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Research Article

A Modified Halpern-Type Iterative Method of a System of Equilibrium Problems and a Fixed Point for a Totally Quasi-ϕ-Asymptotically Nonexpansive Mapping in a Banach Space

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The purpose of this paper is to introduce the modified Halpern-type iterative method by the generalized f-projection operator for finding a common solution of fixed-point problem of a totally quasi-ϕ-asymptotically nonexpansive mapping and a system of equilibrium problems in a uniform smooth and strictly convex Banach space with the Kadec-Klee property. Consequently, we prove the strong convergence for a common solution of above two sets. Our result presented in this paper generalize and improve the result of Chang et al., (2012), and some others.

1. Introduction

In 1953, Mann [1] introduced the following iteration process which is now known as Mann's iteration:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{1.1}$$

where T is nonexpansive, the initial guess element $x_1 \in C$ is arbitrary, and $\{\alpha_n\}$ is a sequence in $[0, 1]$. Mann iteration has been extensively investigated for nonexpansive mappings. In an

infinite-dimensional Hilbert space, Mann iteration can conclude *only weak convergence* (see [2, 3]).

Later, in 1967, Halpern [4] considered the following algorithm:

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_1 + (1-\alpha_n)Tx_n, \quad \forall n \geq 0, \tag{1.2}$$

where T is nonexpansive. He proved the strong convergence theorem of $\{x_n\}$ to a fixed point of T under some control condition $\{\alpha_n\}$. Many authors improved and studied the result of Halpern [4] such as Qin et al. [5], Wang et al. [6], and reference therein.

In 2008-2009, Takahashi and Zembayashi [7, 8] studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of the Banach spaces.

On the other hand, Li et al. [9] introduced the following hybrid iterative scheme for approximation fixed points of relatively nonexpansive mapping using the generalized f-projection operator in a uniformly smooth real Banach space which is also uniformly convex. They obtained strong convergence theorem for finding an element in the fixed point set of T.

Recently, Ofoedu and Shehu [10] extended algorithm of Li et al. [9] to prove a strong convergence theorem for a common solution of a system of equilibrium problem and the set of common fixed points of a pair of relatively quasi-nonexpansive mappings in the Banach spaces by using generalized f-projection operator. Chang et al. [11] extended and improved Qin and Su [12] to obtain a strong convergence theorem for finding a common element of the set of solutions for a generalized equilibrium problem, the set of solutions for a variational inequality problem, and the set of common fixed points for a pair of relatively nonexpansive mappings in a Banach space.

Very recently, Chang et al. [13] extended the results of Qin et al. [5] and Wang et al. [6] to consider a modification to the Halpern-type iteration algorithm for a total quasi-ϕ-asymptotically nonexpansive mapping to have the strong convergence under a limit condition only in the framework of Banach spaces.

The purpose of this paper is to be motivated and inspired by the works mentioned above, we introduce a modified Halpern-type iterative method by using the new hybrid projection algorithm of the generalized f-projection operator for solving the common solution of fixed point for totally quasi-ϕ-asymptoically nonexpansive mappings and the system of equilibrium problems in a uniformly smooth and strictly convex Banach space with the Kadec-Klee property. The results presented in this paper improve and extend the corresponding ones announced by many others.

2. Preliminaries and Definitions

Let E be a real Banach space with dual E^*, and let C be a nonempty closed and convex subset of E. Let $\{\theta_i\}_{i\in\Gamma} : C \times C \to \mathbb{R}$ be a bifunction, where Γ is an arbitrary index set. The *system of equilibrium problems* is to find $x \in C$ such that

$$\theta_i(x,y) \geq 0, \quad i \in \Gamma, \ \forall y \in C. \tag{2.1}$$

If Γ is a singleton, then problem (2.1) reduces to the *equilibrium problem*, which is to find $x \in C$ such that

$$\theta(x, y) \geq 0, \quad \forall y \in C. \tag{2.2}$$

A mapping T from C into itself is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{2.3}$$

T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C. \tag{2.4}$$

T is said to be *total asymptotically nonexpansive* if there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + \nu_n \varphi(\|x - y\|) + \mu_n, \quad \forall x, y \in C, \ \forall n \geq 1. \tag{2.5}$$

A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the fixed point set of T; that is, $F(T) = \{x \in C : Tx = x\}$. A point p in C is called an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The asymptotic fixed point set of T is denoted by $\widehat{F}(T)$.

The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by $J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|\}$. If E is a Hilbert space, then $J = I$, where I is the identity mapping. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \tag{2.6}$$

where J is the normalized duality mapping and $\langle \cdot, \cdot \rangle$ denote the duality pairing of E and E^*. If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$. It is obvious from the definition of ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{2.7}$$

A mapping T from C into itself is said to be *ϕ-nonexpansive* [14, 15] if

$$\phi(Tx, Ty) \leq \phi(x, y), \quad \forall x, y \in C. \tag{2.8}$$

T is said to be *quasi-ϕ-nonexpansive* [14, 15] if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \ p \in F(T). \tag{2.9}$$

T is said to be *asymptotically ϕ-nonexpansive* [15] if there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(T^n x, T^n y) \leq k_n \phi(x,y), \quad \forall x, y \in C. \tag{2.10}$$

T is said to be *quasi-ϕ-asymptotically nonexpansive* [15] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq k_n \phi(p,x), \quad \forall x \in C,\ p \in F(T),\ \forall n \geq 1. \tag{2.11}$$

T is said to be *totally quasi-ϕ-asymptotically nonexpansive*, if $F(T) \neq \emptyset$ and there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(0) = 0$ such that

$$\phi(p, T^n x) \leq \phi(p,x) + \nu_n \varphi(\phi(p,x)) + \mu_n, \quad \forall n \geq 1,\ \forall x \in C,\ p \in F(T). \tag{2.12}$$

A mapping T from C into itself is said to be *closed* if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} T x_n = y_0$, then $T x_0 = y_0$.

Alber [16] introduced the *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution of the minimization problem:

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(y,x). \tag{2.13}$$

The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(y,x)$ and the strict monotonicity of the mapping J (see, e.g., [16–20]). If E is a Hilbert space, then $\phi(x,y) = \|x-y\|^2$ and Π_C becomes the metric projection $P_C : H \to C$. If C is a nonempty, closed, and convex subset of a Hilbert space H, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces, and consequently, it is not available in more general Banach spaces. Later, Wu and Huang [21] introduced a new generalized f-projection operator in the Banach space. They extended the definition of the generalized projection operators and proved some properties of the generalized f-projection operator. Next, we recall the concept of the generalized f-projection operator. Let $G : C \times E^* \to \mathbb{R} \cup \{+\infty\}$ be a functional defined by

$$G(y, \varpi) = \|y\|^2 - 2\langle y, \varpi \rangle + \|\varpi\|^2 + 2\rho f(y), \tag{2.14}$$

where $y \in C$, $\varpi \in E^*$, ρ is positive number, and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous. From the definition of G, Wu and Huang [21] proved the following properties:

(1) $G(y, \varpi)$ is convex and continuous with respect to ϖ when y is fixed;

(2) $G(\mathrm{y}, \varpi)$ is convex and lower semicontinuous with respect to y when ϖ is fixed.

Definition 2.1. Let E be a real Banach space with its dual E^*. Let C be a nonempty, closed, and convex subset of E. We say that $\pi_C^f : E^* \to 2^C$ is a *generalized f-projection operator* if

$$\pi_C^f \varpi = \left\{ u \in C : G(u, \varpi) = \inf_{y \in C} G(y, \varpi),\ \forall \varpi \in E^* \right\}. \tag{2.15}$$

A Banach space E with norm $\|\cdot\|$ is called *strictly convex* if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. A Banach space E is called *smooth* if the limit $\lim_{t \to 0}((\|x+ty\| - \|x\|)/t)$ exists for each $x, y \in U$. It is also called *uniformly smooth* if the limit exists uniformly for all $x, y \in U$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by $\rho_E(t) = \sup\{(\|x + y\| + \|x - y\|)/2 - 1 : \|x\| = 1, \|y\| \leq t\}$. The *modulus of convexity* of E (see [22]) is the function $\delta_E : [0, 2] \to [0, 1]$ defined by $\delta_E(\varepsilon) = \inf\{1 - \|(x + y)/2\| : x, y \in E,\ \|x\| = \|y\| = 1,\ \|x - y\| \geq \varepsilon\}$. In this paper we denote the strong convergence and weak convergence of a sequence $\{x_n\}$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Remark 2.2. The basic properties of E, E^*, J, and J^{-1} (see [18]) are as follows.

(i) If E is an arbitrary Banach space, then J is monotone and bounded.

(ii) If E is a strictly convex, then J is strictly monotone.

(iii) If E is a smooth, then J is single valued and semicontinuous.

(iv) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

(v) If E is reflexive smooth and strictly convex, then the normalized duality mapping J is single valued, one-to-one, and onto.

(vi) If E is a reflexive strictly convex and smooth Banach space and J is the duality mapping from E into E^*, then J^{-1} is also single valued, bijective, and is also the duality mapping from E^* into E, and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$.

(vii) If E is uniformly smooth, then E is smooth and reflexive.

(viii) E is uniformly smooth if and only if E^* is uniformly convex.

(ix) If E is a reflexive and strictly convex Banach space, then J^{-1} is norm-weak*-continuous.

Remark 2.3. If E is a reflexive, strictly convex, and smooth Banach space, then $\phi(x, y) = 0$, if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (2.6), we have $\|x\| = \|y\|$. This implies that $\langle x, Jy \rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$ (see [18, 20, 23] for more details).

Recall that a Banach space E has the Kadec-Klee property [18, 20, 24], if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach space, then E has the Kadec-Klee property.

We also need the following lemmas for the proof of our main results.

Lemma 2.4 (see Change et al. [25]). *Let C be a nonempty, closed, and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. Let $T : C \to C$ be a closed*

and total quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequence ν_n and μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$. If $\mu_1 = 0$, then the fixed point set $F(T)$ is a closed convex subset of C.

Lemma 2.5 (see Wu and Hung [21]). *Let E be a real reflexive Banach space with its dual E^* and C a nonempty, closed, and convex subset of E. The following statement hold:*

(1) *$\pi_C^f \varpi$ is a nonempty, closed and convex subset of C for all $\varpi \in E^*$;*

(2) *if E is smooth, then for all $\varpi \in E^*$, $x \in \pi_C^f \varpi$ if and only if*

$$\langle x - y,\ \varpi - Jx \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C; \tag{2.16}$$

(3) *if E is strictly convex and $f : C \to \mathbb{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$ where $x \in C$), then $\pi_C^f \varpi$ is single-valued mapping.*

Lemma 2.6 (see Fan et al. [26]). *Let E be a real reflexive Banach space with its dual E^* and C be a nonempty, closed and convex subset of E. If E is strictly convex, then $\pi_C^f \varpi$ is single valued.*

Recall that J is single-valued mapping when E is a smooth Banach space. There exists a unique element $\varpi \in E^*$ such that $\varpi = Jx$ where $x \in E$. This substitution in (2.14) gives

$$G(y, Jx) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 + 2\rho f(y). \tag{2.17}$$

Now we consider the second generalized f projection operator in Banach space (see [9]).

Definition 2.7. Let E be a real smooth Banach space, and let C be a nonempty, closed, and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is generalized f-projection operator if

$$\Pi_C^f x = \left\{ u \in C : G(u, Jx) = \inf_{y \in C} G(y, Jx),\ \forall x \in E \right\}. \tag{2.18}$$

Lemma 2.8 (see Deimling [27]). *Let E be a Banach space, and let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function. Then there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, x^* \rangle + \alpha, \quad \forall x \in E. \tag{2.19}$$

Lemma 2.9 (see Li et al. [9]). *Let E be a reflexive smooth Banach space, and let C be a nonempty, closed, and convex subset of E. The following statements hold:*

(1) *$\Pi_C^f x$ is nonempty, closed and convex subset of C for all $x \in E$;*

(2) *for all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if*

$$\langle \hat{x} - y,\ Jx - J\hat{x} \rangle + \rho f(y) - \rho f(\hat{x}) \geq 0, \quad \forall y \in C; \tag{2.20}$$

(3) *if E is strictly convex, then Π_C^f is single-valued mapping.*

Lemma 2.10 (see Li et al. [9]). *Let E be a real reflexive smooth Banach space, let C be a nonempty, closed, and convex subset of E, $x \in E$, and let $\hat{x} \in \Pi_C^f x$. Then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \quad \forall y \in C. \tag{2.21}$$

Remark 2.11. Let E be a uniformly convex and uniformly smooth Banach space and $f(x) = 0$ for all $x \in E$, then Lemma 2.10 reduces to the property of the generalized projection operator considered by Alber [16].

If $f(y) \geq 0$ for all $y \in C$ and $f(0) = 0$, then the definition of totally quasi-ϕ-asymptotically nonexpansive T is equivalent to if $F(T) \neq \emptyset$, and there exist nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that

$$G(p, T^n x) \leq G(p, x) + \nu_n \zeta G(p, x) + \mu_n, \quad \forall n \geq 1, \ \forall x \in C, \quad p \in F(T). \tag{2.22}$$

For solving the equilibrium problem for a bifunction $\theta : C \times C \to \mathbb{R}$, let us assume that θ satisfies the following conditions:

(A1) $\theta(x, x) = 0$ for all $x \in C$;

(A2) θ is monotone; that is, $\theta(x, y) + \theta(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} \theta(tz + (1-t)x, y) \leq \theta(x, y); \tag{2.23}$$

(A4) for each $x \in C$, $y \mapsto \theta(x, y)$ is convex and lower semicontinuous.

For example, let A be a continuous and monotone operator of C into E^* and define

$$\theta(x, y) = \langle Ax, \ y - x \rangle, \quad \forall x, y \in C. \tag{2.24}$$

Then, θ satisfies (A1)–(A4). The following result is in Blum and Oettli [28].

Lemma 2.12 (see Blum and Oettli [28]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$\theta(z, y) + \frac{1}{r} \langle y - z, \ Jz - Jx \rangle \geq 0, \quad \forall y \in C. \tag{2.25}$$

Lemma 2.13 (see Takahashi and Zembayashi [8]). *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E, and let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions (A1)–(A4). For all $r > 0$ and $x \in E$, define a mapping $T_r^\theta : E \to C$ as follows:*

$$T_r^\theta x = \left\{ z \in C : \theta(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C \right\}. \tag{2.26}$$

Then the following hold:

(1) *T_r^θ is single-valued;*

(2) *T_r^θ is a firmly nonexpansive-type mapping [29]; that is, for all $x, y \in E$,*

$$\langle T_r^\theta x - T_r^\theta y, JT_r^\theta x - JT_r^\theta y\rangle \leq \langle T_r^\theta x - T_r^\theta y, Jx - Jy\rangle; \tag{2.27}$$

(3) *$F(T_r^\theta) = \mathrm{EP}(\theta)$;*

(4) *$\mathrm{EP}(\theta)$ is closed and convex.*

Lemma 2.14 (see Takahashi and Zembayashi [8]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $r > 0$. Then, for $x \in E$ and $q \in F(T_r^\theta)$,*

$$\phi\left(q, T_r^\theta x\right) + \phi\left(T_r^\theta x, x\right) \leq \phi(q, x). \tag{2.28}$$

3. Main Result

Theorem 3.1. *Let C be a nonempty, closed, and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. For each $j = 1, 2, \ldots, m$, let θ_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4). Let $S : C \to C$ be a closed totally quasi-ϕ-asymptotically nonexpansive mappings with nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$, and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous function with $C \subset \mathrm{int}\,(D(f))$ such that $f(x) \geq 0$ for all $x \in C$ and $f(0) = 0$. Assume that $\mathcal{F} := F(S) \cap (\cap_{j=1}^m \mathrm{EP}(\theta_j)) \neq \emptyset$. For an initial point $x_1 \in E$ and $C_1 = C$, one define the sequence $\{x_n\}$ by*

$$\begin{gathered} u_n = T_{r_{m,n}}^{\theta_m} T_{r_{m-1,n}}^{\theta_{m-1}} T_{r_{m-2,n}}^{\theta_{m-2}} \cdots T_{r_{1,n}}^{\theta_1} x_n, \\ z_n = J^{-1}(\alpha_n J x_1 + (1 - \alpha_n) J S^n u_n), \\ C_{n+1} = \{v \in C_n : G(v, Jz_n) \leq G(v, Ju_n) \leq G(v, Jx_1) + (1 - \alpha_n) G(v, Jx_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_1, \quad n \in \mathbb{N}, \end{gathered} \tag{3.1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\zeta_n = \nu_n \sup_{q \in \mathcal{F}} \psi(G(q, x_n)) + \mu_n$ and $\{r_{j,n}\} \subset [d, \infty)$ for some $d > 0$. If $\lim_{n \to \infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}}^f x_0$.

Proof. We split the proof into four steps. □

Step 1. First, we show that C_n is closed and convex for all $n \in \mathbb{N}$.

Clearly $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for all $n \in \mathbb{N}$. For any $v \in C_n$, we know that $G(v, Jz_n) \leq G(v, Jx_n) + \zeta_n$ is equivalent to

$$2\langle v, Jx_n - Jz_n \rangle \leq \|x_n\|^2 - \|z_n\|^2 + \zeta_n. \tag{3.2}$$

So, C_{n+1} is closed and convex. Hence by induction C_n is closed and convex for all $n \geq 1$.

Step 2. We will show that the sequence $\{x_n\}$ is well defined.

We will show by induction that $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $\mathcal{F} \subset C_1 = C$. Suppose that $\mathcal{F} \subset C_n$ for some $n \in \mathbb{N}$. Let $q \in \mathcal{F}$, put $u_n = K_n^m x_n$, $K_n^j = T_{r_{j,n}}^{\theta_j} T_{r_{j-1,n}}^{\theta_{j-1}} \dots T_{r_{1,n}}^{\theta_1}$ for all $j = 1, 2, 3, \dots, m$, $K_n^0 = I$, we have that

$$G(q, Ju_n) = \& G(q, JK_n^m x_n) \leq \& G(q, Jx_n). \tag{3.3}$$

From (3.3) and S which is a totally quasi-ϕ asymptotically nonexpansive mappings, it follows that

$$\begin{aligned}
G(q, Jz_n) &= G(q, (\alpha_n Jx_1 + (1-\alpha_n) JS^n u_n)) \\
&= \|q\|^2 - 2\alpha_n \langle q, Jx_1 \rangle - 2(1-\alpha_n)\langle q, JS^n u_n \rangle \\
&\quad + \|\alpha_n Jx_1 + (1-\alpha_n) JS^n u_n\|^2 + 2\rho f(q) \\
&\leq \|q\|^2 - 2\alpha_n \langle q, Jx_1 \rangle - 2(1-\alpha_n)\langle q, JS^n u_n \rangle \\
&\quad + \alpha_n \|Jx_1\|^2 + (1-\alpha_n)\|JS^n u_n\|^2 + 2\rho f(q) \\
&= \alpha_n G(q, Jx_1) + (1-\alpha_n) G(q, JS^n u_n) \\
&\leq \alpha_n G(q, Jx_1) + (1-\alpha_n)(G(q, Ju_n) + \nu_n \psi(G(q, Ju_n)) + \mu_n) \\
&\leq \alpha_n G(q, Jx_1) + (1-\alpha_n) G(q, Jx_n) + \nu_n \sup_{q \in \mathcal{F}} \psi(G(q, Jx_n)) + \mu_n \\
&= \alpha_n G(q, Jx_1) + (1-\alpha_n) G(q, Jx_n) + \zeta_n.
\end{aligned} \tag{3.4}$$

This shows that $q \in C_{n+1}$ which implies that $\mathcal{F} \subset C_{n+1}$, and hence, $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N}$. and the sequence $\{x_n\}$ is well defined. From $x_n = \Pi_{C_n}^f x_1$, we see that

$$\langle x_n - q, Jx_1 - Jx_n \rangle + \rho f(q) - \rho f(x_n) \geq 0, \quad \forall q \in C_n. \tag{3.5}$$

Since $\mathcal{F} \subset C_n$ for each $n \in \mathbb{N}$, we arrive at

$$\langle x_n - q, Jx_1 - Jx_n \rangle + \rho f(q) - \rho f(x_n) \geq 0, \quad \forall q \in \mathcal{F}. \tag{3.6}$$

Hence, the sequence $\{x_n\}$ is well defined.

Step 3. We will show that $x_n \to p \in \mathcal{F} := F(S) \cap (\cap_{j=1}^{m} \mathrm{EP}(\theta_j))$.

Let $f : E \to \mathbb{R}$ is convex and lower semicontinuous function, follows from Lemma 2.8, there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \langle y, x^* \rangle + \alpha, \quad \forall y \in E. \tag{3.7}$$

Since $x_n \in C_n \subset E$, it follows that

$$\begin{aligned} G(x_n, Jx_1) &= \|x_n\|^2 - 2\langle x_n, Jx_1 \rangle + \|x_1\|^2 + 2\rho f(x_n) \\ &\geq \|x_n\|^2 - 2\langle x_n, Jx_1 \rangle + \|x_1\|^2 + 2\rho \langle x_n, x^* \rangle + 2\rho\alpha \\ &= \|x_n\|^2 - 2\langle x_n, Jx_1 - \rho x^* \rangle + \|x_1\|^2 + 2\rho\alpha \\ &\geq \|x_n\|^2 - 2\|x_n\| \|Jx_1 - \rho x^*\| + \|x_1\|^2 + 2\rho\alpha \\ &= (\|x_n\| - \|Jx_1 - \rho x^*\|)^2 + \|x_1\|^2 - \|Jx_1 - \rho x^*\|^2 + 2\rho\alpha. \end{aligned} \tag{3.8}$$

For $q \in \mathcal{F}$ and $x_n = \Pi_{C_n}^f x_1$, we have

$$G(q, Jx_1) \geq G(x_n, Jx_1) \geq (\|x_n\| - \|Jx_1 - \rho x^*\|)^2 + \|x_1\|^2 - \|Jx_1 - \rho x^*\|^2 + 2\rho\alpha. \tag{3.9}$$

This shows that $\{x_n\}$ is bounded and so is $\{G(x_n, Jx_1)\}$. From the fact that $x_{n+1} = \Pi_{C_{n+1}}^f x_1 \in C_{n+1} \subset C_n$ and $x_n = \Pi_{C_n}^f x_1$, it follows from Lemma 2.10 that

$$0 \leq (\|x_{n+1} - \|x_n\|)^2 \leq \phi(x_{n+1}, x_n) \leq G(x_{n+1}, Jx_1) - G(x_n, Jx_1). \tag{3.10}$$

That is, $\{G(x_n, Jx_1)\}$ is nondecreasing. Hence, we obtain that $\lim_{n\to\infty} G(x_n, Jx_1)$ exists. Taking $n \to \infty$, we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.11}$$

Since E is reflexive, $\{x_n\}$ is bounded, and C_n is closed and convex for all $n \in \mathbb{N}$. Without loss of generality, we can assume that $x_n \rightharpoonup p \in C_n$. From the fact that $x_n = \Pi_{C_n}^f x_1$, we get that

$$G(x_n, Jx_1) \leq G(p, Jx_1), \quad \forall n \in \mathbb{N}. \tag{3.12}$$

Since f is convex and lower semicontinuous, we have

$$\begin{aligned} \liminf_{n\to\infty} G(x_n, Jx_1) &= \liminf_{n\to\infty} \left\{ \|x_n\|^2 - 2\langle x_n, Jx_1 \rangle + \|x_1\|^2 + 2\rho f(x_n) \right\} \\ &\geq \|p\|^2 - 2\langle p, Jx_1 \rangle + \|x_1\|^2 + 2\rho f(p) \\ &= G(x_n, Jx_1). \end{aligned} \tag{3.13}$$

By (3.12) and (3.13), we get

$$G(p, Jx_1) \leq \liminf_{n\to\infty} G(x_n, Jx_1) \leq \limsup_{n\to\infty} G(x_n, Jx_1) \leq G(p, Jx_1). \tag{3.14}$$

That is, $\lim_{n\to\infty} G(x_n, Jx_1) = G(p, Jx_1)$; this implies that $\|x_n\| \to \|p\|$; by virtue of the Kadec-Klee property of E, we obtain that

$$\lim_{n\to\infty} x_n = p. \tag{3.15}$$

We also have

$$\lim_{n\to\infty} x_{n+1} = p. \tag{3.16}$$

From (3.15), we get that

$$\lim_{n\to\infty} \zeta_n = \lim_{n\to\infty} \left(\nu_n \sup_{q\in\mathcal{F}} \psi(G(q, x_n)) + \mu_n \right) = 0. \tag{3.17}$$

(a) We show that $p \in \cap_{j=1}^{m} \mathrm{EP}(\theta_j)$.

Since $x_{n+1} = \Pi_{C_{n+1}}^{f} x_1 \in C_{n+1} \subset C_n$ and the definition of C_{n+1}, we have

$$G(x_{n+1}, Ju_n) \leq \alpha_n G(x_{n+1}, Jx_1) + (1-\alpha_n) G(x_{n+1}, Jx_n) + \zeta_n \tag{3.18}$$

is equivalent to

$$\phi(x_{n+1}, u_n) \leq \alpha_n \phi(x_{n+1}, x_1) + (1-\alpha_n)\phi(x_{n+1}, x_n) + \zeta_n. \tag{3.19}$$

From (3.11), (3.15), and (3.17), it follows that

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{3.20}$$

From (2.7), we have

$$(\|x_{n+1}\| - \|u_n\|)^2 \longrightarrow 0. \tag{3.21}$$

Since $\|x_{n+1}\| \to \|p\|$, we have

$$\|u_n\| \longrightarrow \|p\| \quad \text{as } n \longrightarrow \infty. \tag{3.22}$$

It follow that

$$\|Ju_n\| \longrightarrow \|Jp\| \quad \text{as } n \longrightarrow \infty. \tag{3.23}$$

That is, $\{\|Ju_n\|\}$ is bounded in E^* and E^* is reflexive; we assume that $Ju_n \rightharpoonup u^* \in E^*$. In view of $J(E) = E^*$, there exists $u \in E$ such that $Ju = u^*$. It follows that

$$\begin{aligned}\phi(x_{n+1}, u_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|u_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|Ju_n\|^2.\end{aligned} \tag{3.24}$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality above and $\|\cdot\|$ is the weak lower semicontinuous, it yields that

$$\begin{aligned}0 &\geq \|p\|^2 - 2\langle p, u^*\rangle + \|u^*\|^2 \\ &= \|p\|^2 - 2\langle p, Ju\rangle + \|Ju\|^2 \\ &= \|p\|^2 - 2\langle p, Ju\rangle + \|u\|^2 \\ &= \phi(p, u).\end{aligned} \tag{3.25}$$

That is, $p = u$, which implies that $u^* = Jp$. It follows that $Ju_n \rightharpoonup Jp \in E^*$. From (3.23) and the Kadec-Klee property of E^* we have $Ju_n \to Jp$ as $n \to \infty$. Note that $J^{-1} : E^* \to E$ is norm-weak *-continuous; that is, $u_n \rightharpoonup p$. From (3.22) and the Kadec-Klee property of E, we have

$$\lim_{n\to\infty} u_n = p. \tag{3.26}$$

For $q \in F \subset C_n$, by nonexpansiveness, we observe that

$$\begin{aligned}\phi(q, u_n) &= \phi(q, K_n^m x_n) \\ &\leq \phi\left(q, K_n^{m-1} x_n\right) \\ &\leq \phi\left(q, K_n^{m-2} x_n\right) \\ &\vdots \\ &\leq \phi\left(q, K_n^j x_n\right).\end{aligned} \tag{3.27}$$

By Lemma 2.14, we have for $j = 1, 2, 3, \ldots, m$

$$\phi\left(K_n^j x_n, x_n\right) \& \leq \phi(q, x_n) - \phi\left(q, K_n^j x_n\right) \leq \phi(q, x_n) - \phi(q, u_n). \tag{3.28}$$

Since $x_n, u_n \to p$ as $n \to \infty$, we get $\phi(K_n^j x_n, x_n) \to 0$ as $n \to \infty$, for $j = 1, 2, 3, \ldots, m$. From (2.7), it follow that

$$\left(\left\|K_n^j x_n\right\| - \|x_n\|\right)^2 \longrightarrow 0. \tag{3.29}$$

Since $\|x_n\| \to \|p\|$, we also have

$$\left\|K_n^j x_n\right\| \longrightarrow \|p\| \quad \text{as } n \longrightarrow \infty. \tag{3.30}$$

Since $\{K_n^j x_n\}$ is bounded and E is reflexive, without loss of generality we assume that $K_n^j y_n \rightharpoonup h$. We know that C_n is closed and convex for each $n \geq 1$ it is obvious that $h \in C_n$. Again since

$$\phi\left(K_n^j x_n, x_n\right) = \left\|K_n^j x_n\right\|^2 - 2\left\langle K_n^j x_n, J x_n\right\rangle + \|x_n\|^2, \tag{3.31}$$

taking $\liminf_{n\to\infty}$ on the both sides of equality above, we have

$$0\& \geq \|h\|^2 - 2\langle h, Jp\rangle + \|p\|^2 = \phi(h,p). \tag{3.32}$$

That is, $h = p$, for all $j = 1,2,3,\ldots,m$; it follow that

$$K_n^j x_n \rightharpoonup p; \tag{3.33}$$

from (3.30), (3.33), and the Kadec-Klee property, it follows that

$$\lim_{n\to\infty} K_n^j x_n = p, \quad \forall j = 1,2,3,\ldots,m. \tag{3.34}$$

By using triangle inequality, we have

$$\left\|x_n - K_n^j x_n\right\| \leq \|x_n - p\| + \left\|p - K_n^j u_n\right\|. \tag{3.35}$$

Since $x_n, K_n^j x_n \to p$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\left\|x_n - K_n^j x_n\right\| = 0, \quad \forall j = 1,2,3,\ldots,m. \tag{3.36}$$

Again by using triangle inequality, we have

$$\left\|K_n^j x_n - K_n^{j-1} x_n\right\| \leq \left\|K_n^j x_n - x_n\right\| + \left\|x_n - K_n^{j-1} x_n\right\|. \tag{3.37}$$

From (3.36), we also have

$$\lim_{n\to\infty}\left\|K_n^j x_n - K_n^{j-1} x_n\right\| = 0, \quad \forall j = 1,2,3,\ldots,m. \tag{3.38}$$

Since J is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty}\left\|J K_n^j x_n - J K_n^{j-1} x_n\right\| = 0, \quad \forall j = 1,2,3,\ldots,m. \tag{3.39}$$

From $r_{j,n} > 0$, we have $\|JK_n^j x_n - JK_n^{j-1} x_n\|/r_{j,n} \to 0$ as $n \to \infty$ for all $j = 1,2,3,\ldots,m$, and

$$\theta_j\left(K_n^j y_n, y\right) + \frac{1}{r_{j,n}}\langle y - K_n^j x_n, JK_n^j x_n - JK_n^{j-1} x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.40}$$

By (A2), that

$$\begin{aligned}\left\|y - K_n^j y_n\right\| \frac{\left\|JK_n^j y_n - JK_n^{j-1} x_n\right\|}{r_n} &\geq \frac{1}{r_{j,n}}\langle y - K_n^j x_n, JK_n^j y_n - JK_n^{j-1} x_n\rangle \\ &\geq -\theta_j\left(K_n^j x_n, y\right) \\ &\geq \theta_j\left(y, K_n^j x_n\right), \quad \forall y \in C,\end{aligned} \tag{3.41}$$

and $K_n^j x_n \to p$ as $n \to \infty$, we get $\theta_j(y,p) \leq 0$, for all $y \in C$. For $0 < t < 1$, define $y_t = ty + (1-t)p$, then $y_t \in C$ which imply that $\theta_j(y_t, p) \leq 0$. From (A1), we obtain that

$$0 = \theta_j(y_t, y_t) \leq t\theta_j(y_t, y) + (1-t)\theta_j(y_t, p) \leq t\theta_j(y_t, y). \tag{3.42}$$

We have that $\theta_j(y_t, y) \geq 0$. From (A3), we have $\theta_j(p, y) \geq 0$, for all $y \in C$ and $j = 1,2,3,\ldots,m$. That is, $p \in \mathrm{EP}(\theta_j)$, for all $j = 1,2,3,\ldots,m$. This imply that $p \in \cap_{j=1}^m \mathrm{EP}(\theta_j)$.

(b) We show that $p \in F(S)$.

Since $x_{n+1} = \Pi_{C_{n+1}}^f x_1 \in C_{n+1} \subset C_n$ and the definition of C_{n+1}, we have

$$G(x_{n+1}, Jz_n) \leq \alpha_n G(x_{n+1}, Jx_1) + (1-\alpha_n)G(x_{n+1}, Jx_n) + \zeta_n \tag{3.43}$$

is equivalent to

$$\phi(x_{n+1}, z_n) \leq \alpha_n \phi(x_{n+1}, x_1) + (1-\alpha_n)\phi(x_{n+1}, x_n) + \zeta_n. \tag{3.44}$$

Following (3.11), (3.15), and (3.17), we get that

$$\lim_{n\to\infty} \phi(x_{n+1}, z_n) = 0. \tag{3.45}$$

From (2.7), we also have

$$\|z_n\| \longrightarrow \|p\| \quad \text{as } n \longrightarrow \infty. \tag{3.46}$$

It follows that

$$\|Jz_n\| \longrightarrow \|Jp\| \quad \text{as } n \longrightarrow \infty. \tag{3.47}$$

This implies that $\{\|Jz_n\|\}$ is bounded in E^*. Since E is reflexive and E^* is also reflexive, we can assume that $Jz_n \rightharpoonup z^* \in E^*$. In view of the reflexive of E, we see that $J(E) = E^*$. There exists $z \in E$ such that $Jz = z^*$. It follows that

$$\begin{aligned}\phi(x_{n+1}, z_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jz_n\rangle + \|z_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jz_n\rangle + \|Jz_n\|^2.\end{aligned} \tag{3.48}$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality above and in view of the weak lower semicontinuity of norm $\|\cdot\|$, it yields that

$$\begin{aligned}0 &\geq \|p\|^2 - 2\langle p, z^*\rangle + \|z^*\|^2 \\ &= \|p\|^2 - 2\langle p, Jz\rangle + \|Jz\|^2 \\ &= \|p\|^2 - 2\langle p, Jz\rangle + \|z\|^2 \\ &= \phi(p, z);\end{aligned} \tag{3.49}$$

That is $p = z$, which implies that $z^* = Jp$. It follows that $Jz_n \rightharpoonup Jp \in E^*$.From (3.47) and the Kadec-Klee property of E^* we have $Jz_n \to Jp$ as $n \to \infty$. Since $J^{-1} : E^* \to E$ is norm-weak *-continuous, $z_n \rightharpoonup p$ as $n \to \infty$. From (3.46) and the Kadec-Klee property of E, we have

$$\lim_{n\to\infty} z_n = p. \tag{3.50}$$

Since $\{x_n\}$ is bounded, then a mapping S is also bounded. From the condition $\lim_{n\to\infty}\alpha_n = 0$, we have that

$$\|Jz_n - JS^n u_n\| = \lim_{n\to\infty} \alpha_n \|Jx_1 - JS^n u_n\| = 0. \tag{3.51}$$

From (3.47), we get

$$\|JS^n u_n\| \longrightarrow \|Jp\| \quad \text{as } n \longrightarrow \infty. \tag{3.52}$$

Since $J^{-1} : E^* \to E$ is norm-weak*-continuous,

$$S^n u_n \rightharpoonup p \quad \text{as } n \longrightarrow \infty. \tag{3.53}$$

On the other hand, we observe that

$$\big|\|S^n u_n\| - \|p\|\big| = \|J(S^n u_n)\| - \|Jp\| \leq \|J(S^n u_n) - Jp\|. \tag{3.54}$$

In view of (3.52), we obtain $\|S^n u_n\| \to \|p\|$. Since E has the Kadee-Klee property, we get

$$S^n u_n \longrightarrow p \quad \text{for each } n \in \mathbb{N}. \tag{3.55}$$

From $S^n u_n \to p$, we get $S^{n+1}u_n \to p$; that is, $SS^n u_n \to p$. In view of closeness of S, we have $Sp = p$. This implies that $p \in F(S)$. From (a) and (b), it follows that $p \in \cap_{j=1}^{m} \mathrm{EP}(\theta_j) \cap F(S)$.

Step 4. We will show that $p = \Pi_{\mathcal{F}}^{f} x_1$.

Since $\mathcal{F}$ is closed and convex set from Lemma 2.9, we have $\Pi_{\mathcal{F}}^{f} x_1$ which is single valued, denoted by v. By definition $x_n = \Pi_{C_n}^{f} x_1$ and $v \in \mathcal{F} \subset C_n$, we also have

$$G(x_n, Jx_1) \leq \mathrm{G}(v, Jx_1), \quad \forall n \geq 1. \tag{3.56}$$

By the definition of G and f, we know that, for each given x, $G(\xi, Jx)$ is convex and lower semicontinuous with respect to ξ. So

$$G(p, Jx_1) \leq \liminf_{n\to\infty} G(x_n, Jx_1) \leq \limsup_{n\to\infty} G(x_n, Jx_1) \leq G(v, Jx_1). \tag{3.57}$$

From the definition of $\Pi_{\mathcal{F}}^{f} x_1$ and since $p \in \mathcal{F}$, we conclude that $v = p = \Pi_{\mathcal{F}}^{f} x_1$ and $x_n \to p$ as $n \to \infty$. The proof is completed.

Setting $\nu_n \equiv 0$ and $\mu_n \equiv 0$ in Theorem 3.1, then we have the following corollary.

Corollary 3.2. *Let C be a nonempty, closed, and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. For each $j = 1, 2, \ldots, m$, let θ_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4). Let $S : C \to C$ be a closed and quasi-ϕ-asymptotically nonexpansive mappings, and let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous function with $C \subset \mathrm{int}\,(D(f))$ such that $f(x) \geq 0$ for all $x \in C$ and $f(0) = 0$. Assume that $\mathcal{F} = F(S) \cap (\cap_{j=1}^{m} \mathrm{EP}(\theta_j)) \neq \emptyset$. For an initial point $x_1 \in E$ and $C_1 = C$, we define the sequence $\{x_n\}$ by*

$$\begin{gathered} u_n = T_{r_{m,n}}^{\theta_m} T_{r_{m-1,n}}^{\theta_{m-1}} T_{r_{m-2,n}}^{\theta_{m-2}} \cdots T_{r_{1,n}}^{\theta_1} x_n, \\ z_n = J^{-1}(\alpha_n Jx_1 + (1-\alpha_n) JS^n u_n), \\ C_{n+1} = \{v \in C_n : G(v, Jz_n) \leq G(v, Ju_n) \leq G(v, Jx_1) + (1-\alpha_n) G(v, Jx_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}^{f} x_1, \quad n \in \mathbb{N}, \end{gathered} \tag{3.58}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$, $\zeta_n = \nu_n \sup_{q\in\mathcal{F}} \psi(G(q, x_n)) + \mu_n$, and $\{r_{j,n}\} \subset [d, \infty)$ for some $d > 0$. If $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}}^{f} x_1$.

Let E be a real Banach space, and let C be a nonempty closed convex subset of E. Given a mapping $A : C \to E^*$, let $\theta(x, y) = \langle Ax, y - x\rangle$ for all $x, y \in C$. Then $x^* \in \mathrm{EP}(\theta)$ if and only if $\langle Ax^*, y - x^*\rangle \geq 0$ for all $y \in C$; that is, x^* is a solution of *the classical variational inequality problem.* The set of this solution is denoted by $\mathrm{VI}(A, C)$. For each $r > 0$ and $x \in E$, we define

the mapping $T_r^\theta x$ by

$$T_r^\theta x = \left\{ z \in C : \langle Az, y - z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C \right\}. \tag{3.59}$$

Hence, we obtain the following corollary.

Corollary 3.3. *Let C be a nonempty, closed, and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. For each $j = 1, 2, \ldots, m$, let $\{A_j\}$ be a continuous monotone mapping of C into E^*. Let $S : C \to C$ be a closed totally quasi-ϕ-asymptotically nonexpansive mappings with nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$, and let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous function with $C \subset \operatorname{int}(D(f))$ such that $f(x) \geq 0$ for all $x \in C$ and $f(0) = 0$. Assume that $\mathcal{F} = F(S) \cap (\cap_{j=1}^m \mathrm{VI}(\mathrm{A_j}, \mathrm{C})) \neq \emptyset$. For an initial point $x_1 \in E$ and $C_1 = C$, one defines the sequence $\{x_n\}$ by*

$$\begin{gathered} u_n = T_{r_{m,n}}^{\theta_m} T_{r_{m-1,n}}^{\theta_{m-1}} T_{r_{m-2,n}}^{\theta_{m-2}} \cdots T_{r_{1,n}}^{\theta_1} x_n, \\ z_n = J^{-1}(\alpha_n J x_1 + (1 - \alpha_n) J S^n u_n), \\ C_{n+1} = \{v \in C_n : G(v, Jz_n) \leq G(v, Ju_n) \leq G(v, Jx_1) + (1 - \alpha_n) G(v, Jx_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_1, \quad n \in \mathbb{N}, \end{gathered} \tag{3.60}$$

where $\zeta_n = \nu_n \sup_{q \in \mathcal{F}} \psi(G(q, x_n)) + \mu_n$, $\{\alpha_n\}$ is a sequence in $[0, 1]$, and $\{r_{j,n}\} \subset [d, \infty)$ for some $d > 0$. If $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}}^f x_1$.

If $f(x) = 0$ for all $x \in E$, we have $G(\xi, Jx) = \phi(\xi, x)$ and $\Pi_C^f x = \Pi_C x$. From Theorem 3.1, we obtain the following corollary.

Corollary 3.4. *Let C be a nonempty, closed, and convex subset of a uniformly smooth and strictly convex Banach space E with the Kadec-Klee property. For each $j = 1, 2, \ldots, m$, let θ_j be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4). Let $S : C \to C$ be a closed totally quasi-ϕ-asymptotically nonexpansive mappings with nonnegative real sequences ν_n, μ_n with $\nu_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$. Assume that $\mathcal{F} = F(S) \cap (\cap_{j=1}^m \mathrm{EP}(\theta_j)) \neq \emptyset$. For an initial point $x_1 \in E$ and $C_1 = C$, we define the sequence $\{x_n\}$ by*

$$\begin{gathered} u_n = T_{r_{m,n}}^{\theta_m} T_{r_{m-1,n}}^{\theta_{m-1}} T_{r_{m-2,n}}^{\theta_{m-2}} \cdots T_{r_{1,n}}^{\theta_1} x_n, \\ z_n = J^{-1}(\alpha_n J x_1 + (1 - \alpha_n) J S^n u_n), \\ C_{n+1} = \{v \in C_n : G(v, Jz_n) \leq G(v, Ju_n) \leq G(v, Jx_1) + (1 - \alpha_n) G(v, Jx_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad n \in \mathbb{N}, \end{gathered} \tag{3.61}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\zeta_n = \nu_n \sup_{q \in \mathcal{F}} \psi(G(q, x_n)) + \mu_n$, and $\{r_{j,n}\} \subset [d, \infty)$ for some $d > 0$. If $\lim_{n\to\infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

Remark 3.5. Our main result extends and improves the result of Chang et al. [13] in the following sense.

(i) From the algorithm we used new method replace by the generalized f-projection method which is more general than generalized projection.

(ii) For the problem, we extend the result to a common problem of fixed point problems and equilibrium problems.

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