,\ \lambda_1(b\theta_c/(1+k\theta_c))]$. By taking the limit in (4.12), we know that $\tilde{u}$ satisfies the following equation weakly:

$$\Delta\tilde{u}+(a-b\theta_ch)\tilde{u}=0,\quad \|\tilde{u}\|_\infty=1,\ \tilde{u}|_{\partial\Omega}=0. \tag{4.15}$$

By Harnack's inequality, we have $\tilde{u}>0$ in Ω. Since $m_i\|u_i\|_\infty\to\infty$ and $\tilde{u}_i\to\tilde{u}$, then $1/(1+m_iu_i)(1+kv_i)=1/(1+m_i\|u_i\|_\infty\tilde{u}_i)(1+kv_i)\to0$ in any compact subset of Ω. Hence $h\equiv0$ and $a=\lambda_1$, which contradicts the assumption that $a\ge\lambda_1+\epsilon$. Therefore $m_i\|u_i\|_\infty$ is uniformly bounded.

Let $w_i=m_iu_i$. Then w_i satisfies

$$\Delta w_i+\left(a_i-u_i-\frac{bv_i}{(1+w_i)(1+kv_i)}\right)w_i=0,\quad w_i|_{\partial\Omega}=0. \tag{4.16}$$

Since $\|w_i\|_\infty = m_i\|u_i\|_\infty$ is bounded, due to standard elliptic regularity theory and Sobolev embedding theorems, we may assume that $w_i \xrightarrow{C^1} w$. By letting $i \to \infty$ in (4.16), we know that w is a nonnegative solution of (4.12). There are two possibilities as follows.

(i) $a = \lambda_1(b\theta_c/(1 + k\theta_c))$. For this case, $m_i u_i = w_i \to w \equiv 0$. Since any positive solution of (4.12) with $a = a_i$ approaches zero when $a_i \to a$, $m_i u_i$ is certainly close to positive solutions of (4.12) with $a = a_i$.

(ii) $\lambda_1+\epsilon \le a < \lambda_1(b\theta_c/(1 + k\theta_c))$. In this case, we will prove that w is a positive solution of (4.12). If not, by Harnack's inequality, we have $w \equiv 0$. Let $\tilde{w}_i = w_i/\|w_i\|_\infty$. Then we have

$$\Delta\tilde{w}_i + \left(a_i - u_i - \frac{bv_i}{(1 + w_i)(1 + kv_i)}\right)\tilde{w}_i = 0, \quad \tilde{w}_i|_{\partial\Omega} = 0. \tag{4.17}$$

Hence we may assume that $\tilde{w}_i \xrightarrow{C^1} w$. By taking the limit in (4.17), we find

$$\Delta\tilde{w} + \left(a - \frac{b\theta_c}{1 + k\theta_c}\right)\tilde{w} = 0, \quad \tilde{w}|_{\partial\Omega} = 0. \tag{4.18}$$

Since $a < \lambda_1(b\theta_c/(1+k\theta_c))$, we must have $\tilde{w} \equiv 0$, which contradicts $\|\tilde{w}\|_\infty = \lim_{i\to\infty}\|\tilde{w}_i\|_\infty = 1$. The proof is complete. □

The proof of (ii) in Theorem 4.4 is more difficult. To this end, we need several lemmas. First we show that there is no positive solution of (1.2) with small u component if $a \ge \lambda_1(b\theta_c/(1 + k\theta_c))$ and m is large.

Lemma 4.8. *There exists a large M such that if $m \ge M$, then for all $a \ge \lambda_1(b\theta_c/(1 + k\theta_c))$, any positive solution (u, v) of (1.2) satisfies $u \ge \theta_{\tilde{\lambda}}$, where $\tilde{\lambda} = (\lambda_1 + \lambda_1(b\theta_c/(1 + k\theta_c)))/2$.*

Proof. Suppose that our conclusion is not true, then there exist $m_i \to \infty$, $a_i \ge \lambda_1(b\theta_c/(1 + k\theta_c))$, and a positive solution sequence $\{(u_i, v_i)\}$ of (1.2) with $(a, m) = (a_i, m_i)$ such that $u_i \ge \theta_{\tilde{\lambda}}$ does not hold.

First, let $a_i \to a \in (\lambda_1(b\theta_c/(1+k\theta_c)), \infty]$. Since $\theta_c \le v_i \le \theta_{(c+d/m_i)}$, for large i, we obtain

$$-\Delta u_i \ge \left(\frac{a + \lambda_1(b\theta_c/(1 + k\theta_c))}{2} - u_i - \frac{b\theta_{(c+\delta)}}{1 + k\theta_{(c+\delta)}}\right)u_i, \tag{4.19}$$

where $\delta > 0$ is small such that $\lambda_1(b\theta_{(c+\delta)}/(1 + k\theta_{(c+\delta)})) < (a + \lambda_1(b\theta_c/(1 + k\theta_c)))/2$, which is possible when $a > \lambda_1(b\theta_c/(1 + k\theta_c))$. Due to the super- and subsolution method, we obtain $u_i \ge \tilde{w}$, where $\tilde{w}$ is a unique positive solution of

$$-\Delta\tilde{w} = \left(\frac{a + \lambda_1(b\theta_c/(1 + k\theta_c))}{2} - \frac{b\theta_{(c+\delta)}}{1 + k\theta_{(c+\delta)}} - \tilde{w}\right)\tilde{w}, \quad \tilde{w}|_{\partial\Omega} = 0. \tag{4.20}$$

Hence for large i, we get

$$-\Delta u_i \geq \left(\lambda_1\left(\frac{b\theta_c}{1+k\theta_c}\right) - u_i - \frac{b}{m_i}\sup_{\Omega}\frac{\theta_{(c+d/m_i)}}{\tilde{w}}\right)u_i \geq \left(\tilde{\lambda} - u_i\right)u_i. \tag{4.21}$$

Applying the super- and subsolution method again, we obtain $u_i \geq \theta_{\tilde{\lambda}}$, which contradicts the assumption.

Secondly, we consider the case that $a_i \to a = \lambda_1(b\theta_c/(1+k\theta_c))$. Thanks to standard elliptic regularity theory, we may assume that $u_i \to u, v_i \to v$ in C^1 and $1/(1+m_iu_i)(1+kv_i)$ weakly converge to h in L^2 with $0 \leq h \leq 1$. Hence u satisfies the following equation weakly:

$$-\Delta u = \left(\lambda_1\left(\frac{b\theta_c}{1+k\theta_c}\right) - bvh - u\right)u, \quad u|_{\partial\Omega} = 0. \tag{4.22}$$

If $u \geq 0, \not\equiv 0$, by Harnack's inequality, we have $u > 0$ in Ω. Thus $h \equiv 0$ and $u_i \xrightarrow{C^1} \theta_{\lambda_1(b\theta_c/(1+k\theta_c))}$. Since $\theta_{\lambda_1(b\theta_c/(1+k\theta_c))} > \theta_{\tilde{\lambda}}$, we have $u_i \geq \theta_{\tilde{\lambda}}$ for large i. This contradicts our assumption at the beginning of the proof. If $u \equiv 0$, by letting $\hat{u}_i = u_i/\|u_i\|_\infty$, then we know that $\hat{u}_i$ satisfies

$$\Delta\hat{u}_i + \left(a_i - u_i - \frac{bv_i}{(1+m_iu_i)(1+kv_i)}\right)\hat{u}_i = 0, \quad \hat{u}_i|_{\partial\Omega} = 0. \tag{4.23}$$

Thanks to standard elliptic regularity theory, we may assume that $\hat{u}_i \xrightarrow{C^1} \hat{u}$ with $\|\hat{u}\|_\infty = 1$ and $\hat{u} > 0$ in Ω. Since $u_i \xrightarrow{C^1} u \equiv 0$, we have $v_i \xrightarrow{C^1} \theta_c$. By taking the weak limit in (4.23), we have

$$\Delta\hat{u} + \left(\lambda_1\left(\frac{b\theta_c}{1+k\theta_c}\right) - b\theta_c h\right)\hat{u} = 0, \quad \hat{u}|_{\partial\Omega} = 0. \tag{4.24}$$

Let $\Phi > 0$ be the positive solution to

$$-\Delta\Phi + \frac{b\theta_c}{1+k\theta_c}\Phi = \lambda_1\left(\frac{b\theta_c}{1+k\theta_c}\right)\Phi, \quad \Phi|_{\partial\Omega} = 0, \ \|\Phi\|_\infty = 1. \tag{4.25}$$

Multiplying (4.24) by Φ and integrating, we obtain

$$b\int_\Omega \hat{u}\Phi\theta_c\left(\frac{1}{1+k\theta_c} - h\right) = 0. \tag{4.26}$$

Since $0 \leq h \leq 1/(1+k\theta_c)$, we must have $h \equiv 1/(1+k\theta_c)$ and $\hat{u} \equiv \Phi$. Investigate the following equation for u_i:

$$\Delta u_i + \left(a_i - u_i - \frac{bv_i}{(1+m_iu_i)(1+kv_i)}\right)u_i = 0, \quad u_i|_{\partial\Omega} = 0. \tag{4.27}$$

Multiplying (4.27) by Φ and using (4.25), we obtain

$$\left(a_i - \lambda_1\left(\frac{b\theta_c}{1+k\theta_c}\right)\right)\int_\Omega u_i\Phi = \int_\Omega u_i^2\Phi + \int_\Omega \frac{bu_iv_i\Phi}{(1+m_iu_i)(1+kv_i)} - \int_\Omega \frac{b\theta_c u_i\Phi}{1+k\theta_c}. \tag{4.28}$$

After some rearrangements, we obtain

$$\begin{aligned}\left(a_i - \lambda_1\left(\frac{b\theta_c}{1+k\theta_c}\right)\right)\int_\Omega u_i\Phi = &\int_\Omega u_i^2\Phi + b\int_\Omega \frac{u_i(v_i-\theta_c)\Phi}{(1+m_iu_i)(1+kv_i)(1+k\theta_c)} \\ &- bm_i\int_\Omega \frac{\theta_c u_i^2\Phi}{(1+m_iu_i)(1+k\theta_c)}.\end{aligned} \tag{4.29}$$

Set $w_i = (v_i - \theta_c)/\|u_i\|_\infty$. By the definition of v_i and θ_c, we have

$$-\Delta w_i + (-c+2\theta_c)w_i = (\theta_c - v_i)w_i + \frac{dv_i\widehat{u}_i}{(1+m_iu_i)(1+kv_i)}. \tag{4.30}$$

Multiplying (4.30) by w_i and integrating, we obtain

$$\begin{aligned}\lambda_1(-c+2\theta_c)\int_\Omega w_i^2 \le &\int_\Omega \left(|\nabla w_i|^2 + (-c+2\theta_{[c]})w_i^2\right) \\ \le &\|v_i-\theta_c\|_\infty \int_\Omega w_i^2 + d\|\widehat{u}_i\|_\infty\|v_i\|_\infty\int_\Omega w_i.\end{aligned} \tag{4.31}$$

Since $\|v_i - \theta_c\|_\infty \to 0$, $\|\widehat{u}_i\|_\infty$ and $\|v_i\|_\infty$ are bounded, we know that $\|w_i\|_2$ is bounded by Hölder inequality. Therefore from L^p estimate and Sobolev embedding theorem, it is easy to see that $\|w_i\|_\infty$ is bounded. Dividing (4.29) by $\|u_i\|_\infty^2$, we show that

$$\begin{aligned}\frac{a_i - \lambda_1(b\theta_c/(1+k\theta_c))}{\|u_i\|_\infty}\int_\Omega \widehat{u}_i\Phi = &\int_\Omega \widehat{u}_i^2\Phi + b\int_\Omega \frac{\widehat{u}_iw_i\Phi}{(1+m_iu_i)(1+kv_i)(1+k\theta_c)} \\ &- bm_i\int_\Omega \frac{\theta_c\widehat{u}_i^2\Phi}{(1+m_iu_i)(1+k\theta_c)}.\end{aligned} \tag{4.32}$$

Since $\widehat{u}_i \xrightarrow{C^1} \Phi > 0$, $1/(1+m_iu_i)(1+kv_i) \rightharpoonup 1/(1+k\theta_c)$ weakly in L^2 and w_i are uniformly bounded, we obtain

$$\frac{a_i - \lambda_1(b\theta_c/(1+k\theta_c))}{\|u_i\|_\infty} \longrightarrow -\infty \tag{4.33}$$

as $i \to \infty$. Hence $a_i < \lambda_1(b\theta_c/(1+k\theta_c))$ if i is large enough, which contradicts our assumption that $a_i \ge \lambda_1(b\theta_c/(1+k\theta_c))$ for all i. Thus, the proof is complete. □

By Lemma 4.8 and a simple variant of the proof of Lemma 4.6, we immediately get the following result.

Lemma 4.9. *For any given $\overline{A} > \lambda_1(b\theta_c/(1+k\theta_c))$, there exists that $M = M(\overline{A}) > 0$ is large such that if $a \in [\lambda_1(b\theta_c/(1+k\theta_c)), \overline{A}]$ and $m \geq M$; then any positive solution of* (1.2) *is nondegenerate and linearly stable.*

Next we investigate the case that a is large.

Lemma 4.10. *For any $\epsilon > 0$, there exists that $\widetilde{A} = \widetilde{A}(\epsilon) > 0$ is large such that if $m \geq \epsilon, c \leq 1/\epsilon$ and $a > \widetilde{A}$, then any positive solution of* (1.2) *is nondegenerate and linearly stable.*

Proof. Suppose the conclusion is not true. Then there exist some $\epsilon_0 > 0$, $m_i \geq \epsilon_0$, $c_i \leq 1/\epsilon_0$, $a_i \to \infty$, $\operatorname{Re}\eta_i \leq 0$, and $(\phi_i, \varphi_i) \not\equiv (0,0)$ with $\|\phi_i\|_2^2 + \|\varphi_i\|_2^2 = 1$ such that

$$\Delta\phi_i + \left(a_i - 2u_i - \frac{bv_i}{(1+m_iu_i)^2(1+kv_i)}\right)\phi_i - \frac{bu_i}{(1+m_iu_i)(1+kv_i)^2}\varphi_i + \eta_i\phi_i = 0, \tag{4.34}$$

$$\Delta\varphi_i + \left(c_i - 2v_i + \frac{du_i}{(1+m_iu_i)(1+kv_i)^2}\right)\varphi_i + \frac{dv_i}{(1+m_iu_i)^2(1+kv_i)}\phi_i + \eta_i\varphi_i = 0, \tag{4.35}$$

where (u_i, v_i) is a positive solution of (1.2) with $(a, c, m) = (a_i, c_i, m_i)$. Since $c_i > \lambda_1 - d/m_i$, we may assume that $c_i \to c \in [\lambda_1 - d/\epsilon_0, (1/\epsilon_0)]$, $m_i \to m \in [\epsilon_0, \infty]$. Let $\delta = b(1+d)/(\epsilon_0 + k(1+d))$. From $v_i \leq \theta_{(c_i+d/m_i)} \leq (1+d)/\epsilon_0$, we have

$$-\Delta u_i \geq \left(a_i - u_i - \frac{bv_i}{1+kv_i}\right)u_i \geq (a_i - \delta - u_i). \tag{4.36}$$

Hence $u_i \geq \theta_{(a_i-\delta)}$. Due to Kato's inequality, we have

$$\begin{aligned} -\Delta|\phi_i| &\leq -\operatorname{Re}\left(\frac{\overline{\phi}_i}{|\phi_i|\Delta\phi_i}\right) \\ &\leq \left(a_i - 2u_i - \frac{bv_i}{(1+m_iu_i)^2(1+kv_i)}\right)|\phi_i| + \frac{bu_i|\varphi_i|}{(1+m_iu_i)(1+kv_i)^2} + \operatorname{Re}\eta_i|\phi_i| \\ &\leq (a_i - 2u_i)|\phi_i| + \frac{b|\varphi_i|}{m_i}. \end{aligned} \tag{4.37}$$

Multiplying (4.37) by $|\phi_i|$ and integrating by parts, we obtain

$$\begin{aligned} \lambda_1(-a_i + 2\theta_{(a_i-\delta)})\int_\Omega |\phi_i|^2 &\leq \lambda_1(-a_i + 2u_i)\int_\Omega |\phi_i|^2 \\ &\leq \int_\Omega \left(|\nabla|\phi_i||^2 + (-a_i + 2u_i)|\phi_i|^2\right) \leq \frac{b}{m_i}\int_\Omega |\phi_i||\varphi_i| \leq C. \end{aligned} \tag{4.38}$$

By the method of Lemma 2.2 in [18], there exists $k_0 \in (1, 2^{2/3})$ such that

$$\begin{aligned}\lambda_1\left(-a_i + 2\theta_{(a_i-\delta)}\right) &= -a_i + \lambda_1\left(2\theta_{(a_i-\delta)}\right) \geq -a_i + k_0(a_i - \delta) \\ &= (k_0 - 1)a_i - k_0\delta \longrightarrow +\infty.\end{aligned} \tag{4.39}$$

It follows from (4.38) and (4.39) that $\|\phi_i\|_2 \to 0$. Multiplying (4.34) by $\overline{\phi}_i$ and integrating, we have

$$\int_\Omega |\nabla\phi_i|^2 = \int_\Omega \left[a_i - 2u_i - \frac{bv_i}{(1+m_iu_i)^2(1+kv_i)}\right]|\phi_i|^2 - b\int_\Omega \frac{u_i\varphi_i\overline{\phi}_i}{(1+m_iu_i)(1+kv_i)^2} + \eta_i\int_\Omega |\phi_i|^2. \tag{4.40}$$

Multiplying (4.35) by $\overline{\varphi}_i$ and integrating, we have

$$\int_\Omega |\nabla\varphi_i|^2 = \int_\Omega \left[c_i - 2v_i + \frac{du_i}{(1+m_iu_i)(1+kv_i)^2}\right]|\varphi_i|^2 + d\int_\Omega \frac{v_i\phi_i\overline{\varphi}_i}{(1+m_iu_i)^2(1+kv_i)} + \eta_i\int_\Omega |\varphi_i|^2. \tag{4.41}$$

Adding the previous two identities, we have

$$\begin{aligned}\eta_i = &\int_\Omega |\nabla\phi_i|^2 - \int_\Omega \left[a_i - 2u_i - \frac{bv_i}{(1+m_iu_i)^2(1+kv_i)}\right]|\phi_i|^2 + b\int_\Omega \frac{u_i\varphi_i\overline{\phi}_i}{(1+m_iu_i)(1+kv_i)^2} \\ &+ \int_\Omega |\nabla\varphi_i|^2 - \int_\Omega \left[c_i - 2v_i + \frac{du_i}{(1+m_iu_i)(1+kv_i)^2}\right]|\varphi_i|^2 - d\int_\Omega \frac{v_i\phi_i\overline{\varphi}_i}{(1+m_iu_i)^2(1+kv_i)}.\end{aligned} \tag{4.42}$$

It is easy to show that the imaginary part of the right-hand side of the previous identity is bounded. On the other hand, due to (4.38), (4.39) and the fact that $\int_\Omega |(\nabla|\phi_i|)|^2 \leq \int_\Omega |\nabla\phi_i|^2$, we have $\operatorname{Re}\eta_i$ is bounded nextly. Hence η_i is bounded as we assume that $\operatorname{Re}\eta_i \leq 0$. Thus we may assume that $\eta_i \to \eta$ with $\operatorname{Re}\eta \leq 0$. Thanks to (4.35) and standard elliptic regularity theory, $\|\varphi_i\|_{W^{2,2}}$ is bounded. Therefore we may assume that $\varphi_i \to \varphi$ in H_0^1. Since $a_i \to \infty, \theta_{(a_i-\delta)} \leq u_i \leq \theta_{(a_i)}$ and $\theta_{(a_i)}/a_i \to 1$, we have $u_i \to \infty$ and $v_i \to \theta_{(c+d/m)}$ in C^1. By letting $i \to \infty$ in (4.35), we know that φ satisfies the following equation weakly:

$$\Delta\varphi + \left(c + \frac{d}{m} - 2\theta_{(c+d/m)}\right)\varphi + \eta\varphi = 0 \tag{4.43}$$

with $\operatorname{Re}\eta \leq 0$. The self-disjointness of the previous problem implies $\eta \in R$. Since $\lambda_1(\theta_{(c+d/m)} - (c + d/m)) = 0$, we have

$$\eta = \lambda_1\left(2\theta_{(c+d/m)} - \left(c + \frac{d}{m}\right)\right) \geq \lambda_1\left(\theta_{(c+d/m)} - \left(c + \frac{d}{m}\right)\right) = 0. \tag{4.44}$$

Hence $\eta = 0$, which implies $c + d/m = \lambda_1$, $\theta_{(c+d/m)} \equiv 0$ and $\varphi = \beta\Phi_1/\|\Phi_1\|_2$, $|\beta| = 1$. Using Kato's inequality again, we have

$$\begin{aligned} -\Delta|\varphi_i| &\leq -\operatorname{Re}\left(\frac{\overline{\varphi}_i}{|\varphi_i|\Delta\varphi_i}\right) \\ &\leq \left[c_i - 2v_i + \frac{du_i}{(1+m_iu_i)(1+kv_i)^2}\right]|\varphi_i| + \frac{dv_i|\phi_i|}{(1+m_iu_i)^2(1+kv_i)} + \operatorname{Re}\eta_i|\varphi_i|. \end{aligned} \tag{4.45}$$

It follows that v_i satisfies

$$\Delta v_i + \left(c_i - v_i + \frac{du_i}{(1+m_iu_i)(1+kv_i)}\right)v_i = 0, \quad v_i|_{\partial\Omega} = 0. \tag{4.46}$$

Multiplying (4.45) by v_i, integrating by parts, and using (4.46), we obtain

$$\int_\Omega v_i^2|\varphi_i| \leq d\int_\Omega \frac{v_i^2|\phi_i|}{(1+m_iu_i)^2(1+kv_i)} + \operatorname{Re}\eta_i\int_\Omega v_i|\varphi_i| \leq d\int_\Omega v_i^2|\phi_i|. \tag{4.47}$$

Set $\widehat{v}_i = v_i/\|v_i\|_\infty$. Then $\widehat{v}_i$ satisfies

$$\Delta\widehat{v}_i + \left(c_i - v_i + \frac{du_i}{(1+m_iu_i)(1+kv_i)}\right)\widehat{v}_i = 0. \tag{4.48}$$

Thanks to standard elliptic regularity theory, we may assume that $\widehat{v}_i \xrightarrow{C^1} \widehat{v}$. Then $\widehat{v}$ satisfies

$$\Delta\widehat{v} + \left(c + \frac{d}{m} - \theta_{(c+d/m)}\right)\widehat{v} = 0. \tag{4.49}$$

Since $c + d/m = \lambda_1$, we have $\widehat{v} \equiv \Phi_1$. Dividing both sides of (4.47) by $\|v_i\|_\infty^2$, we know that

$$\int_\Omega \left(\frac{v_i}{\|v_i\|_\infty}\right)^2|\varphi_i| \leq d\int_\Omega \left(\frac{v_i}{\|v_i\|_\infty}\right)^2|\phi_i|. \tag{4.50}$$

Since $\|\phi_i\|_2 \to 0$, the right-hand side of (4.50) converges to 0 by Hölder inequality. However, the previous discussion implies that $\int_\Omega (v_i/\|v_i\|_\infty)^2|\varphi_i| \to \int_\Omega(\Phi_1^3/\|\Phi_1\|_2)(i \to \infty)$. The contradiction completes the proof. □

Proof of (ii) of Theorem 4.4. It suffices to prove the uniqueness. Investigate the following system with $t \in [0,1]$:

$$\begin{gathered}\Delta u + u\left(a - u - \frac{tbv}{(1+mu)(1+kv)}\right) = 0, \quad x \in \Omega,\\ \Delta v + v\left(c - v + \frac{tdu}{(1+mu)(1+kv)}\right) = 0, \quad x \in \Omega,\\ u = v = 0, \quad x \in \partial\Omega.\end{gathered} \tag{4.51}$$

Set $S = \{(u,v) \in X \mid \ \theta_{\tilde{\lambda}}/2 < u < a, \theta_c/2 < v < c + ad\}$ and $B_t : \overline{S} \to W$ by

$$B_t(u,v) = (-\Delta + P)^{-1}\begin{pmatrix} u\left(a + P - u - \dfrac{tbv}{(1+mu)(1+kv)}\right) \\ v\left(c + P - v + \dfrac{tdu}{(1+mu)(1+kv)}\right)\end{pmatrix}, \tag{4.52}$$

where $P = \max\{ad,\ b(c+ad)\}$. Thanks to standard regularity results, we can prove that B_t is a completely continuous operator. Clearly, (u,v) is a positive solution of (4.51) if and only if it is a positive fixed point of B_t in S. For $m \geq M$ and $a \geq \lambda_1(b\theta_c/(1+k\theta_c))$, we first show that B_t has no fixed point on ∂S.

We claim that any positive solution (u,v) of (4.51) satisfies $u \geq \theta_{\tilde{\lambda}}$, where $\tilde{\lambda} = (\lambda_1 + \lambda_1(b\theta_c/(1+k\theta_c)))/2$. Suppose that this claim is not true. Then there exist $m_i \to \infty$, $a_i \geq \lambda_1(b\theta_c/(1+k\theta_c))$, $t_i \in [0,1]$, and a positive solution (u_i, v_i) of (4.51) with$(a,m,t) = (a_i, m_i, t_i)$ such that $u_i \geq \theta_{\tilde{\lambda}}$ fails. Since the case that $t_i \to t_0 \equiv 1$ is considered in Lemma 4.8, it remains to discuss the case that $t_i \to t_0 \in [0,1)$. Since $v_i \leq \theta_{(c+d/m_i)}$, we have

$$-\Delta u_i \geq \left(\lambda_1\left(\frac{b\theta_c}{1+k\theta_c}\right) - \frac{t_i b\theta_{(c+d/m_i)}}{1+k\theta_c} - u_i\right)u_i \geq \left(\lambda_1\left(\frac{\bar{t}b\theta_c}{1+k\theta_c}\right) - \frac{t_0 b\theta_c}{1+k\theta_c} - u_i\right)u_i, \tag{4.53}$$

for large i, where $\bar{t} \in (t_0, 1)$. Therefore u_i is a supersolution to

$$-\Delta\omega = \left(\lambda_1\left(\frac{\bar{t}b\theta_c}{1+k\theta_c}\right) - \frac{t_0 b\theta_c}{1+k\theta_c} - \omega\right)\omega, \qquad \omega|_{\partial\Omega} = 0. \tag{4.54}$$

Due to the choice of $\bar{t}$, (4.54) has a unique positive solution ω. Thus we have $u_i \geq \omega$ for all large i. Hence,

$$-\Delta u_i \geq \left(\lambda_1\left(\frac{b\theta_c}{1+k\theta_c}\right) - \frac{b}{m_i}\sup_{\Omega}\frac{\theta_{(c+d/m_i)}}{\omega} - u_i\right)u_i \geq \left(\tilde{\lambda} - u_i\right)u_i. \tag{4.55}$$

By the super- and subsolution method again, we have $u_i \geq \theta_{\tilde{\lambda}}$ for large i. This is a contradiction.

Our claim implies that B_t has no fixed point on ∂S. Hence $\text{index}_W(B_t, S) \equiv \text{constant}$. In particular,

$$\text{index}_W(B_1, S) = \text{index}_W(B_0, S). \tag{4.56}$$

Since B_0 has a unique fixed point (θ_a, θ_c) in S and $\text{index}_W(B_0, (\theta_a, \theta_c)) = 1$, we have $\text{index}_W(B_1, S) = 1$. From Lemmas 4.8–4.10, we see that for $m \geq M$, all fixed points of B_1 fall into S, and they are nondegenerate and linearly stable. Then by compactness, there are at most finitely many fixed points of B_1, which we denote by $\{(u_i, v_i)\}_{i=1}^{l}$. As shown in the proof of part (i), we have $\text{index}_W(B_1, (u_i, v_i)) = 1$. Using the additivity property of the index, we know that

$$1 = \text{index}_W(B_1, S) = \sum_{i=1}^{l} \text{index}_W(B_1, (u_i, v_i)) = l. \tag{4.57}$$

Hence for $m \geq M$ and $a \geq \lambda_1(b\theta_c/(1 + k\theta_c))$, (1.2) has a unique positive solution, and it is stable. □

Our final task is to establish the exact multiplicity and stability results for large m and a close to $\lambda_1 + \varepsilon$ or $\lambda_1(b\theta_c/(1 + k\theta_c))$. Firstly we consider the elliptic equation (4.12), which acts as a limiting problem of (1.2) when $m \to \infty$. Applying the similar method to Lemma 2.7 in paper [18], we have the following conclusion.

Lemma 4.11. *The problem* (4.12) *has a positive solution if and only if* $\lambda_1 < a < \lambda_1(b\theta_c/(1 + k\theta_c))$. *Moreover, all positive solutions of* (4.12) *are unstable. Furthermore, there exists some* $\epsilon_1 > 0$ *such that if* $a \in (\lambda_1, \lambda_2] \cup [\lambda_1(b\theta_c/(1 + k\theta_c)) - \epsilon_1, \lambda_1(b\theta_c/(1 + k\theta_c)))$, *then* (4.12) *has at most one positive solution and it is nondegenerate (if it exists).*

Define $\epsilon_0 = \min\{\lambda_2, \lambda_1(b\theta_c/(1 + k\theta_c)) - \epsilon_1\} - \lambda_1$, *where* ϵ_1 *is defined by Lemma 4.11.*

Theorem 4.12. *For any* $\epsilon \in (0, \epsilon_0)$, *one can find that* $M = M(\epsilon)$ *is large such that if* $a \in (\lambda_1 + \epsilon, \lambda_2 + \epsilon_0] \cup [\lambda_1(b\theta_c/(1 + k\theta_c)) - \epsilon_1, \lambda_1(b\theta_c/(1 + k\theta_c)))$, *and* $m > M(\epsilon)$, *then* (1.2) *has exactly two positive solutions, one asymptotically stable and the other unstable.*

To verify Theorem 4.12, we need some intermediate results. Theorem 4.7 has shown that (1.2) has only two types of positive solutions for when m large and $a \in [\lambda_1(b\theta_c/(1 + k\theta_c)) - \epsilon_1, \lambda_1(b\theta_c/(1 + k\theta_c)))$. In the following lemma, we will prove another result.

Lemma 4.13. *There exist that* $\epsilon_2 > 0$ *is small and* $M_1 > 0$ *is large; both depend only on* b, c, d, *and* k, *such that if* $a \in [\lambda_1(b\theta_c/(1 + k\theta_c)) - \epsilon_2, \lambda_1(b\theta_c/(1 + k\theta_c)))$, *and* $m > M_1$, *then* (1.2) *has exactly two positive solutions, one asymptotically stable and the other unstable.*

Proof. First we prove that for large m, (1.2) has a unique asymptotically stable positive solution of type (i) in Theorem 4.7. In fact, if we choose δ small enough in Theorem 4.7, then any positive solution of (1.2) of type (i) satisfies (4.8). Hence by Lemma 4.6, they are nondegenerate and linearly stable. Now by a simple variant of the proof of part (ii) of Theorem 4.4, we find that there is only one positive solution of (1.2) satisfying type (i), and it is asymptotically stable.

Next we show that (1.2) has a unique unstable positive solution of type (ii). If we can prove this, then by Theorem 4.7, our proof of Lemma 4.13 is complete. Due to Theorem 4.7 and Lemmas 4.11, if any solution $(u;v)$ of (1.1) is close to $(0,\theta_c)$, then mu must be close to w, where w is the unique positive solution of (4.12). Hence to prove uniqueness, it suffices to show that for $a \in [\lambda_1(b\theta_c/(1+k\theta_c)) - \epsilon_2, \lambda_1(b\theta_c/(1+k\theta_c)))$, and $m > M_1$, there is a unique pair (mu,v), $(u;v)$ being a positive solution of (1.2), close to (w,θ_c) for certain ϵ_2 and M_1. Set $U = mu$, $\rho = 1/m$, and discuss

$$\begin{gathered}
-\Delta U = U\left(a - \rho U - \frac{bv}{(1+U)(1+kv)}\right), \quad x \in \Omega, \\
-\Delta v = v\left(c - v + \frac{\rho d U}{(1+U)(1+kv)}\right), \quad x \in \Omega, \\
U = v = 0, \quad x \in \partial\Omega.
\end{gathered} \tag{4.58}$$

Clearly (u,v) solves (1.2) if and only if (mu,v) solves (4.58) with $\rho = 1/m$. Thus it suffices to prove uniqueness for (4.58). For fixed $\rho \geq 0$, regarding a as a parameter, we know that $(\lambda_1(b\theta_c/(1+k\theta_c)),0,\theta_c)$ is a simple bifurcation point of (4.58). Due to a variant of Theorem 1 in [21], there exist $\delta_1 > 0$ and C^1 curves

$$\Gamma_\rho = \{(a(\rho,s), U(\rho,s), v(\rho,s)) : 0 < s \leq \delta_1\}, \quad 0 \leq \rho \leq \delta_1, \tag{4.59}$$

such that, if $0 \leq \rho \leq \delta_1$, then all positive solutions of (4.58) are close to

$$\left(\lambda_1\left(\frac{b\theta_c}{1+k\theta_c}\right), 0, \theta_c\right) = (a(0,0),\ U(0,0),\ v(0,0)) \in \Gamma_\rho. \tag{4.60}$$

Hence we need only to prove that these curves uniformly cover $a-$ Range$[\lambda_1(b\theta_c/(1+k\theta_c)) - \epsilon_2, \lambda_1(b\theta_c/(1+k\theta_c)))$, for suitably chosen ϵ_2, and for fixed ϵ_2 and ρ, Γ_ρ cover the range only once. It is easy to obtain (see Theorem 3.9)

$$\frac{\partial a}{\partial s}(0,0) < 0, \quad 0 \leq \rho \leq \delta_1, \quad 0 < s \leq \delta_1. \tag{4.61}$$

Hence

$$\lambda_1\left(\frac{b\theta_c}{1+k\theta_c}\right) - a(0,\delta_1) = a(0,0) - a(0,\delta_1) > 0. \tag{4.62}$$

By the continuity of $a(\rho,s)$, there exist $\delta \in (0,\delta_1]$ such that

$$\epsilon_2 = \min_{0\leq\rho\leq\delta}\left[\lambda_1\left(\frac{b\theta_c}{1+k\theta_c}\right) - a(\rho,\delta_1)\right] > 0. \tag{4.63}$$

Therefore, if $a \geq \lambda_1(b\theta_c/(1+k\theta_c)) - \epsilon_2$, then for any $\rho \in (0,\delta]$, $a(\rho,\delta_1) \leq a$. This implies that for $\rho \in (0,\delta]$, Γ_ρ covers the $a-$range$[\lambda_1(b\theta_c/(1+k\theta_c)) - \epsilon_2, \lambda_1(b\theta_c/(1+k\theta_c)))$. Moreover, since

$\partial a/\partial s \neq 0$ for $0 \leq \rho, s \leq \delta_1$, each curve covers the range only once. By choosing $M_1 = 1/\delta$, we get that for $m > M_1$ and $a \in ([\lambda_1(b\theta_c/(1+k\theta_c)) - \epsilon_2, \lambda_1(b\theta_c/(1+k\theta_c)))$, (1.2) has exactly one positive solution of type (ii) in Theorem 4.7.

It remains to show that the positive solution of (1.2) close to $(0, \theta_c)$ is unstable. In fact, when m is sufficiently large, applying the method of the proof in Theorem 3.9, we can show that the positive solution of (1.2) close to $(0, \theta_c)$ is unstable. we omit the proof procedure. □

Proof of Theorem 4.12. By Lemma 4.13, it suffices to establish the exact multiplicity and stability when $a \in I = [\lambda_1 + \epsilon, \lambda_1 + \epsilon_0] \cup [\lambda_1(b\theta_c/(1+k\theta_c)) - \epsilon_1, \lambda_1(b\theta_c/(1+k\theta_c)) - \epsilon_2)$ for any given $\epsilon \in (0, \epsilon_0)$, where ϵ_2 is defined in Lemma 4.13

From Theorem 4.7 we know that the solutions of (1.2) for $a \in [\lambda_1 + \epsilon, \lambda_1(b\theta_c/(1+k\theta_c)))$ and m large are of two types, that is, types (i) or (ii). As in the proof of Lemma 4.13, we can prove that there is a unique asymptotically stable positive solution of type (i). Thus to end the proof, we need only to show that there is a unique unstable positive solution of (1.2) close to $(0, \theta_c)$ if $a \in I$ and m is large. Again by Lemma 4.11, it suffices to prove that there is a unique unstable positive solution (u, v) of (1.2) such that (mu, v) is close to (w_a, θ_c), where w_a is the unique positive solution of (4.12) as shown in Lemma 4.11. In this connection, we investigate (4.58) with $a \in I$ and ρ small. Let $a^* \in I$. Since the unique solution w_{a^*} of (4.12) with $a = a^*$ is nondegenerate, then (w_{a^*}, θ_c) is a nondegenerate solution of (4.58) with $(a, \rho) = (a^*, 0)$. Clearly, (4.58) with $\rho > 0$ small is a regular perturbation of (4.58) with $\rho = 0$, and the perturbation is uniform for a in the compact set I. Thus it follows from the implicit function theorem that there exist δ, $\overline{\epsilon} > 0$ small such that for any $a \in I, 0 \leq \rho \leq \overline{\epsilon}$, (4.58) possesses a unique positive solution $(\overline{u}_a, v_a)$ which satisfies

$$\|\overline{u}_a - w_a\| + \|v_a - \theta_c\| \leq \delta. \tag{4.64}$$

Set $M = \max\{1/\epsilon, M(\epsilon, \delta)\}$, where $M(\epsilon, \delta)$ is defined in Lemma 4.11. It is easy to see that for any $\epsilon \in (0, \epsilon_0)$, there exists $M = M(\epsilon)$ such that if $m \geq M$ and $a \in I$, then (1.2) has a unique positive solution of type (ii).

It remains to prove the instability for the unique positive solution of (1.2) of type (ii). Define T and $T_0 : C_0^{2,\alpha}(\overline{\Omega}) \times C_0^{2,\alpha}(\overline{\Omega}) \to C^\alpha(\overline{\Omega}) \times C^\alpha(\overline{\Omega})$ by

$$T\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix} \Delta\xi + \left(a - 2u - \dfrac{bv}{(1+mu)^2(1+kv)}\right)\xi - \dfrac{bu}{(1+mu)(1+kv)^2}\eta \\ \Delta\eta + \left(c - 2v + \dfrac{du}{(1+mu)(1+kv)^2}\right)\eta + \dfrac{dv}{(1+mu)^2(1+kv)}\xi \end{pmatrix},$$
$$T_0\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix} \Delta\xi + \left(a - \dfrac{b\theta_c}{(1+w_a)^2(1+k\theta_c)}\right)\xi \\ \Delta\eta + (c - 2\theta_c)\eta + \dfrac{d\theta_c}{(1+w_a)^2(1+k\theta_c)}\xi \end{pmatrix}. \tag{4.65}$$

It is easy to show that, as $m \to \infty$, $T \to T_0$ in the operator norm uniformly for (u, v) approaches $(0, \theta_c)$ with mu close to w_a and $a \in [\lambda_1 + \epsilon, \lambda_1 + \epsilon_0]$. Since 0 belongs to the resolvent set of T_0 and

$$\mu_0 = \lambda_1 \left(-a + \frac{b\theta_c}{(1 + w_a)^2 (1 + k\theta_c)} \right) < 0 \tag{4.66}$$

is an eigenvalue of T_0. Due to standard perturbation theory, it is easy to get that 0 also belongs to the resolvent set of T and that T has an eigenvalue μ close to μ_0. In particular, $\operatorname{Re} \mu < 0$. This shows that for all large m, the positive solution of (1.2) close to $(0, \theta_c)$ is nondegenerate and unstable. Thus, the proof of Theorem 4.12 is complete. □

Acknowledgments

The authors would like to thank the referees for their valuable suggestions. The project supported by the Scientific Research Plan Projects of Shaanxi Education Department (no. 12JK0865) and the President Fund of Xi'an Technological University (XAGDXJJ1136).

References

[1] A. D. Bazykin, *Nonlinear Dynamics of interacting Populations*, World Scientific, Singapore, 1998.
[2] A. J. Lotka, *Elements of Physical Biology*, Williams and Wilkins Co., Baltimore, Md, USA, 1925.
[3] V. Volterra, "Variations and fluctuations of the number of individuals in animal species living together," in *Animal Ecology*, R. N. Chapman, Ed., pp. 409–448, 1931.
[4] H. Nie and J. Wu, "Multiplicity and stability of a predator-prey model with non-monotonic conversion rate," *Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal*, vol. 10, no. 1, pp. 154–171, 2009.
[5] J. Wu and G. Wei, "Coexistence states for cooperative model with diffusion," *Computers & Mathematics with Applications*, vol. 43, no. 10-11, pp. 1277–1290, 2002.
[6] L. Li, "Coexistence theorems of steady states for predator-prey interacting systems," *Transactions of the American Mathematical Society*, vol. 305, no. 1, pp. 143–166, 1988.
[7] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, NY, USA, 1992.
[8] E. N. Dancer, "On positive solutions of some pairs of differential equations. II," *Journal of Differential Equations*, vol. 60, no. 2, pp. 236–258, 1985.
[9] J. López-Gómez and R. Pardo, "Existence and uniqueness of coexistence states for the predator-prey model with diffusion: the scalar case," *Differential and Integral Equations*, vol. 6, no. 5, pp. 1025–1031, 1993.
[10] E. N. Dancer, "On uniqueness and stability for solutions of singularly perturbed predator-prey type equations with diffusion," *Journal of Differential Equations*, vol. 102, no. 1, pp. 1–32, 1993.
[11] N. Lakos, "Existence of steady-state solutions for a one-predator–two-prey system," *SIAM Journal on Mathematical Analysis*, vol. 21, no. 3, pp. 647–659, 1990.
[12] J. Blat and K. J. Brown, "Global bifurcation of positive solutions in some systems of elliptic equations," *SIAM Journal on Mathematical Analysis*, vol. 17, no. 6, pp. 1339–1353, 1986.
[13] Y. H. Du and Y. Lou, "S-shaped global bifurcation curve and Hopf bifurcation of positive solutions to a predator-prey model," *Journal of Differential Equations*, vol. 144, no. 2, pp. 390–440, 1998.
[14] Y. H. Du and Y. Lou, "Qualitative behaviour of positive solutions of a predator-prey model: effects of saturation," *Proceedings of the Royal Society of Edinburgh A*, vol. 131, no. 2, pp. 321–349, 2001.
[15] J. H. Wu, "Global bifurcation of coexistence state for the competition model in the chemostat," *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal A*, vol. 39, no. 7, pp. 817–835, 2000.
[16] H. Nie and J. Wu, "Positive solutions of a competition model for two resources in the unstirred chemostat," *Journal of Mathematical Analysis and Applications*, vol. 355, no. 1, pp. 231–242, 2009.

[17] H. Nie and J. Wu, "Asymptotic behavior of an unstirred chemostat model with internal inhibitor," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 2, pp. 889–908, 2007.
[18] Y. H. Du and Y. Lou, "Some uniqueness and exact multiplicity results for a predator-prey model," *Transactions of the American Mathematical Society*, vol. 349, no. 6, pp. 2443–2475, 1997.
[19] M. X. Wang and Q. Wu, "Positive solutions of a prey-predator model with predator saturation and competition," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 2, pp. 708–718, 2008.
[20] E. N. Dancer, "On the indices of fixed points of mappings in cones and applications," *Journal of Mathematical Analysis and Applications*, vol. 91, no. 1, pp. 131–151, 1983.
[21] M. G. Crandall and P. H. Rabinowitz, "Bifurcation from simple eigenvalues," *Journal of Functional Analysis*, vol. 8, no. 2, pp. 321–340, 1971.
[22] M. G. Crandall and P. H. Rabinowitz, "Bifurcation, perturbation of simple eigenvalues and linearized stability," *Archive for Rational Mechanics and Analysis*, vol. 52, pp. 161–180, 1973.
[23] Q. X. Ye and Z. Y. Li, *Introduction to Reaction-Diffusion Equations*, Science Press, Beijing, China, 1990.
[24] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, NY, USA, 1966.
[25] K. Wonlyul and R. Kimun, "Coexistence states of a predator-prey system with non-monotonic functional response," *Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal*, vol. 8, no. 3, pp. 769–786, 2007.